Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 404 (2005) 380–388
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Ullemar's formula for the moment map, II ☆

V.G. Tkachev

Matematiska Institutionen, Kungliga Tekniska Högskolan, Lindstedtsvägen 25, 10044 Stockholm, Sweden

Received 2 December 2004; accepted 1 March 2005
Available online 22 April 2005
Submitted by H. Schneider

Abstract

We prove the complex analogue of Ullemar's formula for the Jacobian of the complex moment mapping. This formula was previously established in the real case.

Keywords: Moment map; Resultants; Jacobian; Univalent polynomials

1. Introduction

Consider the 'moment map'

$$\mu : \Omega \to (\mu_0, \mu_1, \mu_2, \ldots),$$

where

$$\mu_k = \frac{\mathrm{i}}{2\pi} \iint_{\Omega} \zeta^k \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}, \tag{1}$$

and Ω is a bounded domain in $\mathbb{C}$. If Ω is a simply-connected domain, we can uniformize it as the image $\Omega = \phi(\mathbb{D})$, where ϕ is a unique function which is holomorphic in the unit disk $\mathbb{D}$ and normalized by the following conditions:

$$\phi(0) = 0, \qquad \phi'(0) > 0. \tag{2}$$

☆ The paper was supported by grant RFBR no. 03-01-00304 and grant of the Royal Swedish Academy of Sciences.

E-mail address: tkatchev@math.kth.se
URL: http://www.math.kth.se/~tkatchev

0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.03.012

Then (1) takes the form

$$\mu_k(\phi) = \frac{\mathrm{i}}{2\pi} \iint_{\mathbb{D}} \phi^k(z)|\phi'(z)|^2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{3}$$

In general, when the function ϕ is not globally univalent in $\mathbb{D}$ and satisfies (2), we use the previous formula as the definition for the complex (or analytic) moments of ϕ [2]; then Ω is regarded as a Riemannian surface over $\mathbb{C}$.

This notion appears in several problems of complex analysis and its applications. In particular, the sequence $(\mu_k)_{k\geqslant 1}$ constitutes an infinite family of invariants of the Hele–Shaw problem [4] of the cell Ω and can be used as a canonic coordinate system in the corresponding Laplacian growth model [3] (see also [6] for the functional analysis interpretation).

In what follows, we consider the special case when the moment map μ is restricted to the set $\mathfrak{S}_n \subset \mathbb{C}[z]$ of polynomials of degree n normalized by (2). It is easy to verify that $\mu_k = 0$ for all $k \geqslant n$ (see also (13)), so only the first n moments are of interest for $P \in \mathfrak{S}_n$. We consider the induced finite dimensional map

$$\mu : P = a_1 z + a_2 z^2 + \cdots + a_n z^n \to (\mu_0(P), \ldots, \mu_{n-1}(P)). \tag{4}$$

Since $\mu_0(P) > 0$,

$$\mu : \mathfrak{S}_n \to \mathbb{R}^+ \times \mathbb{C}^{n-1}.$$

Notice that

$$\dim_{\mathbb{R}} \mathbb{R}^+ \times \mathbb{C}^{n-1} = \dim_{\mathbb{R}} \mathfrak{S}_n = 2n - 1,$$

where $\mathfrak{S}_n$ is regarded as an open subset of a linear space. In [2] Gustafsson proved that the Fréchet derivative $\mathrm{d}\mu$ is non-singular at any P which is a locally univalent polynomial.

On the other hand, notice that the subset $\mathfrak{S}_n^{\mathbb{R}}$ which consists of the polynomials with real coefficients is an invariant set of μ in the sense that all $\mu_k(P)$ are real (cf. (14)). Hence, in a similar way the moment map induces the map $\mu_{\mathbb{R}} : \mathfrak{S}_n^{\mathbb{R}} \to \mathbb{R}^n$.

In [7], Ullemar conjectured the following formula for the Jacobian of $\mu_{\mathbb{R}}$:

$$J_{\mathbb{R}}(P) := \frac{\partial(\mu_0, \ldots, \mu_{n-1})}{\partial(a_1, \ldots, a_n)}(P) = 2^{-\frac{n(n-3)}{2}} a_1^{\frac{n(n-1)}{2}} P'(1)P'(-1)\Delta(P'^*(z)), \tag{5}$$

where Δ stands for the principal Hurwitz determinant [1, §15.715], and Q^* denotes the mirror conjugate image of polynomial Q, i.e.

$$Q^*(z) := z^m \bar{Q}(1/z) = \bar{q}_m + \bar{q}_{m-1} z + \cdots + \bar{q}_0 z^m, \quad m = \deg Q, \tag{6}$$

where $\bar{Q}(z) = \overline{Q(\bar{z})}$ is the conjugate polynomial.

The above expression for the Jacobian was recently proved in [5] as a consequence of the following identity:

$$J_{\mathbb{R}}^2(P) = 4(-1)^{n-1} a_1^{n(n-1)} \mathscr{R}(P', P'^*) \cdot P'(-1)P'(1), \tag{7}$$

where $\mathscr{R}(\cdot, \cdot)$ denotes the resultant of the corresponding polynomials.

In this paper we generalize formula (7) for polynomials with arbitrary complex coefficients.

Theorem 1. *The Jacobian of the moment map is expressed as follows*:

$$J_{\mathbb{C}}(P) := \frac{\partial(\bar{\mu}_{n-1}, \ldots, \bar{\mu}_1, \mu_0, \mu_1, \ldots, \mu_{n-1})}{\partial(\bar{a}_n, \ldots, \bar{a}_2, a_1, a_2, \ldots, a_n)} = 2a_1^{n^2-n+1}\mathscr{R}(P', P'^*). \quad (8)$$

A well known theorem of Sylvester (see Section 2.2) allows us to compute the above resultant as the determinant of a matrix of size $2n-2$, whose entries are 0 or a coefficient of either P' or P'^*. In particular, the resultant is homogeneous in the coefficients of P' and P'^* separately, with respective degree $n-1$.

On the other hand, geometrically, the hypersurface

$$\{(a, \bar{a}) : \mathscr{R}(P', P'^*) = 0\},$$

i.e. the critical set of the Jacobian, is the projection of the incidence variety

$$\left\{(a, \bar{a}, z) : \sum_{k=1}^{n} k a_k z^{k-1} = \sum_{k=0}^{n-1} (n-k)\bar{a}_{n-k} z^k\right\},$$

that is to say, the set of $(a, \bar{a})$ which appear above for some z.

The following assertion is a direct consequence of the definition (15) of the resultant and formula (6) above, and it characterizes the set of critical points of $\mathrm{d}\mu$.

Corollary 2. *The moment map is degenerate at P if and only if the derivative P' has two roots α_i and α_j such that $\alpha_i\bar{\alpha}_j = 1$ (the case $i = j$ is permitted).*

Note that for a locally univalent polynomial in the closed unit disk we have $|\alpha_j| < 1$ for all the roots of its derivative. Hence, we obtain another proof of the above result due to Gustafsson [2].

Corollary 3. *The moment map is locally injective on the set of all locally univalent polynomials in the closed unit disk.*

2. Preliminaries

2.1. Complex moments

Using the Stokes formula, we obtain

$$\mu_k = \frac{\mathrm{i}}{2\pi(k+1)} \int_{\partial\Omega} \zeta^{k+1}\,\mathrm{d}\bar{\zeta} = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} \zeta^k \bar{\zeta}\,\mathrm{d}\zeta, \quad (9)$$

which implies

$$\mu_k(\phi) = \frac{\mathrm{i}}{2\pi(k+1)} \int_{\mathbb{T}} \phi^{k+1}(z)\bar{\phi}'(\bar{z})\,\mathrm{d}\bar{z} = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \phi^k(z)\bar{\phi}(\bar{z})\phi'(z)\,\mathrm{d}z,$$

where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. Hence, using the identity $\bar{z} = 1/z$ which holds everywhere in $\mathbb{T}$, we get

$$\begin{aligned}\mu_k(\phi) &= \frac{1}{2\pi\mathrm{i}(k+1)} \int_{\mathbb{T}} \phi^{k+1}(z)\bar{\phi}'(1/z)\frac{\mathrm{d}z}{z^2} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \phi^k(z)\bar{\phi}(1/z)\phi'(z)\,\mathrm{d}z. \end{aligned} \tag{10}$$

Given a function which is analytic in a neighborhood of $\mathbb{T}$, let us denote by $\lambda_s(f)$ the sth Laurent coefficient of f, i.e.

$$f(z) = \sum_{s=-\infty}^{\infty} \lambda_s(f)z^s,$$

hence

$$\mu_k(\phi) = \frac{1}{k+1}\lambda_1(\phi^{k+1}(z)\bar{\phi}'(1/z)) = \lambda_{-1}(\phi^k(z)\phi'(z)\bar{\phi}(1/z)). \tag{11}$$

Now, let P be an arbitrary polynomial in $\mathfrak{S}_n$. Then $\bar{P}'(1/z) = P^{'*}(z)z^{1-n}$ and $\bar{P}(1/z) = P^*(z)z^{-n}$, which by virtue of (11) yields

$$\mu_k(P) = \frac{1}{k+1}\lambda_n(P^{k+1}P^{'*}) = \lambda_{n-1}(P'P^kP^*). \tag{12}$$

It follows from the first identity in (12) and $P(0) = 0$ that

$$\mu_k(P) = 0, \quad k \geqslant n. \tag{13}$$

On the other hand, the second identity in (12) yields the so-called *Richardson formula*

$$\mu_k(P) = \sum s_1 a_{s_1} \cdots a_{s_{k+1}} \bar{a}_{s_1+\cdots+s_{k+1}}, \tag{14}$$

where the sum is taken over all possible sets of indices $s_1, \ldots, s_k \geqslant 1$. It is assumed that $a_j = 0$ for $j \geqslant n+1$. These formulae are easy to use for straightforward manipulations with the complex moments and it follows also that $\mu_k(P)$ is a *polynomial* mapping.

It is convenient to identify $\mathfrak{S}_n$ with the corresponding coefficient subset in $\mathbb{R}^+ \times \mathbb{C}^{n-1}$ in a standard way:

$$a \sim P := a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Since,

$$\mu_0(P) = \sum_{s=1}^{n} s|a_s|^2 > 0, \quad \mu_{n-1}(P) = na_1^n \bar{a}_n \neq 0,$$

the moment map (4) is well defined as an automorphism of $\mathfrak{S}_n$ into itself.

2.2. Resultants

Here we review some basic facts about the resultant; see [8] for a detailed introduction.

The *resultant* of two polynomials

$$A(z) = a_m \prod_{j=1}^{m} (z - \alpha_j), \quad B(z) = b_k \prod_{j=1}^{k} (z - \beta_j)$$

with respect to z is the polynomial

$$\mathscr{R}(A, B) = a_m^k b_k^m \prod_{i,j=1} (\alpha_i - \beta_j). \tag{15}$$

The resultant vanishes iff A and B have a common root. It can be evaluated as the determinant of the *Sylvester matrix*, which is the following $m + k$ by $m + k$ matrix:

$$\begin{pmatrix} a_0 & a_1 & \dots & \dots & a_m & & & & \\ & a_0 & a_1 & \dots & \dots & a_m & & & \\ & & & & \vdots & & & & \\ & & & a_0 & a_1 & \dots & \dots & a_m \\ b_0 & b_1 & \dots & b_k & & & & \\ & b_0 & b_1 & \dots & b_k & & & \\ & & & & \vdots & & & \\ & & & b_0 & b_1 & \dots & \dots & b_k \end{pmatrix}$$

in which the first k rows are the coefficients of A, the next m rows are the coefficients of B, and the elements not shown are all zero. The following are some useful elementary properties we will use below.

$$\begin{aligned} &\mathscr{R}(A, B) = (-1)^{km} \mathscr{R}(B, A), \\ &\mathscr{R}(A_1 A_2, B) = \mathscr{R}(A_1, B)\mathscr{R}(A_2, B), \\ &\mathscr{R}(z^n, A) = A^n(0). \end{aligned} \tag{16}$$

Next, given a polynomial $A(z)$ of degree n, we define its mirror conjugate image as

$$A^*(z) := z^n \bar{A}(1/z) = \bar{a}_n + \bar{a}_{n-1} z + \dots + \bar{a}_0 z^n,$$

where $\bar{A}(z) = \overline{A(\bar{z})}$ is the conjugate polynomial. We have for their roots: $\alpha_j^* = (\bar{\alpha_j})^{-1}$ and the corresponding resultant takes the following form:

$$\mathscr{R}(A, A^*) = \det \begin{pmatrix} a_0 & a_1 & \dots & \dots & a_n & & & & \\ & a_0 & a_1 & \dots & \dots & a_n & & & \\ & & & & \vdots & & & & \\ & & & a_0 & a_1 & \dots & \dots & a_n & \\ \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_0 & & & & \\ & \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_0 & & & \\ & & & & \vdots & & & & \\ & & & \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_0 & \end{pmatrix}. \quad (17)$$

Remark 4. We wish to point out that the latter form, $\mathscr{R}(A, A^*)$, is irreducible as a polynomial of $(a, \bar{a})$ over $\mathbb{C}$. The proof is given in [5, Theorem 6].

3. Proof of the Theorem

First, we evaluate the partial derivative of the moment map. Namely, we have for all $k = 0, \dots, n-1$, $j = 1, \dots, n$, except for $j = 1, k = 0$,

$$\begin{aligned} \frac{\partial \mu_k(P)}{\partial a_j} &= \lambda_{n-j}(P'^* P^k), \\ \frac{\partial \mu_k(P)}{\partial \bar{a}_j} &= \lambda_{j-1}(P' P^k). \end{aligned} \quad (18)$$

In fact, let j be an integer from $\{2, \dots, n\}$. Then by the first identity in (12) we have for

$$\frac{\partial \mu_k(P)}{\partial a_j} = \frac{1}{k+1} \lambda_n \left(P'^* \frac{\partial P^{k+1}}{\partial a_j} \right) = \lambda_n (P'^* P^k z^j) = \lambda_{n-j}(P'^* P^k).$$

Similarly, using $P^* = \bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_2 z^{n-2} + a_1 z^{n-1}$ and the second identity in (12) we obtain

$$\frac{\partial \mu_k(P)}{\partial \bar{a}_j} = \lambda_{n-1} \left(P' P^k \frac{\partial P^*}{\partial \bar{a}_j} \right) = \lambda_{n-1}(P' P^k z^{n-j}) = \lambda_{j-1}(P' P^k).$$

Finally, for $j = 1$ we have by the first identity in (12)

$$\begin{aligned} \frac{\partial \mu_k(P)}{\partial a_1} &= \frac{1}{k+1} \lambda_n \left(P'^* \frac{\partial P^{k+1}}{\partial a_1} + P^{k+1} \frac{\partial P'^*}{\partial a_1} \right) \\ &= \lambda_{n-k}(P'^* P^k) + \frac{1}{k+1} \lambda_1 (P^{k+1}). \end{aligned}$$

But $\lambda_1(P^{k+1}) = 0$ for $k \geqslant 1$, hence the desired assertion follows.

We will make use of the following notation

$$\nabla f := \left(\frac{\partial f}{\partial a_n}, \frac{\partial f}{\partial a_{n-1}}, \dots, \frac{\partial f}{\partial a_2}, \frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial \bar{a}_2}, \dots, \frac{\partial f}{\partial \bar{a}_{n-1}}, \frac{\partial f}{\partial \bar{a}_n}\right),$$

and by

$$q_{j-1} = ja_j$$

we denote the coefficients of the derivative $Q := P'$. Then

$$\nabla\mu_0 = (\bar{q}_{n-1}, \dots, \bar{q}_1, 2q_0, q_1, \dots, q_{n-1}), \tag{19}$$

and for all $k = 1, \dots, n-1$ we have from (18)

$$\begin{aligned}\nabla\mu_k = \big(&\lambda_0(Q^*P^k), \dots, \lambda_{n-2}(Q^*P^k), \lambda_{n-1}(Q^*P^k),\\ &\lambda_1(QP^k), \dots, \lambda_{n-1}(Q^*P^k)\big).\end{aligned} \tag{20}$$

Let $\mathbf{Y}_0 = \nabla\mu_0$ and for $k \geqslant 1$ write

$$\begin{aligned}\mathbf{Y}_k := \big(&\lambda_0(Q^*z^k), \dots, \lambda_{n-2}(Q^*z^k), \lambda_{n-1}(Q^*z^k),\\ &\lambda_1(Qz^k), \dots, \lambda_{n-1}(Q^*z^k)\big).\end{aligned}$$

As a direct consequence of the above formula we conclude that

$$\mathbf{Y}_k = \mathbf{0}, \quad k \geqslant n.$$

Then it follows from (20) and

$$P = z(a_1 + \cdots + a_n z^{n-1})$$

that for all $k \geqslant 1$

$$\nabla\mu_k = a_1^k \mathbf{Y}_k + \sum_{j=k+1}^{n-1} w_{k,j}\mathbf{Y}_j.$$

Thus,

$$\nabla\mu_0 \wedge \nabla\mu_1 \wedge \cdots \wedge \nabla\mu_{n-1} = a_1^N \mathbf{Y}_0 \wedge \mathbf{Y}_1 \wedge \cdots \wedge \mathbf{Y}_{n-1}, \tag{21}$$

where $N = (n-1)n/2$.

On the other hand, for all $k \geqslant 1$ we have

$$\mathbf{Y}_k := (0, \dots, 0, \bar{q}_{n-1}, \dots, \bar{q}_k, 0, \dots, 0, q_0, q_1, \dots, q_{k-1}),$$

where the zeroes groups contain k and $k-1$ items respectively.

Now we treat the conjugate moments. We have $\bar{\mu}_k(P) = \mu_k(\overline{P})$, whence

$$\nabla\bar{\mu}_k = (\nabla\mu_k)^*,$$

where by $\mathbf{X}^*$ we denote the mirror conjugate image of vector $\mathbf{X} = (x_1, x_2, \dots, x_{2n-1})$, i.e.

$$\mathbf{X}^* = (\bar{x}_{2n-1}, \ldots, \bar{x}_2, \bar{x}_1).$$

Repeating the above argument for the conjugate expressions yields

$$\nabla\bar{\mu}_{n-1} \wedge \nabla\bar{\mu}_{n-2} \wedge \cdots \wedge \nabla\bar{\mu}_1 = a_1^N \mathbf{Y}^*_{n-1} \wedge \mathbf{Y}^*_{n-2} \wedge \cdots \wedge \mathbf{Y}^*_1, \tag{22}$$

hence

$$\begin{aligned} &\nabla\bar{\mu}_{n-1} \wedge \cdots \wedge \nabla\bar{\mu}_1 \wedge \nabla\mu_0 \wedge \nabla\mu_1 \wedge \cdots \wedge \nabla\mu_{n-1} \\ &\quad = a_1^{n^2-n} \mathbf{Y}^*_{n-1} \wedge \cdots \wedge \mathbf{Y}^*_1 \wedge \mathbf{Y}_0 \wedge \mathbf{Y}_1 \wedge \cdots \wedge \mathbf{Y}_{n-1}. \end{aligned} \tag{23}$$

We rewrite the latter identity in terms of determinants which gives the following expression for the Jacobian:

$$J_{\mathbb{C}}(P) = \frac{\partial(\bar{\mu}_{n-1}, \ldots, \bar{\mu}_1, \mu_0, \mu_1, \ldots, \mu_{n-1})}{\partial(\bar{a}_{n-1}, \ldots, \bar{a}_1, a_0, a_1, \ldots, a_{n-1})} = a_1^{n^2-n} \det \mathbf{Y}, \tag{24}$$

where

$$\mathbf{Y} = \begin{pmatrix} q_0 & & & & q_{n-1} & & & & & \\ \bar{q}_1 & q_0 & & & q_{n-2} & q_{n-1} & & & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & & \\ \bar{q}_{n-2} & \bar{q}_{n-3} & \cdots & \bar{q}_0 & q_1 & q_2 & q_3 & \cdots & q_{n-1} & \\ \bar{q}_{n-1} & \bar{q}_{n-2} & \cdots & \bar{q}_1 & 2q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \bar{q}_{n-1} & \cdots & \bar{q}_2 & \bar{q}_1 & q_0 & q_1 & \cdots & q_{n-3} & q_{n-2} \\ & & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ & & & \bar{q}_{n-1} & \bar{q}_{n-2} & & & & q_0 & q_1 \\ & & & & \bar{q}_{n-1} & & & & & q_0 \end{pmatrix},$$

and the elements not shown are all zero.

Now, let $\mathbf{X}_j$ denote the jth column in $\mathbf{Y}$. We have for $j = 1, \ldots, n-1$

$$\mathbf{X}_j = (0, \ldots, 0, q_0, \bar{q}_1, \ldots, \bar{q}_{n-1}, 0, \ldots, 0)^{\mathrm{T}},$$

with $j-1$ first zeroes, and for $j = n+1, \ldots, 2n-1$:

$$\mathbf{X}_j = (0, \ldots, 0, q_{n-1}, \ldots, q_1, q_0, 0, \ldots, 0)^{\mathrm{T}},$$

with $j-n$ first zeroes, and

$$\mathbf{X}_n = (q_{n-1}, \ldots, q_1, 2q_0, \bar{q}_1, \ldots, \bar{q}_{n-1})^{\mathrm{T}}.$$

One can readily verify that

$$\mathbf{X}_n - \sum_{j=1}^{n-1} \frac{q_{n-j}}{q_0} \mathbf{X}_j + \sum_{j=n+1}^{2n-1} \frac{\overline{q}_{j-n}}{q_0} \mathbf{X}_j = (0, \ldots, 0, 2q_0, 2\bar{q}_1, \ldots, 2\bar{q}_{n-1})^{\mathrm{T}},$$

which yields for the determinant

$$\det \mathbf{Y} = 2 \det \begin{pmatrix} q_0 & & & & & & & & & \\ \bar{q}_1 & q_0 & & & & q_{n-1} & & & & \\ \vdots & \vdots & \ddots & & & \vdots & & \ddots & & \\ \bar{q}_{n-1} & \bar{q}_{n-2} & \cdots & \bar{q}_1 & q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \bar{q}_{n-1} & \cdots & \bar{q}_2 & \bar{q}_1 & q_0 & q_1 & \cdots & q_{n-3} & q_{n-2} \\ & & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ & & & \bar{q}_{n-1} & \bar{q}_{n-2} & & & & q_0 & q_1 \\ & & & & \bar{q}_{n-1} & & & & & q_0 \end{pmatrix}. \qquad (25)$$

The latter is the transposed Sylvester matrix of $Q^*(z)$ and $zQ(z)$, hence by (16)

$$\begin{aligned} \det \mathbf{Y} &= 2\mathscr{R}(Q^*, zQ) = 2(-1)^{n(n-1)}\mathscr{R}(zQ, Q^*) \\ &= 2\mathscr{R}(z, Q^*)\mathscr{R}(Q, Q^*) = 2Q(0)\mathscr{R}(Q, Q^*). \end{aligned} \qquad (26)$$

Thus, using our notation $Q = P'$ we arrive at

$$J_{\mathbb{C}}(P) = 2a_1^{n^2-n+1}\mathscr{R}(P', P'^*),$$

which completes the proof. □

Acknowledgments

The author wish to thank anonymous referees for valuable comments and suggestions.

References

[1] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products, sixth ed., Academic Press, San Diego, CA, 2000.
[2] B. Gustafsson, On a differential equation arising in a Hele–Shaw flow moving boundary problem, Ark. för Mat. 22 (1984) 251–268.
[3] I. Krichever, M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin, Laplacian growth and Whitham equations of soliton theory (arXiv: nlin.SI/0311005).
[4] S. Richardson, Hele–Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, J. Fluid Mech. 56 (1972) 609–618.
[5] O.S. Kuznetsova, V.G. Tkachev, Ullemar's formula for the Jacobian of the complex moment mapping, Complex Variables Appl. 49 (1) (2004) 55–72.
[6] M. Putinar, Linear analysis of quadrature domains, III, J. Math. Anal. Appl. 239 (1999) 101–117.
[7] C. Ullemar, Uniqueness theorem for domains satisfying quadrature identity for analytic functions, TRITA-MAT 1980-37, Mathematics, 1980 (Preprint of Royal Inst. of Technology, Stockholm).
[8] B.L. van der Waerden, Modern Algebra, vol. 1, Springer, Berlin, 1971.