Existence of equilibria in articulated bearings in presence of cavity

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Abstract

The existence of equilibrium solutions for a lubricated system consisting of an articulated body sliding over a flat plate is considered. Here we consider the case when cavity can occur.

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1. Introduction

This work is a continuation of a recent paper [1] treating the existence of equilibrium in a tilting-pad thrust bearings with cavitation effects disregarded.

Articulated sliders have two degrees of freedom: The first one is the vertical displacement $a$ under the effect of a force $F$, applied at a position $x_0$, and of the pressure load $\int_\Omega p \, dx$ where $p = p(x)$ is the pressure of the fluid between the surfaces of the slider. The second degree of freedom is the tilt (or pitch) angle $\theta$ (see Fig. 1) (see [1] for more explanations on the physical model).
In [1] we treated the case in which \( h \), the non-dimensional distance between the surfaces, is non-increasing with \( x \). This guarantees positivity of the pressure, \( p \), on all the domain \( \Omega \). Thus \( p \) satisfies the Reynolds equation
\[
\nabla \cdot \left[ h^3(x) \nabla p \right] = \frac{\partial h}{\partial x_1}, \quad x \in \Omega,
\]
\[
p = 0, \quad x \in \partial \Omega.
\]
(1.1)

In the present work we consider that the system can have a situation with \( h \) non-increasing firstly and then non-decreasing (convergent–divergent case). In such situation the solution of (1.1) is not always non-negative and we must replace Eq. (1.1) by the corresponding variational inequation. In all this paper we restrict our study to the one-dimensional case.

Thus the equilibrium problem becomes: find \((p, a, \theta) \in K \times \mathbb{R} \times \mathbb{R}\) satisfying
\[
\int_{\Omega} h^3 \frac{dp}{dx} \frac{d}{dx} \left( \varphi - p \right) \geq \int_{\Omega} h \frac{d}{dx} \left( \varphi - p \right) \frac{d}{dx} \left( \varphi - p \right), \quad \forall \varphi \in K,
\]
(1.2)
\[
\int_{\Omega} p = F,
\]
(1.3)
\[
\int_{\Omega} xp = F x_0,
\]
(1.4)
\[
h(x) = h_0(x) + a + \theta x,
\]
(1.5)
with \( \Omega = ]-1, 1[ \), \( K = \{ \varphi \in H^1_0(\Omega), \ \varphi \geq 0 \} \), \( x_0 \in \Omega \), \( F > 0 \) and \( h_0 : \Omega \to \mathbb{R} \) a given function.

We consider the following hypothesis on the function \( h_0 \):
\[
\begin{aligned}
& h_0 \in C^2(\hat{\Omega}), \quad h_0'' > 0, \quad \forall x \in \Omega, \\
& h_0'' \text{ is a non-decreasing function,} \\
& h_0' \leq 0, \quad \forall x \in \hat{\Omega}, \quad h_0(0) = h_0'(0) = 0.
\end{aligned}
\]
(1.6)
A typical example of such a function is \( h_0(x) = (1 - x)^2 \).

In [1] we proved the existence of at least a solution of (1.2)–(1.5) with \( \theta < 0 \) in which case the variational inequality (1.2) becomes the Reynolds equation. The result proved in [1] says that for any \( F > 0 \) there exists a solution of (1.2)–(1.5) with \( \theta < 0 \) provided that the articulation point \( x_0 \) is situated not far from the right end side \( (x = 1) \) of \( \Omega \).

We then can hope to improve the result by enlarging the domain where we search for the unknown \( \theta \); it is clear that for \( \theta \geq -h'_0(-1) \) the unique solution \( p \) of (1.2) is \( p \equiv 0 \) then (1.3) and (1.4) are impossible.

In the present paper we prove that for any \( F > 0 \) and any \( x_0 \in \Omega \) there exists at least a solution of the problem (1.2)–(1.5) with \( \theta < -h'_0(-1) \); this improves the result in [1] in the one-dimensional case.

Nevertheless, this stronger result is obtained under the stronger hypothesis (1.6) than that of [1].

In Section 2 we give some preliminary results on the behaviour of the solution of a variational inequality with respect to a parameter.

In Section 3 we give the desired existence result by reducing our problem to an algebraic equation with an unknown \( \theta \).

2. Preliminaries

In this section we shall give some preliminary results concerning the behaviour of the mapping: \( G : A \in ]0, +\infty[ \rightarrow \int_{\Omega} P(x) \, dx \in ]0, +\infty[ \) with \( P = P(A) \) is the unique solution of the Reynolds variational inequality

\[
\begin{cases}
P \in K, \\
\int_{\Omega} (H_0 + A)^3 \frac{d}{dx} \frac{d}{dx} (\varphi - P) \geq \int_{\Omega} H_0 \frac{d}{dx} (\varphi - P), \quad \forall \varphi \in K,
\end{cases}
\]

where \( H_0 : \Omega \rightarrow ]0, +\infty[ \) is a given bounded non-negative function.

We shall consider two principal situations:

- The function \( H_0 \) is assumed decreasing in \( \Omega \). In this case the variational inequality becomes an equality (Reynolds equation (1.1) with \( h \) replaced by \( H_0 + A \)).
- The function \( H_0 \) is assumed decreasing in \([-1, d]\) and increasing in \([d, 1]\) for some number \( d \in \Omega \). In this case we must consider the general Reynolds variational inequality.

We give now the first general result on the behaviour of \( G(A) \) when \( G \) goes to infinity.

**Proposition 2.1.** Suppose that \( H_0 \) is bounded and non-negative. Then

\[
\lim_{A \to +\infty} G(A) = 0.
\]

More precisely, we have

\[
G(A) \leq \frac{\sqrt{2}}{2} \frac{\|H_0\|_{L^2(\Omega)}}{A^3}.
\]
Proof. Taking $\varphi = 0$ in (2.1) and since $H_0$ is non-negative we obtain

$$A^3 \int_{\Omega} \left( \frac{dP}{dx} \right)^2 \leq \|H_0\|_{L^2(\Omega)} \left\| \frac{dP}{dx} \right\|_{L^2(\Omega)}$$

which gives easily the result by Poincaré inequality. \qed

In this section we are mainly interested on the behaviour of $G(A)$ when $A$ tends to 0 and on the monotony of $G$. These kind of problems have been considered in [1,2] in the context of Reynolds equation only. We recall these results in the next subsection and then we extend them to the case of Reynolds variational inequality.

2.1. Equation case

In this subsection the following hypothesis on $H_0$ will be considered:

$$\begin{align*}
H_0 &\in W^{1,\infty}(\Omega), \quad \min_{x \in \bar{\Omega}} H_0(x) = 0, \\
H_0'(x) &\leq 0 \quad \text{a.e. } x \in \Omega, \quad \text{mes}\{x \in \Omega: H_0'(x) < 0\} > 0
\end{align*}$$

and we remark that in this case the problem (2.1) becomes

$$\begin{align*}
\frac{d}{dx} \left[ (H_0 + A)^3 \frac{dP}{dx} \right] &= \frac{dH_0}{dx}, \quad x \in \Omega = ]-1, 1[, \\
P(-1) &= P(1) = 0.
\end{align*}$$

We now recall a result proved in [1,2]. This result claims that $G(A)$ goes to infinity when $A$ goes to 0 if $-H_0'(x)$ is equivalent to $(1 - x)^{\alpha}$ with $\alpha > 0$ in a neighbourhood of $x = 1$.

Proposition 2.2. Assume, in addition to hypothesis (2.2), that there exist $0 < M_0 < M_1$, $\alpha \geq 0$ and $\delta_0 \in ]0, 1[$ such that

$$M_0(1-x)^{\alpha} \leq -H_0'(x) \leq M_1(1-x)^{\alpha} \quad \text{a.e. } x \in ]1 - 2\delta_0, 1[. \quad (2.4)$$

Then there exists $c = c(M_0, M_1, \delta_0, \alpha)$ such that the solution $P$ of (2.3) satisfies

$$\int_{1-B}^1 P(x) \, dx \geq c \left( \log \left( 1 + M_1 \frac{B}{A} \right) - 2 \right), \quad \forall B \in ]0, \delta_0], \; \forall A > 0, \text{ if } \alpha = 0,$n

and

$$\int_{1-B}^1 P(x) \, dx \geq cA^{\frac{2}{\alpha}-2}, \quad \forall B \in ]0, \delta_0], \; \forall A < B^{\alpha}, \text{ if } \alpha > 0.$$

Corollary 2.3. Under hypotheses of Proposition 2.2, we have

$$\lim_{A \to 0} G(A) = +\infty.$$

We now give the following technical result.
Lemma 2.4. Under the hypothesis (2.2) and in addition
\[ \Theta + 2M_0(1 - x) \leq -H_0'(x) \leq \Theta + 2M_1(1 - x), \quad \forall x \in [1 - 2\delta_0, 1], \] (2.5)
with \( 0 < \Theta \leq 1, \) \( 0 < M_0 < M_1 \) and \( \delta_0 \in ]0, 1[, \) then there exists \( C = C(M_0, M_1, \delta_0) > 0 \) such that
\[ \int_{\Omega} P \, dx \geq \frac{C}{A + \Theta^2} \quad \text{for all } A > 0, \quad \text{such that } A + \Theta^2 < \delta_0^2. \]

Proof. We follow the lines of the proof of Proposition 2.2.

Step 1: We take \( \phi \in C^2(\hat{\Omega}) \) with \( \phi \equiv 1 \) on \([1 - \delta_0, 1] \) and \( \phi(1 - 2\delta_0) = 0. \) We first show that there exists \( C_1 > 0 \) depending only on \( M_0, M_1 \) and \( \delta_0 \) such that
\[ P(x) \succeq C_1 q_1(x) \phi(x), \quad \forall x \in [1 - 2\delta_0, 1], \]
with
\[ q_1(x) = \frac{\Theta(1 - x)^2 + M_0(1 - x)^3}{[\Theta(1 - x) + M_1(1 - x)^2 + A]^3}. \]

Proof of Step 1. We apply the maximum principle on \([1 - 2\delta_0, 1] \). Since \( q_1(1) = 0 \) and \( \phi(1 - 2\delta_0) = 0 \), it suffices to prove that
\[ -C_1 \frac{d}{dx} \left[ (A + H_0)^3 \frac{d(q_1\phi)}{dx} \right] \leq -H_0'(x), \quad \forall x \in [1 - 2\delta_0, 1]. \]
According to (2.5), it is enough to show the existence of \( C_2 > 0 \) dependent on \( M_0, M_1 \) and \( \delta_0 \) such that
\[ -\frac{d}{dx} \left[ (A + H_0)^3 \frac{d(q_1\phi)}{dx} \right] \leq C_2(\Theta + 2M_0(1 - x)). \]
By developing, it is enough to show
\[ I_k \leq \frac{C_2}{5} (\Theta + 2M_0(1 - x)), \quad k = 1, 2, \ldots, 5, \] (2.6)
with
\[
\begin{align*}
I_1 &= 3 |(A + H_0)^2 H_0' q_1' \phi|, \\
I_2 &= 3 |(A + H_0)^2 H_0' \phi' q_1|, \\
I_3 &= |(A + H_0)^3 q_1'' \phi|, \\
I_4 &= 2 |(A + H_0)^3 q_1' \phi'|, \\
I_5 &= |(A + H_0)^3 q_1 \phi''|. 
\end{align*}
\]
Some elementary calculations yields
\[
\begin{align*}
|q_1'(x)| &\leq C_3 \frac{\Theta(1 - x) + M_0(1 - x)^2}{[\Theta(1 - x) + M_1(1 - x)^2 + A]^3}, \quad \forall x \in [1 - \delta_0, 1[, \quad (2.7) \\
|q_1''(x)| &\leq C_4 \frac{\Theta + 1 - x}{[\Theta(1 - x) + M_1(1 - x)^2 + A]^3}, \quad \forall x \in [1 - \delta_0, 1[, \quad (2.8)
\end{align*}
\]
with \( C_3, C_4 > 0 \) depending only on \( M_0 \) and \( M_1 \).
Now, integrating (2.5) between $x$ and 1 with $x \in ]-1, 1[$ and adding $A$ we obtain
\[ A + \Theta(1 - x) + M_0(1 - x)^2 \leq A + H_0(x) \leq A + \Theta(1 - x) + M_1(1 - x)^2, \]
$\forall x \in [1 - 2\delta_0, 1[$. (2.9)

Using now the expression of $q_1$, the inequalities (2.7)–(2.9) and the hypothesis (2.5), we easily deduce (2.6) which ends the proof of Step 1. □

**Step 2:** We prove the desired inequality. Using the elementary inequality $\Theta(1 - x) \leq (1 - x)^2 + \Theta^2$, we deduce
\[ \Theta(1 - x) + M_1(1 - x)^2 + A \leq (M_1 + 1)(1 - x)^2 + A + \Theta^2. \]

From the non-negativity of $P$ and Step 1 we deduce
\[ \int_{\Omega} P \geq \int_{1 - \delta_0}^{1} P \geq C_1 \int_{1 - \delta_0}^{1} q_1(x) \, dx, \]
then
\[ \int_{\Omega} P \geq C_1 M_0 \int_{1 - \delta_0}^{1} \frac{(1 - x)^3 \, dx}{((M_1 + 1)(1 - x)^2 + A + \Theta^2)^3}. \]

Let $\bar{x} \in ]1 - \delta_0, 1[$ such that $1 - \bar{x} = \sqrt{A + \Theta^2}$. Then
\[ \int_{\Omega} P \geq \frac{C_1 M_0}{(M_1 + 2)^3(A + \Theta^2)^3} \int_{\bar{x}}^{1} (1 - x)^3 \, dx \]
\[ = \frac{C_1 M_0}{4(M_1 + 2)^3(A + \Theta^2)^3} (A + \Theta^2)^2, \]
which ends the proof. □

We finally give the regularity and monotonicity result which is proved in [1].

**Proposition 2.5.** The function $G$ is of class $C^\infty$ and $G'(A) < 0$, $\forall A > 0$.

**2.2. Inequality case**

In all this subsection the following hypothesis on the given function $H_0$ is considered
\[ \begin{cases} H_0 \in C^2(\tilde{\Omega}) & \text{and} & H_0''(x) > 0, \quad \forall x \in \Omega, \\ \text{and there exists } d \in ]-1, 1[ & \text{such that} & H_0(d) = H_0'(d) = 0, \\ H_0'(x) < 0 & \text{for } x < d & \text{and} \quad H_0'(x) > 0 & \text{for } x > d. \end{cases} \] (2.10)

The hypothesis (2.10) implies $H_0(x) > 0$ for any $x \in \Omega - \{d\}$. We recall that we consider $P$ (depending on $A$) the solution of (2.1). The analogous of Proposition 2.2 is the following
Proposition 2.6. Let $d_0 \in ]-1, 1[$. Then there exists $C > 0$ depending only on $d_0$, $\min_{x \in \Omega} H_0''(x)$ and $\max_{x \in \Omega} H_0''(x)$ such that for any $d > d_0$, we have

$$\int_{\Omega} P(x) \, dx \geq \frac{C}{A} \text{ for any } A \text{ with } 0 < A < \left( \frac{d_0 + 1}{2} \right)^2.$$  

Proof. From results of [3], we have that $P > 0$ on $]-1, d[$, and $P$ satisfies

$$\frac{d}{dx} \left( (H_0 + A)^3 \frac{dP}{dx} \right) = \frac{dH_0}{dx}, \quad \forall x \in ]-1, d[.$$  

Since $P(-1) = 0$ and $P(d) \geq 0$, we have by maximum principle

$$P \geq q, \quad \forall x \in [-1, d],$$  

with $q$ the solution of the problem

$$\begin{cases}
\frac{d}{dx} \left( (H_0 + A)^3 \frac{dq}{dx} \right) = \frac{dH_0}{dx}, & \forall x \in ]-1, d[, \\
q(-1) = q(d) = 0.
\end{cases}$$  

(2.11)

Using now the Taylor development of $H_0'(x)$ around $d$ we easily obtain

$$\frac{1}{2} \min_{y \in [-1,d[} H_0''(y)(d-x) \leq H_0'(x) \leq \frac{1}{2} \min_{y \in [-1,d[} H_0''(y)(d-x) \text{ for any } x \in ]-1, d[.$$  

(2.12)

By an affine a change of variables:

$$x' \in ]-1, 1[ \rightarrow x \in ]-1, d[,$$

$$x' \rightarrow x = \frac{d + 1}{2} x' + \frac{d - 1}{2},$$  

(2.13)

the problem (2.11) can be written in $]-1, 1[$ and we can easily see that the hypothesis (2.4) is satisfied with $\alpha = 1$. Applying Proposition 2.2 we deduce the result since $P \geq 0$ on $\Omega$. □

Corollary 2.7.

$$\lim_{A \to 0} G(A) = +\infty.$$  

We now study the monotony of $G(A)$. From the results of [3] and assumptions (2.10), we have two possibilities:

- $P(x) > 0, \forall x \in \Omega$, and $P$ satisfies

$$\frac{d}{dx} \left( (H_0 + A)^3 \frac{dP}{dx} \right) = \frac{dH_0}{dx}, \quad \forall x \in \Omega,$$

$$P(-1) = P(1) = 0.$$  

(2.14)

- There exists $\beta \in ]d, 1[$ such that $P(x) > 0, \forall x \in ]-1, \beta[$, $P(\beta) = P'(\beta) = 0$ and $P(x) = 0$, $\forall x \in [\beta, 1]$. 

In this case \( P \) satisfies the problem
\[
\frac{d}{dx} \left( (H_0 + A)^3 \frac{dP}{dx} \right) = \frac{dH_0}{dx}, \quad \forall x \in ]-1, \beta[,
\]
\[
P(-1) = P(\beta) = 0,
\]
\[
P'(\beta) = 0,
\]
\[
P(x) = 0, \quad \forall x \in [\beta, 1].
\]
(2.15)

We remark that \( \beta \) is here an unknown. By integrating (2.15), we obtain
\[
(H_0 + A)^3 \frac{dP}{dx} = H_0 - C
\]
and \( P'(\beta) = 0 \) gives \( C = H_0(\beta) \).

We then deduce
\[
P'(x) = \frac{1}{(H_0(x) + A)^2} - \frac{H_0(\beta) + A}{(H_0(x) + A)^3}.
\]

Due to \( P(-1) = 0 \) we obtain
\[
P(x) = \int_{-1}^{x} \frac{ds}{(H_0(s) + A)^2} - \left( H_0(\beta) + A \right) \int_{-1}^{x} \frac{ds}{(H_0(s) + A)^3}.
\]
(2.16)

Finally the condition \( P(\beta) = 0 \), allows us to obtain a scalar equation satisfied by \( \beta \). It is then natural to introduce the following application:
\[
\psi : [-1, 1] \times ]0, +\infty[ \to \mathbb{R},
\]
defined by
\[
\psi(\beta, A) = \int_{-1}^{\beta} \frac{ds}{(H_0(s) + A)^2} - \left( H_0(\beta) + A \right) \int_{-1}^{\beta} \frac{ds}{(H_0(s) + A)^3}.
\]
(2.17)

Then for any \( A \), the problem (2.15) reduces to the existence of \( \beta \in ]d, 1[ \) such that \( \psi(\beta, A) = 0 \).

It is clear that \( \psi \in C^2([-1, 1[ \times ]0, +\infty[) \) and we have
\[
\frac{\partial \psi}{\partial \beta}(\beta, A) = -H'_0(\beta) \int_{-1}^{\beta} \frac{ds}{(H_0(s) + A)^3}.
\]
(2.18)

Hypotheses (2.10) imply that for any \( A \), \( \max_{\beta \in [-1, 1]} \psi(\beta, A) = \psi(d, A) \), \( \frac{\partial \psi}{\partial \beta} > 0 \) for \( \beta \in ]-1, d[ \) and \( \frac{\partial \psi}{\partial \beta} < 0 \) for \( \beta \in ]d, 1[ \).

Since \( \psi(-1, A) = 0 \) and from the above considerations, for any \( A > 0 \) the existence and uniqueness of a solution of (2.15) is equivalent to the condition
\[
\psi(1, A) < 0.
\]

Then we have proved:

**Proposition 2.8.** If \( \psi(1, A) < 0 \), then there exists \( \beta = \beta(A) \in ]d, 1[ \) such that \( P \) satisfies (2.15).

If \( \psi(1, A) \geq 0 \), then \( P \) satisfies (2.14).
The next lemma is useful in the study of the monotony of $G$.

**Lemma 2.9.** Assume that $H_0''$ is non-decreasing. Let $v \in ]d, 1]$ such that $H_0(-1) > H_0(v)$ and $\phi : [0, +\infty[ \to ]0, +\infty[$ an arbitrary continuous function.

Let $q$ be the solution of the problem

$$
\begin{align*}
\frac{d}{dx} \left( \phi(H_0) \frac{dq}{dx} \right) &= \frac{dH_0}{dx}, \quad \forall x \in ]-1, v[,
q(-1) &= q(v) = 0.
\end{align*}
$$

Then

$$
\int_{-1}^{v} q(x) \, dx > 0.
$$

**Remark 2.10.** This result can be seen as a kind of “maximum principle” in the sense: if $\int_{-1}^{1} \frac{dH_0}{dx} < 0$, then $\int_{-1}^{1} q \, dx > 0$.

**Proof.** By integrating (2.19) there exists $c \in \mathbb{R}$ such that

$$
\phi(H_0)q' = H_0 - c. \tag{2.20}
$$

Due to the assumptions (2.10), three different cases are possible:

**Case 1:** $q'(x) \leq 0$, $\forall x \in ]-1, v[$ or $q'(x) \geq 0$, $\forall x \in ]-1, v[$ with $\text{mes}\{x: q'(x) = 0\} = 0$; this is impossible because $q(-1) = q(v) = 0$.

**Case 2:** There exists $v_1 \in ]-1, v[$ such that

$$
\begin{align*}
q'(x) &= 0 \quad \text{for} \quad x \in ]-1, v_1[,
q'(v_1) &= 0.
\end{align*}
$$

It is clear that $q(x) > 0$, $\forall x \in ]-1, v[$ which proves the result.

**Case 3:** There exists $v_1 \in ]-1, d[ \text{ and } v_2 \in ]d, v[ \text{ such that }

$$
\begin{align*}
q'(x) &= 0 \quad \text{if} \quad x \in ]-1, v_1[ \cup ]v_2, v[,
q'(v_1) &= q'(v_2).
\end{align*}
$$

Let $v_* \in ]-1, v[$ such that $H_0(v_*) = H_0(v)$. It is clear that $v_* < v_1$.

We have:

$$
\int_{-1}^{v} H_0''(x)q(x) \, dx = - \int_{-1}^{v} H_0'q' \, dx = - \int_{-1}^{v_*} H_0'q' \, dx - \int_{v_*}^{v} H_0'q' \, dx. \tag{2.21}
$$

From (2.20) and using also $H_0(v_*) = H_0(v)$ we deduce

$$
\int_{v_*}^{v} H_0'q' \, dx = 0.
$$
We then easily obtain from (2.21)
\[ \int_{-1}^{v} H''_0 q(x) \, dx > 0. \]  
(2.22)

Let \( v_3 \in ]v_1, v_2[ \) such that \( q(v_3) = 0, \) \( q(x) > 0 \) if \( x \in ]-1, v_3[ \) and \( q(x) < 0 \) if \( x \in ]v_3, 1[ \).

Since \( H''_0 \) is non-decreasing, we have
\[ \int_{-1}^{v} q(x) \, dx > \frac{1}{H''_0(v_3)} \int_{-1}^{v_3} H''_0(x)q(x) \, dx + \frac{1}{H''_0(v_3)} \int_{v_3}^{v} H''_0(x)q(x) \, dx. \]

Together with (2.22) this proves the claimed result. \( \square \)

We have a first result concerning the monotony of \( G \).

**Proposition 2.11.** Suppose that \( H''_0 \) is non-decreasing and \( A > 0 \) such that \( \psi(1, A) \neq 0 \). Then \( G \) is derivable in \( A \) and we have
\[ \frac{\partial G}{\partial A}(A) < 0. \]

**Proof.** Due to Proposition 2.8, we distinguish two cases following the sign of \( \psi(1, A) \).

**Case 1:** \( \psi(1, A) > 0. \) Under this hypothesis \( P \) satisfies (2.14) (equation case).

The continuity of \( \psi \) implies
\[ \psi(1, A') > 0 \quad \text{for any } A' \text{ in a neighborhood of } A. \]

From the Implicit Function Theorem, the solution \( P \) is derivable in \( A \).

Denoting \( q_1 = \frac{\partial P}{\partial A} \), we have \( G'(A) = \int_{\Omega} q_1(x) \, dx \) where \( q_1 \) is the solution of the problem
\[ \frac{d}{dx}\left((H_0 + A)^3 \frac{dq_1}{dx}\right) = -3 \frac{d}{dx}\left((H_0 + A)^2 \frac{dP}{dx}\right), \quad \forall x \in \Omega, \]
\[ q_1(-1) = q_1(1) = 0. \]  
(2.23)

By integrating (2.23) there exists a constant \( C_1(A) \in \mathbb{R} \) such that
\[ (H_0 + A)^3 \frac{dq_1}{dx} = -3(H_0 + A)^2 \frac{dP}{dx} + C_1(A). \]  
(2.24)

Multiplying (2.24) by \( H_0 + A \) and differentiating in \( x \) we obtain
\[ \frac{d}{dx}\left((H_0 + A)^4 \frac{dq_1}{dx}\right) = -3 \frac{d}{dx}\left((H_0 + A)^3 \frac{dP}{dx}\right) + C_1(A) \frac{dH_0}{dx}. \]

Using (2.14) we deduce
\[ \frac{d}{dx}\left((H_0 + A)^4 \frac{dq_1}{dx}\right) = (C_1(A) - 3) \frac{dH_0}{dx}. \]

By linearity we can write
\[ q_1 = (C_1(A) - 3) \hat{q}_1 \]  
(2.25)

with \( \hat{q}_1 \) solution of
\[
\frac{d}{dx} \left( (H_0 + A)^4 \frac{d\hat{q}_1}{dx} \right) = \frac{dH_0}{dx}, \quad \forall x \in \Omega,
\]
\[
\hat{q}_1(-1) = \hat{q}_1(1) = 0.
\]

(2.26)

We now need to know the sign of \( C_1(A) - 3 \) and of \( \int_{-1}^{1} \hat{q}_1(x) \, dx \).

Due to (2.24) we obtain
\[
\frac{dq_1}{dx} = -3 \frac{dP}{dx} + \frac{C_1(A)}{(H_0 + A)^3}
\]
and due to \( q_1(-1) = q_1(1) = 0 \), we have
\[
C_1(A) = 3 \frac{\int_{-1}^{1} \frac{dP}{dx} (H_0 + A)^{-1}}{\int_{-1}^{1} (H_0 + A)^{-3}}.
\]

(2.27)

Integrating (2.14) there exists \( C_2(A) \in \mathbb{R} \) such that
\[
(H_0 + A)^3 \frac{dP}{dx} = H_0 + A - C_2(A).
\]

(2.28)

Dividing by \( (H_0 + A)^4 \) and integrating we have
\[
\int_{-1}^{1} \frac{dP}{dx} (H_0 + A)^{-1} \, dx = \int_{-1}^{1} (H_0 + A)^{-3} - C_2(A) \int_{-1}^{1} (H_0 + A)^{-4}.
\]

Substituting in (2.27) we obtain
\[
C_1(A) - 3 = -3C_2(A) \frac{\int_{-1}^{1} (H_0 + A)^{-4}}{\int_{-1}^{1} (H_0 + A)^{-3}}.
\]

Dividing (2.28) by \( (H_0 + A)^3 \), integrating in \( \Omega \) and using \( P(-1) = P(1) = 0 \) we deduce \( C_2(A) > 0 \) which gives
\[
C_1(A) < 3.
\]

(2.29)

Let us now prove by absurd that
\[
H_0(1) < H_0(-1).
\]

(2.30)

Assuming that \( \max_{s \in [-1,1]} H_0(s) = H_0(1) \) we deduce
\[
(H_0(1) + A) \int_{-1}^{1} \frac{ds}{(H_0(s) + A)^3} \geq \int_{-1}^{1} \frac{ds}{(H_0(s) + A)^2}
\]
which will contradict hypothesis \( \psi(1, A) > 0 \). This implies (2.30).

We now apply Lemma 2.9 with \( \nu = 1 \) and \( \phi(z) = (z + A)^4 \) and we deduce that the solution \( \hat{q}_1 \) of (2.26) satisfies
\[
\int_{\Omega} \hat{q}_1 \, dx > 0.
\]

With (2.25) and (2.29) we obtain the result in this case.
Case 2: \( \psi(1, A) < 0 \). We have \( \psi(1, A') < 0 \) for any \( A' \) in a neighborhood of \( A \). We deduce that \( P = P(A') \) satisfies (2.15) with \( A \) replaced by \( A' \) and \( \beta = \beta(A') \in ]d, 1[ \).

From (2.18) we obtain \( \frac{d\psi}{d\beta}(\beta(A), A) < 0 \) because \( \beta(A) > d \).

Then the Implicit Function Theorem implies that the application \( \beta \) is of class \( C^1 \) in \( A \).

An elementary calculus gives

\[
\frac{d\psi}{dA}(\beta, A) = -3 \int_{-1}^{\beta} \frac{ds}{(H_0(s) + A)^3} + 3(H_0(\beta) + A) \int_{-1}^{\beta} \frac{ds}{(H_0(s) + A)^4} \tag{2.31}
\]

and

\[
\beta'(A) = 3 \frac{(H_0(\beta) + A) \int_{-1}^{\beta} (H_0(s) + A)^{-4} ds - \int_{-1}^{\beta} (H_0(s) + A)^{-3} ds}{H_0'(\beta) \int_{-1}^{\beta} (H_0(s) + A)^{-3} ds}. \tag{2.32}
\]

On the other hand we have

\[
G(A) = \int_{-1}^{\beta} P(x) dx
\]

with \( P = P(A) \) given by (2.16) and \( \beta = \beta(A) \).

It is clear that \( G \) is derivable in \( A \) and since \( P(\beta) = 0 \), an elementary calculus gives

\[
G'(A) = \int_{-1}^{\beta} q_2(x) dx \tag{2.33}
\]

with \( q_2 : [-1, \beta] \to \mathbb{R} \) given by

\[
q_2(x) = -(3 + H_0'(\beta) \beta'(A)) \int_{-1}^{x} (H_0(s) + A)^{-3} ds + 3(H_0(\beta) + A) \int_{-1}^{x} (H_0(s) + A)^{-4} ds.
\]

Differentiating in \( x \) and multiplying by \((H_0(s) + A)^4\) we obtain that \( q_2 \) satisfies

\[
\begin{cases}
\frac{d}{dx} \left[ (H_0 + A)^4 \frac{dq_2}{dx} \right] = -\left[ 3 + H_0'(\beta) \beta'(A) \right] \frac{dH_0}{dx}, \\
q_2(-1) = q_2(\beta) = 0,
\end{cases} \tag{2.34}
\]

where \( q_2(\beta) = 0 \) is a direct consequence of (2.32).

By linearity we have as in Case 1

\[
q_2 = -\left[ 3 + H_0'(\beta) \beta'(A) \right] \hat{q}_2, \tag{2.35}
\]

where \( \hat{q}_2 \) satisfies

\[
\begin{cases}
\frac{d}{dx} \left[ (H_0 + A)^4 \frac{d\hat{q}_2}{dx} \right] = \frac{dH_0}{dx}, & \forall x \in ]-1, \beta[, \\
\hat{q}_2(-1) = \hat{q}_2(\beta) = 0.
\end{cases} \tag{2.36}
\]
On the other hand, from (2.32) we obtain
\[ 3 + H_{0}'(\beta)\beta'(A) = 3\left(H_{0}(\beta) + A\right)\frac{\int_{-1}^{\beta}(H_{0} + A)^{-4}}{\int_{-1}^{\beta}(H_{0} + A)^{-3}} \]
which implies
\[ 3 + H_{0}'(\beta)\beta'(A) > 0. \tag{2.37} \]
We now use Lemma 2.9 in order to find the sign of \( \int_{-1}^{1} \hat{q}_{2}(x) \, dx \).
Let us first prove by absurd that
\[ H_{0}(\beta) < H_{0}(-1). \tag{2.38} \]
Assume that \( \max\{1,\beta\} \, H_{0} = H_{0}(\beta) \). This implies
\[ (H_{0}(\beta) + A) \int_{-1}^{\beta} \frac{ds}{(H_{0}(s) + A)^{3}} > \int_{-1}^{\beta} \frac{ds}{(H_{0}(s) + A)^{2}} \]
which contradicts the equality that \( \psi(\beta, A) = 0 \), which shows (2.38).
Applying Lemma 2.9 to problem (2.36) with \( \nu = \beta \) and \( \phi(z) = (z + A)^{4} \), we obtain
\[ \int_{-1}^{\beta} \hat{q}_{2} \, dx > 0. \tag{2.39} \]
Combining (2.39), (2.35) and (2.37) the claimed result is proved. \( \square \)

**Proposition 2.12.** If \( A > 0 \) satisfies \( \psi(1, A) = 0 \), then \( \frac{\partial \psi}{\partial A}(1, A) > 0. \)

**Proof.** From the definition (2.17) of \( \psi \) we have by elementary calculus
\[ \frac{\partial \psi}{\partial A}(1, A) = 3 \int_{-1}^{1} \frac{H_{0}(1) - H_{0}(x)}{(H_{0}(x) + A)^{4}} \, dx. \]
Exactly as in (2.30), we prove that \( H_{0}(1) < H_{0}(-1) \), which implies the existence of \( v \in ]-1, d[ \) such that \( H_{0}(v) = H_{0}(1) \).
We now write
\[ \frac{\partial \psi}{\partial A}(1, A) = 3 \int_{-1}^{v} \frac{\phi(x)}{H_{0}(x) + A} \, dx + 3 \int_{v}^{1} \frac{\phi(x)}{H_{0}(x) + A} \, dx \]
with \( \phi(x) = \frac{H_{0}(1) - H_{0}(x)}{(H_{0}(x) + A)^{3}}. \)
It is clear that \( \psi(1, A) = 0 \) means
\[ \int_{-1}^{1} \phi(x) \, dx = 0. \tag{2.40} \]
Since we have
\[ \phi(x) < 0 \quad \text{for} \quad x \in ]-1, v[ \quad \text{and} \quad \phi(x) > 0 \quad \text{for} \quad x \in ]v, 1[, \]

we deduce

\[ \frac{\partial \psi}{\partial A} (1, A) > \frac{3}{H_0(1) + A} \int_{-1}^{v} \phi(x) \, dx + \frac{3}{H_0(1) + A} \int_{v}^{1} \phi(x) \, dx \]

which gives the result by using (2.40). \( \Box \)

We are now in a position to state the main result of this section.

**Proposition 2.13.** Assume in addition to hypothesis (2.10) that \( H''_0 \) is non-decreasing. Then if the function \( G \) is continuous, then it is strictly decreasing.

**Proof.** From Proposition 2.12, we deduce that it can exists at most a single point \( A > 0 \) such that \( \psi(1, A) = 0 \). Then Proposition 2.11 gives the claimed result. \( \Box \)

3. The existence result

We point out that we want to show the existence of a solution \((p, a, \theta)\) of the system (1.2)–(1.5). We will search the unknown \((a, \theta)\) in the bidimensional set

\[ Q = \{ (a, \theta) \in \mathbb{R}^2 : \theta < -h'_0(-1), \quad a > -\min_{x \in \Omega} (h_0(x) + \theta x) \}. \]

Let us introduce the following applications \( g_1, g_2 : Q \to \mathbb{R} \) defined by

\[ g_1(a, \theta) = \int_{\Omega} p(x) \, dx - F, \]
\[ g_2(a, \theta) = \int_{\Omega} xp(x) \, dx - Fx_0 \]

with \( p = p(a, \theta) \) the unique solution of the variational inequality (1.2) where \( h \) is given by (1.5).

It is clear that the original problem (1.2)–(1.4) is reduced to find \((a, \theta) \in Q\) solution of the system

\[
\begin{cases}
  g_1(a, \theta) = 0, \\
  g_2(a, \theta) = 0.
\end{cases}
\]  \( \text{(3.1)} \)

The regularity of \( g_1 \) and \( g_2 \) are not obvious since we cannot use there the Implicit Function Theorem. In the next proposition, we show the continuity of \( g_1 \) and \( g_2 \) by a direct method.

**Proposition 3.1.** The functions \( g_1 \) and \( g_2 \) are continuous on \( Q \).

**Proof.** Let \((a, \theta) \) in \( Q \) and \((a_n, \theta_n) \to (a, \theta)\) such that there exists \( b_0 > 0 \) with

\[ h_0(x) + \theta_n x + a_n \geq b_0, \quad \forall x \in \Omega. \]  \( \text{(3.2)} \)

Let \( p_n = p(a_n, \theta_n) \). Taking \( \phi = 0 \) in (1.2) and using (3.2) we deduce

\[ \| p_n \|_{H_0^1(\Omega)} \leq C \]

with \( C \) independent of \( n \). 

We deduce the existence of $p \in K$ and of a subsequence of $p_n$ denoted also by $p_n$ such that $p_n \rightharpoonup p$ in $H^1_0(\Omega)$-weak.

We also have for any $\varphi \in K$,

$$
\int_{\Omega} (h_0 + a_n + \theta_n x)^3 \frac{dp_n}{dx} d\varphi dx d\varphi dx dx \\
\geq \int_{\Omega} (h_0 + a_n + \theta_n x)^3 \left( \frac{dp_n}{dx} \right)^2 + \int_{\Omega} (h_0 + a_n + \theta_n x) \frac{d(\varphi - p_n)}{dx} dx.
$$

(3.3)

Using

$$h_0 + a_n + \theta_n x \to h_0 + a + \theta x \text{ in } L^\infty(\Omega) \text{ strongly}$$

and

$$\liminf_{n \to +\infty} \int_{\Omega} (h_0 + a_n + \theta_n x)^3 \left( \frac{dp_n}{dx} \right)^2 \geq \int_{\Omega} (h_0 + a + \theta x)^3 \left( \frac{dp}{dx} \right)^2$$

and passing to the limit in (3.3), we obtain $p = p(a, \theta)$.

Since $p$ is unique, we deduce that the entire sequence $p_n$ converges to $p$ in $H^1_0(\Omega)$ weakly which easily gives the result. □

Now in order to reduce (3.1) to a single equation, we need the following result:

**Proposition 3.2.** For any $\theta < -h_0'(1)$ there exists an unique solution $a$ of the equation $g_1(a, \theta) = 0$ such that $(a, \theta) \in Q$.

**Proof.** We distinguish the two cases.

**Case 1:** $\theta \leq 0$. We apply the results of Section 2.1 which $H_0(x) = h_0(x) + \theta x - \theta$, which satisfies hypothesis (2.2) and we classically deduce the result as a consequence of Propositions 2.1, 2.5 and Corollary 2.3 with $\alpha = 0$ for $\theta < 0$ and $\alpha = 1$ for $\theta = 0$.

**Case 2:** $\theta \in ]0, -h_0'(1)[$. Since $h_0'$ is increasing (because $h_0'' > 0$) there exists a unique solution $d_\theta \in ]-1, 1[$ of the equation

$$h_0'(d_\theta) = -\theta$$

(3.4)

and we have that the function $x \to h_0(x) + a + \theta x$ is decreasing on $]-1, d_\theta[$ and increasing on $]d_\theta, 1[$.

It is convenient to write $h(x) = h_0(x) + a + \theta x$ in the form $h(x) = b + r(x)$ with

$$b = h_0(d_\theta) + a + \theta d_\theta$$

(3.5)

and

$$r(x) = h_0(x) - h_0(d_\theta) - \frac{\partial h_0}{\partial x}(d_\theta)(x - d_\theta).$$

(3.6)

It is clear that
\( r \in C^2(\bar{\Omega}), \quad r(d_\theta) = r'(d_\theta) = 0, \)
\( r'(x) < 0 \quad \text{for} \quad x < d_\theta, \)
\( r'(x) > 0 \quad \text{for} \quad x > d_\theta, \)
\( r''(x) = h''_0(x) > 0, \quad \forall x \in \Omega. \)

It is also clear that the required problem becomes: show the existence of a unique \( b > 0 \) and \( p \in K \), satisfying:

\[
\int_\Omega (b + r(r')^3 \frac{d(\varphi - p)}{dx}) dx \geq \int_\Omega (b + r) \frac{d(\varphi - p)}{dx} dx, \quad \forall \varphi \in K, \tag{3.7}
\]

\[
\int_\Omega p = F. \tag{3.8}
\]

It is natural to introduce the application \( b \in ]0, +\infty[ \rightarrow G_1(b) = \int_\Omega p dx \in ]0, +\infty[ \) with \( p \) the solution of (3.7) and the problem (3.7)–(3.8) is equivalent to: find \( b > 0 \) such that

\[
G_1(b) = F. \tag{3.9}
\]

(In fact \( G_1(b) = g_1(b - h_0(d_\theta) - \theta d_\theta, \theta) \).

From (3.5) and Proposition 3.1 the function \( G_1 \) is continue since \( \theta \) is fixed.

We apply the results of Section 2.2 with \( H_0 = r \) and \( A = b \). Now the result is a consequence of Proposition 2.1, Corollary 2.7 and Proposition 2.13.

Now for any \( \theta < -h'_0(-1) \), we denote by \( a = a(\theta) \) the unique solution of

\[
g_1(a, \theta) = 0.\]

We introduce the application \( S : ]-\infty, -h'_0(-1)[ \rightarrow \mathbb{R} \) defined by

\[
S(\theta) = g_2(a(\theta), \theta)
\]

and it is clear that the problem (1.2)–(1.5) or (3.1) is reduced to search for at least a zero of \( S \).

We remark that the restriction of \( S \) on \( ]-\infty, 0[ \) is exactly the function \( S \) introduced in [1] (since the variational inequality becomes equation in this case). We proved in [1] (and the result is of course valid here):

\[
\lim_{\theta \rightarrow -\infty} S(\theta) > 0. \tag{3.10}
\]

The next result proves that \( S \) takes a negative value in at least a point.

We obtain the following result.

**Proposition 3.3.** There exists \( \theta_0 \in ]0, -h'_0(-1)[ \) such that

\[
S(\theta_0) < 0.\]

**Proof.** Using the notations of the proof of Case 2 of Proposition 3.2, we have

\[
S(\theta) = \int_\Omega p(x - x_0) \tag{3.11}
\]

with \( p(x), b > 0 \) the unique solution of (3.7)–(3.8), with \( r \) given by (3.6).
It is clear that if \( \theta \to -h'(0)(-1) \), then \( d_\theta \to -1 \) and we obtain \( r(-1) \to 0 \). Since there exists \( \theta \) in a neighborhood of \( h'(0) \) such that

\[
 r(-1) < r(1) \tag{3.12}
\]

we deduce the existence of \( \beta_\theta \in ]-1, 1[ \) such that the solution \( p \) of (3.7) satisfies

\[
 \frac{d}{dx} \left( (r + b)^3 \frac{dp}{dx} \right) = \frac{dr}{dx} \quad \text{on } ]-1, \beta_\theta[,
\]

\[
p(-1) = p(\beta_\theta) = 0, \tag{3.14}
\]

\[
p'(\beta_\theta) = 0, \tag{3.15}
\]

\[
p(x) = 0 \quad \text{on } ]\beta_\theta, 1[ \tag{3.16}
\]

(if not one would deduce exactly as for (2.38) that \( r(1) < r(-1) \) which contradict (3.12)).

We then have exactly as in (2.38) that

\[
r(\beta_\theta) < r(-1) \tag{3.17}
\]

which implies that \( r(\beta_\theta) \to 0 \) when \( \theta \to h'(0)(-1) \).

On the other hand, from (3.6), we easily deduce

\[
r(\beta_\theta) < r(-1)
\]

and the positivity of \( h'' \) implies \( \beta_\theta - d_\theta \to 0 \). We then deduce \( \beta \to -1 \) when \( \theta \to h'(0)(-1) \).

Then for any \( x_0 \in ]-1, 1[ \), we can choose \( \theta \) in a neighborhood of \( -h'(0)(-1) \) such that

\[
 \beta_\theta < x_0. \tag{3.18}
\]

Since \( p = 0 \) for \( x \geq \beta_\theta \), we have

\[
 S(\theta) = \int_{-1}^{\beta_\theta} p(x - x_0) \beta_\theta
\]

and finally the positivity of \( p \) on \( ]-1, \beta_\theta[ \) and (3.18) give the claimed result. \( \square \)

It now remains to show the continuity of the function \( S \).

**Proposition 3.4.** The function \( S \) is continuous on \( ]-\infty, -h'(0)(-1)[ \).

**Proof.** The difficulty of the proof of the continuity of \( S \) comes from the change in the nature of problem satisfied by \( (a, p) \) when \( \theta \) varies in the mentioned interval.

We fixed \( \theta \in ]-\infty, -h'(0)(-1)[ \) and we consider three cases.

**Case 1:** \( \theta_0 < 0 \). The continuity of \( S \) in \( \theta_0 \) was proved in [1] (see Proposition 3.3).

**Case 2:** \( \theta_0 \in ]0, -h'(0)(-1)[ \). Let \( \theta_n \to \theta_0 \) and we can suppose that \( \theta_n \in [0, \theta_1] \) with \( 0 < \theta_1 < -h'(0)(-1) \) fixed.

We denote by \( d_{\theta_n} \) the solution of (3.4) with \( \theta = \theta_n \) and

\[
r_n(x) = h_0(x) - h_0(d_{\theta_n}) - h_0'(d_{\theta_n})(x - d_{\theta_n}).
\]

We also denote \( b_n > 0 \), \( p_n(x) \) the unique solution of (3.7)--(3.8) with \( r = r_n \).
We prove by absurd that there exists a compact \( \hat{K} \subset ]0, +\infty[ \) independent of \( n \) such that
\[
b_n \in \hat{K}, \quad \forall n \in \mathbb{N}.
\] (3.19)

Let us suppose the contrary, then we have two cases:

**Case 2.1:** There exists a subsequence denoted by \( b_n \) such that \( b_n \to +\infty \). Then by Proposition 2.1 with \( H_0 = r_n \) and \( A = b_n \), we deduce that \( \int_{\Omega} p_n \, dx \to 0 \) which contradicts \( \int_{\Omega} p_n \, dx = F \).

**Case 2.2:** There exists a subsequence denoted by \( b_n \) such that \( b_n \to 0 \).
We apply Proposition 2.6 with \( H_0 = r_n, d_0 = d_{\theta_0} \) and \( A = b_n \) (notice that \( r_n''(x) = h_0''(x) \) is independent of \( n \)) and we deduce that \( \int_{\Omega} p_n \, dx \to +\infty \) which also contradicts \( \int_{\Omega} p_n = F \).

Then we proved (3.19).

Then there is a subsequence denoted also by \( b_n \) and an element \( b \in \hat{K} \) such that \( b_n \to b \).

By a similar proof to that of Proposition 3.1 we deduce the existence of a subsequence denoted by \( p_n \) and of an element \( p \in K \) such that \( p_n \to p \) weakly in \( H^1(\Omega) \) with \( p \) and \( b \) satisfying (3.7)–(3.8) with \( r(x) = h_0(x) - h_0(d_{\theta_0}) - h_0'(d_{\theta_0})(x - d_{\theta_0}) \) (since \( r_n \to r \) strongly in \( L^\infty(\Omega) \)).

By uniqueness of the solution of the (3.7)–(3.8), we deduce that all the sequence \((b_n, p_n)\) converges to the limit.

Then
\[
S(\theta_n) \to S(\theta_0).
\]

**Case 3:** \( \theta_0 = 0 \). We also take a sequence \( \theta_n \to 0 \) and we can consider two cases:

**Case 3.1:** \( \theta_n > 0 \). We show exactly as in Case 2 that \( S(\theta_n) \to S(0) \).

**Case 3.2:** \( \theta_n < 0 \). We are in the hypothesis of Section 2.1, so
\[
S(\theta_n) = \int_{\Omega} p_n(x - x_0) \, dx
\]
with \( b_n > 0, p_n(x) \) solution of
\[
\frac{d}{dx} \left[ \left( b_n + h_0(x) - \theta_n(1 - x) \right) \frac{d p_n}{dx} \right] = \frac{d}{dx} \left[ h_0(x) - \theta_n(1 - x) \right],
\]
\[
p_n(-1) = p_n(1) = 0,
\]
\[
\int_{\Omega} p_n \, dx = F.
\]

Exactly as in Case 2.1 we prove that no subsequence \( b_n \to \infty \) can exist. Let us now suppose that a subsequence of \( b_n \) satisfies \( b_n \to 0 \).
We apply Lemma 2.4 with \( H_0(x) = h_0(x) - \theta_n(1 - x) \) and \( A = b_n \). We have
\[
-H_0'(x) = -\theta_n - h_0'(x)
\]
and the assumptions of Lemma 2.4 are satisfied with \( M_0 = \min_{x \in \tilde{\Omega}} h_0'' \), \( M_1 = \max_{x \in \tilde{\Omega}} h_0'' \), \( \delta_0 = 1 \) and \( \Theta = -\theta_n \).

Then we obtain \( \int_{\Omega} p_n \, dx \to +\infty \) which contradicts \( \int_{\Omega} p_n \, dx = F \). We then proved that there is no subsequence of \( b_n \) which tends to 0. We deduce the existence of a compact \( \hat{K} \subset ]0, +\infty[ \) such that \( b_n \in \hat{K}, \forall n \in \mathbb{N} \). The end of the proof is as in Case 2. \( \square \)
Now applying the intermediate value theorem, we obtain from (3.10), Propositions 3.3 and 3.4 and the main result of this paper.

**Theorem 3.5.** For any \( x_0 \in ]-1, 1[ \) and \( F > 0 \) there exists at least a solution \((p, a, \theta)\) of (1.2)–(1.5).

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**References**

