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An Erdős–Ko–Rado theorem for the derangement graph of PGL(2, q) acting on the projective line

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ABSTRACT

Let $G = \mathrm{PGL}(2, q)$ be the projective general linear group acting on the projective line $\mathbb{P}_q$. A subset S of G is intersecting if for any pair of permutations π, σ in S, there is a projective point $p \in \mathbb{P}_q$ such that $p^\pi = p^\sigma$. We prove that if S is intersecting, then $|S| \leqslant q(q-1)$. Also, we prove that the only sets S that meet this bound are the cosets of the stabilizer of a point of $\mathbb{P}_q$.

1. Introduction

The Erdős–Ko–Rado theorem [5] is a very important result in extremal combinatorics. There are many different proofs and extensions of this theorem and we refer the reader to [4] for a full account. There are also many applications of the Erdős–Ko–Rado theorem, for example to qualitatively independent sets [12,13], problems in geometry [6] and in statistics [16].

In this paper, we are concerned with an extension of the Erdős–Ko–Rado theorem to permutation groups. Let G be a permutation group on Ω. We let $\mathrm{fix}(g)$ denote the number of fixed points of the permutation g of G. A subset S of G is said to be *intersecting* if $\mathrm{fix}(g^{-1}h) \neq 0$, for every $g, h \in S$. As with the Erdős–Ko–Rado theorem, we are interested in finding the size of the largest intersecting set of G and classifying the sets that attain this bound. This problem can be formulated in a graph-theoretic terminology. We denote by Γ_G the *derangement graph* of G, the vertices of this graph are the elements of G and the edges are the pairs $\{g, h\}$ such that $\mathrm{fix}(g^{-1}h) = 0$. An intersecting set of G is simply an *independent set* of Γ_G. In [3,15] the natural extension of the Erdős–Ko–Rado theorem for the symmetric group $\mathrm{Sym}(n)$ was independently proven. Indeed, it was shown that every independent set of permutations in $\Gamma_{\mathrm{Sym}(n)}$ has size at most $(n-1)!$. Also, the only sets that meet this bound are

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the cosets of the stabilizer of a point. The same result was also proved in [8] using the character theory of $\mathrm{Sym}(n)$.

In this paper, inspired by the approach used in [8], we prove a similar result for the permutation group $G = \mathrm{PGL}(2,q)$ acting on the projective line $\mathbb{P}_q$. We show that an independent set S in Γ_G has size at most $q(q-1)$. Also, we prove that the sets S that meet this bound are the cosets of the stabilizer of a point of $\mathbb{P}_q$. In particular, these results are a natural q-analogue of the result for the symmetric group $\mathrm{Sym}(n)$. Here we report the main theorem.

Theorem 1. *Every independent set S of the derangement graph of* $\mathrm{PGL}(2,q)$ *acting on the projective line $\mathbb{P}_q$ has size at most $q(q-1)$. Equality is met if and only if S is the coset of the stabilizer of a point.*

Theorem 1 indicates that the argument described in [8] will likely be particularly fruitful for studying the derangement graph of a permutation group G whose character theory is well understood.

In Section 2 we recall some general results on the eigenvalues and the independent sets of the graph Γ_G that will be used throughout the paper. In Section 3 the character table of $\mathrm{PGL}(2,q)$ is described. Section 4 includes technical lemmas that are used in Section 5 to prove Theorem 1. Finally, in Section 6, we conclude with some general remarks on the derangement graphs of 2-transitive and 3-transitive groups.

2. General results

Let G be a permutation group on Ω and Γ_G its derangement graph. Since the right regular representation of G is a subgroup of $\mathrm{Aut}(\Gamma_G)$, we see that Γ_G is a Cayley graph. Namely, if D is the set of *derangements* of G (i.e. the fixed-point-free permutations of G), then Γ_G is the Cayley graph on G with connection set D, i.e. $\Gamma_G = \mathrm{Cay}(G,D)$. Clearly, D is a union of G-conjugacy classes, so Γ_G is a normal Cayley graph.

As usual, we simply say that the complex number ξ is an *eigenvalue* of the graph Γ if ξ is an eigenvalue of the adjacency matrix of Γ.

In this paper, we use $\mathrm{Irr}(G)$ to denote the *irreducible complex characters* of a group G and given $\chi \in \mathrm{Irr}(G)$ and a subset S of G we write $\chi(S)$ for $\sum_{s\in S}\chi(s)$. In the following lemma we recall that the eigenvalues of Γ_G are determined by the irreducible complex characters of the group G.

Lemma 2. *Let G be a permutation group on Ω and D the set of derangements of G. The spectrum of the graph Γ_G is $\{\chi(D)/\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Also, if τ is an eigenvalue of Γ_G and $\chi_1,\dots,\chi_s$ are the irreducible characters of G such that $\tau = \chi_i(D)/\chi_i(1)$, then the dimension of the τ eigenspace of Γ_G is $\sum_{i=1}^{s}\chi_i(1)^2$.*

Proof. Since Γ_G is a normal Cayley graph, the result follows from [1]. □

Since Γ_G is a Cayley graph, the $\mathbb{C}$-vector space with basis elements labelled by the vertices of Γ_G is the $\mathbb{C}$-vector space underlying the group algebra $\mathbb{C}G$. If S is a subset of G, we simply write v_S for the vector $\sum_{g\in S} g$ of $\mathbb{C}G$ (so v_S is the *characteristic vector* for the set S). In particular, v_G is the all-1 vector of $\mathbb{C}G$. Finally, we recall Hoffman's bound for the size of an independent set of Γ_G and a consequence for when equality holds in this bound.

Lemma 3. *Let G be a permutation group on Ω. Let S be an independent set of Γ_G and τ be the minimum eigenvalue of Γ_G. Assume that the valency of Γ_G is d. Then $|S| \leqslant |G|/(1-\frac{d}{\tau})$. If the equality is met, then $v_S - \frac{|S|}{|G|}v_G$ is an eigenvector of Γ_G with eigenvalue τ.*

Proof. Set $v = |G|$, and $M = A - \tau I - (d-\tau)/v\,J$, where A is the adjacency matrix of Γ_G and J is the all-1 matrix. By definition of τ and by construction, M is a positive semidefinite symmetric matrix.

Hence, given a set $S \subseteq G$ of size s, it is true that

$$0 \leqslant v_S^T M v_S = v_S^T A v_S - \tau v_S^T v_S - \frac{d-\tau}{v} v_S^T J v_S = v_S^T A v_S - \tau s - \frac{d-\tau}{v} s^2.$$

If S is an independent set of Γ_G, then $v_S^T A v_S = 0$ and the previous inequality yields the first part of the lemma.

Now, suppose that equality holds. Then $v_S^T M v_S = 0$. Since M is positive semidefinite, we obtain $M v_S = 0$. Therefore

$$A\left(v_S - \frac{s}{v} v_G\right) = \tau\left(v_S - \frac{s}{v} v_G\right),$$

and the proof is completed. □

3. PGL(2, q)

We recall that the character table of PGL$(2,q)$ was computed by Jordan and Schur in 1907 and can be found in many textbooks, see [11]. Also, the character tables of PGL$(3,q)$ and PGL$(4,q)$ were found by R. Steinberg and the character table of PGL(n,q) was finally determined by J. Green in 1955 in the celebrated paper [9].

We give the character table of PGL$(2,q)$, first for q even and second for q odd. We note that by abuse of terminology we often refer to the elements of PGL$(2,q)$ as matrices.

We briefly explain the notation used, but refer the reader to [11] for full details. We start by describing the conjugacy classes. The elements of PGL$(2,q)$ can be collected in four sets: the set consisting only of the identity element, the set consisting of the non-scalar matrices with only one eigenvalue, the set consisting of the matrices with two distinct eigenvalues in $\mathbb{F}_q$ and the set of matrices with no eigenvalues in $\mathbb{F}_q$. The non-scalar matrices with only one eigenvalue form a conjugacy class of size q^2-1 and are represented by a unipotent matrix u. Every matrix with two distinct eigenvalues in $\mathbb{F}_q$ is conjugate to a diagonal matrix d_x, with x and 1 along the main diagonal. Now, d_x and d_y are conjugate if and only if $y = x$ or $y = x^{-1}$. So, the label x in the character table of PGL$(2,q)$ for d_x represents an element of $\mathbb{F}_q \setminus \{0,1\}$ up to inversion. Now, for q odd, let $i \in \mathbb{F}_{q^2}$ be an element of order 2 in $\mathbb{F}_{q^2}^*/\mathbb{F}_q^*$. The matrices with no eigenvalues in $\mathbb{F}_q$ are conjugate in PGL$(2,q^2)$ to a diagonal matrix, with r and r^q along the main diagonal (for $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$). Each of these matrices is conjugate in PGL$(2,q)$ to the matrix v_r depicted below. Also, v_x and v_y are conjugate if and only if $y\mathbb{F}_q^* = x\mathbb{F}_q^*$ or $y\mathbb{F}_q^* = x^{-1}\mathbb{F}_q^*$. So, the label r in the character table of PGL$(2,q)$ for v_r represents an element of $\mathbb{F}_{q^2}^*/\mathbb{F}_q^*$ up to inversion. So,

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad d_x = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, \qquad v_r = \begin{bmatrix} 0 & 1 \\ -r^{1+q} & r + r^q \end{bmatrix}.$$

We note here that in the permutation group PGL$(2,q)$ acting on the projective line $\mathbb{P}_q$, the permutations in the conjugacy class represented by u fix exactly one point, the permutations in the conjugacy classes represented by d_x fix exactly 2 points while the permutations in the conjugacy classes represented by v_r fix no point.

Next we describe the characters of PGL$(2,q)$ by describing the maps ε, γ and η. The map ε is defined by $\varepsilon(x) = 1$ if $d_x \in \mathrm{PSL}(2,q)$ and $\varepsilon(x) = -1$ otherwise. Similarly, $\varepsilon(r) = 1$ if $v_r \in \mathrm{PSL}(2,q)$ and $\varepsilon(r) = -1$ otherwise. So, $\lambda_{-1} : \mathrm{PGL}(2,q) \to \{\pm 1\}$ is the non-principal linear character. The letter γ represents a group homomorphism $\gamma : \mathbb{F}_q^* \to \mathbb{C}$ of order greater than 2. Also, $\nu_{\gamma_1} = \nu_{\gamma_2}$ if and only if $\gamma_2 = \gamma_1$ or $\gamma_2 = \gamma_1^{-1}$. So, the label γ runs through the homomorphisms $\gamma : \mathbb{F}_q^* \to \mathbb{C}$ of order greater than 2 up to inversion. The letter β stands for a group homomorphism $\beta : \mathbb{F}_{q^2}^*/\mathbb{F}_q^* \to \mathbb{C}$ of order greater than 2. Also, $\eta_{\beta_1} = \eta_{\beta_2}$ if and only if $\beta_2 = \beta_1$ or $\beta_2 = \beta_1^{-1}$. So, the label β runs through the homomorphisms $\beta : \mathbb{F}_{q^2}^*/\mathbb{F}_q^* \to \mathbb{C}$ of order greater than 2 up to inversion. Note that given a non-principal linear character $\psi : C \to \mathbb{C}$ of a cyclic group C, we get $\sum_{c \in C} \psi(c) = 0$. In particular, given γ and β as in Tables 1 and 2, we obtain $\sum_{x \in \mathbb{F}_q^*} \gamma(x) = 0$ and $\sum_{r\mathbb{F}_q^* \in \mathbb{F}_{q^2}^*/\mathbb{F}_q^*} \beta(r) = 0$.

Table 1
Character table of $\mathrm{PGL}(2,q)$, for q even.

	Name	1	u	d_x	v_r
	No.	1	1	$\frac{q}{2}-1$	$\frac{q}{2}$
	Size	1	q^2-1	$q(q+1)$	$q(q-1)$
Name	No.				
λ_1	1	1	1	1	1
ψ_1	1	q	0	1	-1
η_β	$\frac{q}{2}$	$q-1$	-1	0	$-\beta(r)-\beta(r^{-1})$
ν_γ	$\frac{q}{2}-1$	$q+1$	1	$\gamma(x)+\gamma(x^{-1})$	0

Table 2
Character table of $\mathrm{PGL}(2,q)$, for q odd.

	Name	1	u	d_x	d_{-1}	v_r	v_i
	No.	1	1	$\frac{q-3}{2}$	1	$\frac{q-1}{2}$	1
	Size	1	q^2-1	$q(q+1)$	$\frac{q(q+1)}{2}$	$q(q-1)$	$\frac{q(q-1)}{2}$
Name	No.						
λ_1	1	1	1	1	1	1	1
λ_{-1}	1	1	1	$\varepsilon(x)$	$\varepsilon(-1)$	$\varepsilon(r)$	$\varepsilon(i)$
ψ_1	1	q	0	1	1	-1	-1
ψ_{-1}	1	q	0	$\varepsilon(x)$	$\varepsilon(-1)$	$-\varepsilon(r)$	$-\varepsilon(i)$
η_β	$\frac{q-1}{2}$	$q-1$	-1	0	0	$-\beta(r)-\beta(r^{-1})$	$-2\beta(i)$
ν_γ	$\frac{q-3}{2}$	$q+1$	1	$\gamma(x)+\gamma(x^{-1})$	$2\gamma(-1)$	0	0

In the following we simply denote by G_q the permutation group $\mathrm{PGL}(2,q)$ acting on the projective line $\mathbb{P}_q$, and D the set of derangements of G_q.

Using Table 1 (for q even), Table 2 (for q odd) and the fact that the elements in the conjugacy classes labelled by v_r are the derangements of G_q, it is possible to compute $\chi(D)/\chi(1)$, for all $\chi\in\mathrm{Irr}(G_q)$. For example, if $\chi=\lambda_1$ then we easily see that

$$\frac{\lambda_1(D)}{\lambda_1(1)}=\frac{q(q-1)(q-1)}{2}+\frac{q(q-1)}{2}=\frac{q^2(q-1)}{2},$$

which is equal to $|D|$, is an eigenvalue of Γ_{G_q}. The only cases where this is not trivial are when $\chi=\eta_\beta$ and when $\chi=\lambda_{-1}$. If $\chi=\eta_\beta$, then we have

$$\frac{\chi(D)}{\chi(1)}=\frac{q(q-1)}{q-1}\Bigg(\sum_{\substack{r\mathbb{F}_q^*\in\mathbb{F}_{q^2}^*/\mathbb{F}_q^*\\ r\notin\mathbb{F}_q^*}}-\beta(r)\Bigg)=-q\Bigg(\sum_{\substack{r\mathbb{F}_q^*\in\mathbb{F}_{q^2}^*/\mathbb{F}_q^*\\ r\notin\mathbb{F}_q^*}}\beta(r)\Bigg)=q$$

since $\mathbb{F}_{q^2}^*/\mathbb{F}_q^*$ is a cyclic group. By direct calculation or using the description of the conjugacy classes of $\mathrm{PSL}(2,q)$ given in [11], we see that $\mathrm{PSL}(2,q)$ contains $q(q-1)^2/4$ derangements. As $|D|=q^2(q-1)/2$, we get that $\mathrm{PGL}(2,q)\setminus\mathrm{PSL}(2,q)$ contains $q(q^2-1)/4$ derangements. So, if $\chi=\lambda_{-1}$, we have

$$\frac{\lambda_{-1}(D)}{\lambda_{-1}(1)}=\frac{q(q-1)^2}{4}-\frac{q(q^2-1)}{4}=-\frac{q(q-1)}{2}.$$

In particular, we obtain Table 3.

In summary, the valency of the graph Γ_{G_q} is $q^2(q-1)/2$ and the minimum eigenvalue τ is $-q(q-1)/2$. If q is even, then ψ_1 is the only character χ such that $\tau=\chi(D)/\chi(1)$. If q is odd, then λ_{-1} and ψ_1 are the only characters χ such that $\tau=\chi(D)/\chi(1)$.

Table 3
Eigenvalues of Γ_{G_q}.

Character	λ_1	λ_{-1}	ψ_1	ψ_{-1}	η_β	ν_γ
Eigenvalue	$\frac{q^2(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{q-1}{2}$	q	0
Dimension (if q even)	1	–	q^2	–	$\frac{q(q-1)^2}{2}$	$\frac{(q+1)^2(q-2)}{2}$
Dimension (if q odd)	1	1	q^2	q^2	$\frac{(q-1)^3}{2}$	$\frac{(q+1)^2(q-3)}{2}$

Lemma 4. *An independent set of maximal size in Γ_{G_q} has size $q(q-1)$.*

Proof. The coset of the stabilizer of a point of G_q is an independent set in Γ_{G_q} of size $|G_q|/(q+1) = q(q-1)$. From the eigenvalues of Γ_{G_q} and Lemma 3 we see that such an independent set is an independent set of maximal size. □

Similar to the case of the symmetric group (and also the standard Erdős–Ko–Rado theorem for sets), finding the bound in Theorem 1 is not difficult, but it is in the characterization of the sets that meet this bound that the difficulty lies. Indeed, it is not difficult to also establish a similar bound on the size of the independent sets in the derangement graph for the group $\mathrm{PSL}(2,q)$.

Lemma 5. *An independent set of maximal size in $\Gamma_{\mathrm{PSL}(2,q)}$ has size $q(q-1)/2$.*

Proof. This can be proved using Lemma 3 with the information on the character table of $\mathrm{PSL}(2,q)$ in [11] and the fact that a point-stabilizer in $\mathrm{PSL}(2,q)$ has size $q(q-1)/2$. □

The next lemma will be used in Lemma 7 to limit the search of independent sets of maximal size in Γ_{G_q}.

Lemma 6. *Assume q odd. If S is an independent set of maximal size of Γ_{G_q}, then $\lambda_{-1}(S) = 0$.*

Proof. By Lemma 4, we have $|S| = q(q-1)$. Consider the two sets $S_+ = S \cap \mathrm{PSL}(2,q)$ and $S_- = S \setminus S_+$. Clearly S_+ is an independent set in $\Gamma_{\mathrm{PSL}(2,q)}$ and, for $g \in G_q \setminus \mathrm{PSL}(2,q)$, the set gS_- is also an independent set in $\Gamma_{\mathrm{PSL}(2,q)}$. Thus we obtain that $|S_+| = |S_-| = q(q-1)/2$ and in particular $\lambda_{-1}(S) = |S_+| - |S_-| = 0$. □

4. Auxiliary lemmas

Consider the $\{0,1\}$-matrix A, where the rows are indexed by the elements of G_q, the columns are indexed by the ordered pairs of points of $\mathbb{P}_q$ and $A_{g,(p_1,p_2)} = 1$ if and only if $p_1^g = p_2$. In particular, A has $|G_q| = q(q^2-1)$ rows and $|\mathbb{P}_q|^2 = (q+1)^2$ columns.

The entry $(A^TA)_{(p_1,q_1),(p_2,q_2)}$ equals the number of permutations of G_q mapping p_1 into q_1 and p_2 into q_2. Since G_q is 2-transitive, we get by a simple counting argument that

$$\left(A^TA\right)_{(p_1,q_1),(p_2,q_2)} = \begin{cases} q(q-1) & \text{if } p_1 = p_2 \text{ and } q_1 = q_2, \\ q-1 & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that with the proper ordering of the columns of A,

$$A^TA = q(q-1)I_{(q+1)^2} + (q-1)(J_{q+1} - I_{q+1}) \otimes (J_{q+1} - I_{q+1})$$

(in here I_{q+1}, J_{q+1} denote the identity matrix and the all-1 matrix of size $q+1$, respectively). The matrix J_{q+1} has eigenvalue 0 (with multiplicity q) and $q+1$ (with multiplicity 1). Hence $(J_{q+1} - I_{q+1}) \otimes (J_{q+1} - I_{q+1})$ has eigenvalue 1 (with multiplicity q^2), $-q$ (with multiplicity $2q$), and

q^2 (with multiplicity 1). So, A^TA is diagonalizable with eigenvalues $q(q-1)+(q-1)q^2=|G_q|$ (with multiplicity 1), $q(q-1)+(q-1)=q^2-1$ (with multiplicity q^2) and $q(q-1)-(q-1)q=0$ (with multiplicity $2q$). This shows that the kernel of A^TA has dimension $2q$. We now determine the kernel of A.

Let V be the $\mathbb{C}$-vector space whose basis consists of all $e_{(x,y)}$, where (x,y) is an ordered pair of elements of $\mathbb{P}_q$. Consider the following two subspaces of V

$$V_1=\left\langle \sum_{x\in\mathbb{P}_q}(e_{(p_1,x)}-e_{(p_2,x)}) \;\middle|\; p_1,p_2\in\mathbb{P}_q\right\rangle,$$

$$V_2=\left\langle \sum_{x\in\mathbb{P}_q}(e_{(x,p_1)}-e_{(x,p_2)}) \;\middle|\; p_1,p_2\in\mathbb{P}_q\right\rangle.$$

Note that by construction, V_1 and V_2 have dimension q and are G_q-modules. As V_1 and V_2 are orthogonal, we have $V_1\cap V_2=0$. Using the definition of A, it is easy to check that $V_1\oplus V_2$ is contained in the kernel of A. Since the kernel of A^TA has dimension $2q$, we obtain that $V_1\oplus V_2$ is the kernel of A. In particular, we proved the first part of the following lemma.

Lemma 7. *The matrix A has rank q^2+1 and the kernel of A is $V_1\oplus V_2$. Also, the vector space spanned by the columns of A equals the vector space spanned by the characteristic vectors of the independent sets of size $q(q-1)$ of Γ_{G_q}.*

Proof. The claims on the rank and on the kernel follow from the previous discussion.

Note that the columns of A are the characteristic vectors of cosets of stabilizers of points of G_q. In particular, these columns are characteristic vectors of independent sets of size $q(q-1)$ in Γ_{G_q}. By taking the sum of all the columns in A, we see that the all-1 vector is also in the column space of A.

Let S be any independent set of Γ_{G_q} of size $q(q-1)$. By Lemmas 3 and 4 the characteristic vector of S is in the direct sum of the $q^2(q-1)/2$-eigenspace and the $-q(q-1)/2$-eigenspace.

If q is even, from the multiplicities of the eigenvalues in Table 3 it is clear that the vector space spanned by the characteristic vectors of independent sets of size $q(q-1)$ has dimension at most q^2+1. So the lemma follows.

If q is odd then Lemma 6 yields that v_S is orthogonal to the eigenspace arising from the character λ_{-1}. So v_S must lie in the direct sum of the eigenspaces arising from λ_1 and ψ_1. Again, the multiplicities of the eigenvalues in Table 3 show that the vector space spanned by the characteristic vectors of independent sets of size $q(q-1)$ has dimension at most q^2+1 and the lemma follows. □

Now we fix a particular ordering of the rows of A so that the first row is labelled by the identity element of G_q, then we label the next $q^2(q-1)/2$ rows by the derangements of G_q and the final $(q^2-2)(q+1)/2$ rows are labelled by the remaining permutations. Similarly, we fix a particular ordering of the columns of A so that the first $q+1$ columns are labelled by the ordered pairs of the form (p,p) and then the last $q(q+1)$ columns are labelled by the ordered pairs of the form (p_1,p_2), with $p_1\neq p_2$. With this ordering, we get that the matrix A is a block matrix. Namely,

$$A=\begin{pmatrix}1 & 0\\ 0 & M\\ B & C\end{pmatrix}.$$

In particular, the rows of the submatrix M are labelled by the derangements of G_q and the columns of M are labelled by the ordered pairs of distinct elements of $\mathbb{P}_q$.

Now, set $N=M^TM$. In particular, N is a square $(q+1)q$ matrix whose rows and columns are indexed by the ordered pairs of distinct points.

From now on, we identify the elements of $\mathbb{P}_q$ with the elements of the set $\mathbb{F}_q\cup\{\infty\}$. Namely, the point $[1,a]$ corresponds to the element a of $\mathbb{F}_q$ and the point $[0,1]$ corresponds to the element ∞.

Now, given four distinct points $\alpha, \beta, \gamma, \delta$, we recall that the *cross-ratio*, denoted by $\mathrm{crr}(\alpha, \delta, \gamma, \beta)$, is defined by

$$\frac{\alpha-\gamma}{\alpha-\beta}\,\frac{\delta-\beta}{\delta-\gamma}.$$

We recall that the cross-ratio is G_q-invariant, i.e. if $g \in G_q$ and $\alpha, \beta, \gamma, \delta \in \mathbb{P}_q$, then $\mathrm{crr}(\alpha^g, \delta^g, \gamma^g, \beta^g) = \mathrm{crr}(\alpha, \delta, \gamma, \beta)$.

In the following proposition we prove that the entries of the matrix N are determined by the cross-ratio.

Proposition 8. *If q is even, then*

$$N_{(\alpha,\beta),(\gamma,\delta)} = \begin{cases} q(q-1)/2 & \text{if } \alpha=\gamma \text{ and } \beta=\delta,\\ 0 & \text{if } \alpha=\gamma \text{ and } \beta\neq\delta,\\ 0 & \text{if } \alpha\neq\gamma \text{ and } \beta=\delta,\\ 0 & \text{if } \alpha=\delta \text{ and } \beta=\gamma,\\ q/2 & \text{otherwise.} \end{cases}$$

If q is odd, then

$$N_{(\alpha,\beta),(\gamma,\delta)} = \begin{cases} q(q-1)/2 & \text{if } \alpha=\gamma \text{ and } \beta=\delta,\\ 0 & \text{if } \alpha=\gamma \text{ and } \beta\neq\delta,\\ 0 & \text{if } \alpha\neq\gamma \text{ and } \beta=\delta,\\ (q-1)/2 & \text{if } \mathrm{crr}(\alpha,\delta,\gamma,\beta) \text{ is a square in } \mathbb{F}_q,\\ (q+1)/2 & \text{if } \mathrm{crr}(\alpha,\delta,\gamma,\beta) \text{ is not a square in } \mathbb{F}_q. \end{cases}$$

Proof. Note that $N_{(\alpha,\beta),(\gamma,\delta)}$ is the number of derangements mapping α to β and γ to δ. Write n for $N_{(\alpha,\beta),(\gamma,\delta)}$. If $\alpha=\gamma$ and $\beta=\delta$, then n is the number of derangements mapping α to β. Since G_q is transitive of degree $q+1$ and since G_q contains $q^2(q-1)/2$ derangements, we have $n = q^2(q-1)/2q = q(q-1)/2$.

If $\alpha=\gamma$ and $\beta\neq\delta$ or if $\alpha\neq\gamma$ and $\beta=\delta$, then clearly $n=0$.

From now on we can assume that $\alpha\neq\gamma$ and $\beta\neq\delta$. Assume $\alpha=\delta$ and $\beta=\gamma$. Since G_q is 2-transitive, without loss of generality, we can assume that $\alpha=0$ and $\beta=\infty$. In this case, the elements g such that $0^g=\infty$ and $\infty^g=0$ are the matrices of the form

$$g = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix} \quad \text{with } \lambda \in \mathbb{F}_q^*.$$

Further, g is a derangement if and only if g has no eigenvalue in $\mathbb{F}_q$, i.e. the characteristic polynomial $p_\lambda(t) = t^2-\lambda$ of g is irreducible over $\mathbb{F}_q$. If q is even, then $p_\lambda(t)$ is reducible for every value of λ, and so $n=0$. If q is odd, then $\mathbb{F}_q^*$ has $(q-1)/2$ non-square elements. Thence there exist $(q-1)/2$ values of λ such that $p_\lambda(t)$ is irreducible over $\mathbb{F}_q$, and so $n=(q-1)/2$. Note that $\mathrm{crr}(\alpha,\delta,\gamma,\beta) = \mathrm{crr}(0,0,\infty,\infty) = 1$ is a square in $\mathbb{F}_q$.

From now on we can assume that $|\{\alpha,\beta,\gamma,\delta\}| \geqslant 3$.

As N is symmetric, up to interchanging the pairs (α,β), (γ,δ), we may assume that $\beta\neq\gamma$. Since G_q is 3-transitive, without loss of generality, we may assume that $\alpha=0$, $\beta=1$, $\gamma=\infty$ and $\delta=d$, for some $d \in \mathbb{F}_q \setminus \{1\}$. The elements g such that $0^g=1$ and $\infty^g=d$ are the matrices of the form

$$\begin{bmatrix} 1 & 1 \\ \lambda & \lambda d \end{bmatrix} \quad \text{with } \lambda \in \mathbb{F}_q^*.$$

The permutation g is a derangement if and only if g has no eigenvalue in $\mathbb{F}_q$, i.e. the characteristic polynomial $p_\lambda(t) = t^2-(1+\lambda d)t+\lambda d-\lambda$ of g is irreducible over $\mathbb{F}_q$. Now, we determine the number of values of λ such that $p_\lambda(t)$ is irreducible. Assume that $p_\lambda(t)$ is reducible over $\mathbb{F}_q$ with roots $a,b\in\mathbb{F}_q$. So, $p_\lambda(t)=(t-a)(t-b)$. This yields $a+b=1+\lambda d$ and $ab=\lambda d-\lambda$. We get $b=(1+\lambda-a)/(1-a)$ (note that a cannot be 1, because otherwise $\lambda d-\lambda=b=\lambda d$, which yields $\lambda=0$). From this we obtain $\lambda=(a-a^2)/(d-1-ad)$ (note that $d-1-ad\neq 0$, because otherwise $a\in\{0,1\}$, which yields $\lambda=0$). Consider the function $\varphi:\mathbb{F}_q\backslash X\to\mathbb{F}_q$, where $\varphi(a)=(a-a^2)/(d-1-ad)$ and $X=\{a\mid d-1-ad=0\}$. Note that $X=\{1-d^{-1}\}$ if $d\neq 0$, and $X=\emptyset$ if $d=0$.

We have proved so far that $p_\lambda(t)$ is reducible if and only if λ lies in the image of φ. We now compute the size of $\operatorname{Im}\varphi$. It is easy to check that $\varphi(a_1)=\varphi(a_2)$ if and only if $a_2=a_1$ or $a_2=(1-a_1)(d-1)/(d-1-a_1d)$. This shows that the fiber of $\varphi(a_1)$ contains two points if $a_1\neq(1-a_1)(d-1)/(d-1-a_1d)$ and only one point if $a_1=(1-a_1)(d-1)/(d-1-a_1d)$. Note that $a_1=(1-a_1)(d-1)/(d-1-a_1d)$ if and only if $a_1^2d-2(d-1)a_1+(d-1)=0$.

If q is even and $d\neq 0$, then this happens if $a_1^2=d^{-1}-1$ (i.e. for a unique value of a_1). Thence the image of φ contains $(q-1-1)/2+1=q/2$ elements, and $n=|\mathbb{F}_q\setminus\operatorname{Im}\varphi|=q/2$. Similarly, if q is even and $d=0$, the image of φ contains $q/2$ elements, and $n=|\mathbb{F}_q\setminus\operatorname{Im}\varphi|=q/2$.

If q is odd, then $a_1^2d-2(d-1)a_1+(d-1)=0$ if the discriminant $(d-1)^2-d(d-1)=(1-d)$ is a square (in this case there are two distinct solutions for a_1). So, if $1-d$ is a square, then the image of φ contains $(q-1-2)/2+2=(q+1)/2$ elements and so $n=|\mathbb{F}_q\setminus\operatorname{Im}\varphi|=(q-1)/2$. But, if $1-d$ is not a square, then the image of φ contains $(q-1)/2$ elements and so $n=|\mathbb{F}_q\setminus\operatorname{Im}\varphi|=(q+1)/2$. Finally, we note that $\operatorname{crr}(0,d,\infty,1)=1-d$. □

In the next proposition we use the character table of G_q to find the rank of M.

Proposition 9. *The matrix M has rank $q(q-1)$.*

Proof. Let Ω be the set of ordered pairs of distinct elements of $\mathbb{P}_q$ and V be the vector space with basis $\{e_\omega\}_{\omega\in\Omega}$. Clearly, V is a G_q-module. Namely, V is the permutation module of the action of G_q on Ω. Let π be the character afforded by V, so

$$\pi(g)=\bigl|\bigl\{\omega\in\Omega\bigm|\omega^g=\omega\bigr\}\bigr|.$$

We have $\pi(1_{G_q})=q(q+1)$, $\pi(g)=2$ for every element g conjugate to d_x (for some x) and $\pi(g)=0$ otherwise. As $\pi=\sum_{\chi\in\operatorname{Irr}(G_q)}\langle\chi,\pi\rangle\chi$, by direct calculation of $\langle\chi,\pi\rangle$ with Tables 1 and 2, we get that

$$\pi=\lambda_1+2\psi_1+\sum_\beta\eta_\beta+\sum_\gamma\nu_\gamma,\quad\text{for } q \text{ even},$$

$$\pi=\lambda_1+2\psi_1+\psi_{-1}+\sum_\beta\eta_\beta+\sum_\gamma\nu_\gamma,\quad\text{for } q \text{ odd}.$$

Let $C\subseteq\operatorname{Irr}(G_q)$ be the set of constituents of π. Then $V=\bigoplus_{\chi\in C}V_\chi$, where V_χ is an irreducible G_q-submodule of V, unless $\chi=\psi_1$, and V_{ψ_1} is the sum of two isomorphic irreducible G_q-submodules of V of dimension q. Clearly, $V_{\psi_1}\cong V_1\oplus V_2$ (see Lemma 7).

Again we use the matrix $N=M^TM$ and in order to prove that M has rank $q(q-1)$ it suffices to prove that N has rank $q(q-1)$. By Lemma 7, we get that V_{ψ_1} is contained in the kernel of N. Also, as $N_{\omega_1^g,\omega_2^g}=N_{\omega_1,\omega_2}$ for every $g\in G_q$, we obtain that every eigenspace of N is a G_q-submodule of V. Therefore, since for $\chi\neq\psi_1$ the module V_χ is irreducible, we get that V_χ is an eigenspace of N. Thus, to conclude the proof it suffices to show that the eigenvalue s_χ of the eigenspace V_χ is not zero, for all $\chi\neq\psi_1$.

By Wedderburn's theorem [10], we get that $\mathbb{C}G_q=\bigoplus_{\chi\in\operatorname{Irr}(G)}I_\chi$, where I_χ are minimal two-sided ideals of the semisimple algebra $\mathbb{C}G_q$. Also, each I_χ is generated (as an ideal) by the idempotent $E_\chi=\frac{\chi(1)}{|G_q|}\sum_{g\in G}\chi(g^{-1})g$. Set $v_\chi=\sum_{g\in G_q}\chi(g^{-1})e_{(0^g,\infty^g)}$. As $V_\chi=VI_\chi$ and $v_\chi=\frac{|G_q|}{\chi(1)}e_{(0,\infty)}E_\chi\in V_\chi$, we obtain that v_χ is an eigenvector of N with eigenvalue s_χ. Note that, given $\chi\in C$, we have

$$(\dagger)\quad (Nv_\chi)_{(0,\infty)} = \sum_{(a,b)\in\Omega} N_{(0,\infty),(a,b)}(v_\chi)_{(a,b)} = \sum_{(a,b)\in\Omega} \sum_{\substack{g \text{ s.t.}\\ 0^g=a,\ \infty^g=b}} \chi\left(g^{-1}\right) N_{(0,\infty),(a,b)},$$

$$(\ddagger)\quad (v_\chi)_{(0,\infty)} = \sum_{\substack{g \text{ s.t.}\\ 0^g=0,\ \infty^g=\infty}} \chi\left(g^{-1}\right) = (q-1)\left\langle \mathrm{Res}_T(\chi), 1\right\rangle = (q-1)\langle \chi, \pi\rangle = q-1,$$

where T is the stabilizer in G_q of $0,\infty$ and $\mathrm{Res}_T(\chi)$ is the restriction of χ to T (note that in the fourth equality in (‡) we are using Frobenius Reciprocity).

In the rest of the proof, we do not determine (for q odd) the eigenvalue s_χ of v_χ, but we simply prove that $s_\chi > 0$, for $\chi \neq \psi_1$. If $\chi = \lambda_1$, then by (‡) the vector v_χ is $(q-1)$ times the all-1 vector. By Proposition 8, N is a stochastic matrix with row sum $q(q^2-1)/2$, so $s_{\lambda_1} = q(q^2-1)/2$. Now, for the remaining characters in C we distinguish two cases depending on whether q is even or q is odd.

Assume q even. Let $\chi \in C$, with $\chi \neq \lambda_1, \psi_1$. Now, as

$$\sum_{g\in G_q} \chi\left(g^{-1}\right) = 0,$$

subtracting $q/2\sum_{g\in G_q}\chi(g^{-1})$ from (†) and using Proposition 8, we get

$$(Nv_\chi)_{(0,\infty)} = \frac{q^2}{2}\sum_{g\in T}\chi\left(g^{-1}\right) - \frac{q}{2}\left(\sum_{\substack{g \text{ s.t.}\\ 0^g=0}}\chi\left(g^{-1}\right) + \sum_{\substack{g \text{ s.t.}\\ \infty^g=\infty}}\chi\left(g^{-1}\right) + \sum_{\substack{g \text{ s.t.}\\ 0^g=\infty\\ \infty^g=0}}\chi\left(g^{-1}\right)\right). \tag{1}$$

On the right-hand side of Eq. (1), the first, the second and the third summands are

$$\frac{q^2}{2}(q-1)\left\langle \mathrm{Res}_T(\chi), 1\right\rangle = \frac{q^2(q-1)}{2}\langle \chi, \pi\rangle = \frac{q^2(q-1)}{2},$$
$$\frac{q}{2}q(q-1)\left\langle \mathrm{Res}_{(G_q)_0}(\chi), 1\right\rangle = \frac{q^2(q-1)}{2}\langle \chi, \lambda_1+\psi_1\rangle = 0,$$
$$\frac{q}{2}q(q-1)\left\langle \mathrm{Res}_{(G_q)_\infty}(\chi), 1\right\rangle = \frac{q^2(q-1)}{2}\langle \chi, \lambda_1+\psi_1\rangle = 0.$$

Also, if g is a permutation such that $0^g = \infty$ and $\infty^g = 0$, then g has order 2. Therefore g is conjugate to u and so the fourth summand in Eq. (1) is $\frac{q(q-1)}{2}\chi(u)$. Summing up, we obtain

$$(N_{v_\chi})_{(0,\infty)} = \begin{cases} \frac{(q^2-1)q}{2} & \text{if } \chi = \eta_\beta, \\ \frac{(q-1)^2 q}{2} & \text{if } \chi = \nu_\gamma. \end{cases}$$

Hence, (‡) yields $s_{\nu_\gamma} = q(q-1)/2 > 0$ (for every γ) and $s_{\eta_\beta} = q(q+1)/2 > 0$ (for every β).

We point out that the matrix N has only 4 eigenvalues and actually N is the matrix of an association scheme of rank 4, see [17] for more details.

Assume q odd. Let $\chi \in C$, with $\chi \neq \lambda_1, \psi_1$. Now, as $\sum_{g\in G_q}\chi(g^{-1}) = 0$, subtracting $(q-1)/2\sum_{g\in G_q}\chi(g^{-1})$ from (†) and using Proposition 8, we get

$$(Nv_\chi)_{(0,\infty)} = \frac{q^2-1}{2}\sum_{g\in T}\chi\left(g^{-1}\right) - \frac{q-1}{2}\left(\sum_{\substack{g \text{ s.t.}\\ 0^g=0}}\chi\left(g^{-1}\right) + \sum_{\substack{g \text{ s.t.}\\ \infty^g=\infty}}\chi\left(g^{-1}\right)\right) + \sum_{\substack{(a,b)\in\Omega\\ \mathrm{crr}(0,b,a,\infty)\text{ not square}}} \sum_{\substack{g \text{ s.t.}\\ 0^g=a,\ \infty^g=b}} \chi\left(g^{-1}\right). \tag{2}$$

Arguing as in the case of q even, we get that the first three summands in Eq. (2) are $(q^2-1)(q-1)/2$, 0 and 0. Now, consider the subset $\Delta = \{(a,b) \in \Omega \mid \mathrm{crr}(0,b,a,\infty) \text{ not a square}\}$ of Ω and the subset $S = \{g \in G_q \mid (0,\infty)^g \in \Delta\}$ of G_q. Since there are $(q-1)^2/2$ elements in Δ we have that $|S| = (q-1)^3/2$.

If $\chi = \psi_{-1}$, then $|\chi(g^{-1})| \in \{0,1\}$, for every $g \in S$. From Eq. (2) and the previous paragraph, we have $(Nv_\chi)_{(0,\infty)} \geqslant (q^2-1)(q-1)/2 - |S| > 0$, so $s_{\psi_{-1}} > 0$.

If (a,b) is in Δ and g_{ab} is in G_q such that $(0,\infty)^{g_{ab}} = (a,b)$, then the set of elements of G_q mapping $(0,\infty)$ to (a,b) is the coset Tg_{ab}. Since, $\mathrm{crr}(0,b,\infty,a) = \mathrm{crr}(0,b,a,\infty)^{-1}$ and $(a,b) \in \Delta$, we see that $\mathrm{crr}(0,b,\infty,a)$ is not a square. So, by Proposition 8, we get that Tg_{ab} contains exactly $(q+1)/2$ elements conjugate to v_r, for some r. Therefore Tg_{ab} contains at most $(q-3)/2$ elements conjugate to d_x, for some x.

Assume $\chi = \nu_\gamma$. Since $\chi(g) = 0$ if g is conjugate to v_r (for some r), $|\chi(g)| \leqslant 2$ if g is conjugate to d_x (for some x) and $|\Delta| = (q-1)^2/2$, we obtain that the last summand on the right-hand side of Eq. (2) is greater than or equal to

$$\frac{(q-1)^2}{2} \cdot \frac{q-3}{2} \cdot (-2) = -\frac{(q-1)^2(q-3)}{2} > -\frac{(q^2-1)(q-1)}{2}.$$

Hence $s_{\nu_\gamma} > 0$.

Assume $\chi = \eta_\beta$. Now $|\chi(g)| \leqslant 2$ if g is conjugate to v_r (for some r), $\chi(g) = 0$ if g is conjugate to d_x (for some x) and $|\Delta| = (q-1)^2/2$. Checking Table 2, we see that $-2 = \chi(v_r) = -\beta(r) - \beta(r^{-1})$ if and only if $r\mathbb{F}_q^* \in \mathrm{Ker}\,\beta$. Since β has order greater than 2, we get $|\mathrm{Ker}\,\beta| < (q+1)/2$. Hence the last summand on the right-hand side of Eq. (2) is greater than

$$\frac{(q-1)^2}{2} \cdot \frac{q+1}{2} \cdot (-2) = -\frac{(q^2-1)(q-1)}{2}.$$

So, $s_{\eta_\beta} > 0$. □

Now, we construct a submatrix $\overline{A}$ of A. In the submatrix $\overline{A}$, we keep all the rows of A and we delete the columns indexed by the ordered pairs (∞,t), (t,∞), for every $t \in \mathbb{F}_q$. In particular, we get

$$\overline{A} = \begin{pmatrix} 1 & 0 \\ 0 & \overline{M} \\ B & \overline{C} \end{pmatrix},$$

where the matrix $\overline{M}$ and $\overline{C}$ are obtained by deleting the appropriate columns of M and C. The matrix $\overline{A}$ has q^2+1 columns and $\overline{M}$ has $q(q-1)$ columns.

Proposition 10. *We have* $\mathrm{rank}(A) = \mathrm{rank}(\overline{A})$ *and* $\overline{M}$ *has full column rank.*

Proof. We start by proving that the columns indexed by (∞,t), (t,∞) (for $t \in \mathbb{F}_q$) of A are a linear combination of the columns of $\overline{A}$. We denote by a_{xy} the column of A indexed by the ordered pair (x,y), for $x,y \in \mathbb{P}_q$. Since G_q is 2-transitive, it suffices to prove that $a_{0\infty}$ is a linear combination of the columns of $\overline{A}$. Set $v = \sum_{x\neq 0,\infty}\sum_{y\neq\infty} a_{xy}$ and $w = (q-2)\sum_{x\neq\infty} a_{0x} + a_{\infty\infty}$. By construction, v and w are a linear combination of the columns of $\overline{A}$. Also, it is easy to check that

$$v_g = \begin{cases} q-1 & \text{if } 0^g = \infty \text{ or } \infty^g = \infty, \\ q-2 & \text{otherwise,} \end{cases} \qquad w_g = \begin{cases} 0 & \text{if } 0^g = \infty, \\ q-1 & \text{if } \infty^g = \infty, \\ q-2 & \text{otherwise.} \end{cases}$$

Hence $(q-1)a_{0\infty} = v - w$ and we get that $a_{0\infty}$ is a linear combination of the columns of $\overline{A}$. Thence $\mathrm{rank}(A) = \mathrm{rank}(\overline{A})$.

Since $\overline{M}$ has $q(q-1)$ columns, Proposition 9 shows that $\overline{M}$ has full column rank. □

5. Proof of Theorem 1

At this point, we have all the tools to conclude the proof of Theorem 1. By Lemma 4, it remains to prove that if S is an independent set of maximal size of Γ_{G_q}, then S is the coset of the stabilizer of a point. Up to multiplication of S by a suitable element of G_q, we may assume that the identity element of G_q is in S. In particular, we have to prove that S is the stabilizer of a point. By Lemma 7 and Proposition 10, we have that the characteristic vector v_S of S is a linear combination of the columns of $\overline{A}$. Hence

$$\begin{pmatrix} 1 & 0 \\ 0 & \overline{M} \\ B & \overline{C} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = v_S,$$

for some vectors v, w. As the identity element of G_q is in S and by the ordering of the rows of $\overline{A}$, we get

$$v_S = \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}.$$

So, $1^T v = 1$, $\overline{M} w = 0$ and $Bv + \overline{C} w = t$. By Proposition 10, the matrix $\overline{M}$ has full column rank. Thence $w = 0$ and $Bv = t$.

Now, for every point x of $\mathbb{P}_q$, there exists a permutation g_x of G_q fixing x and acting fixed-point-freely on $\mathbb{P}_q \setminus \{x\}$ (indeed, g_x can be chosen any non-identity unipotent matrix of G_q fixing x). Order the rows of B so that the first $q+1$ rows are labelled by the permutations $\{g_x\}_x$. In particular, up to permuting the rows of B, we get

$$B = \begin{pmatrix} I_{q+1} \\ B' \end{pmatrix} \quad \text{and} \quad Bv = \begin{pmatrix} v \\ B'v \end{pmatrix}.$$

Since Bv is equal to the $\{0,1\}$-vector t, we obtain that v is a $\{0,1\}$-vector. But $1^T v = 1$ and so v must be the characteristic vector of a point p of $\mathbb{P}_q$. This shows that v_S is the stabilizer of the point p and the proof is complete.

6. Comments

Theorem 1 proves that in the derangement graph Γ_G, where G is the group $\mathrm{PGL}(2,q)$, the independent sets of maximal size are the cosets of the stabilizer of a point. The same result holds if G is the symmetric group [3,15] or if G is the alternating group [14]. It is interesting to ask for which other permutation groups does this result hold, and is there a way to characterize the permutation groups that have this property?

In Lemma 5, we saw that the cosets of the stabilizer of a point are independent sets of maximal size in the derangement graph of $\mathrm{PSL}(2,q)$. We further conjecture that, similar to the case for $\mathrm{PGL}(2,q)$, $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, all independent sets of maximal size in $\Gamma_{\mathrm{PSL}(2,q)}$ are cosets of the stabilizer of a point.

Conjecture 1. *Every independent set S of the derangement graph of* $\mathrm{PSL}(2,q)$ *acting on the projective line $\mathbb{P}_q$ has size at most $q(q-1)/2$. Equality is met if and only if S is the coset of the stabilizer of a point.*

It seems likely that the methods used in this paper could be applied to $\mathrm{PSL}(2,q)$, since the character table of this group is well understood.

In our proof of Theorem 1, we used the fact that the group $\mathrm{PGL}(2,q)$ is 2-transitive (for example, in the proofs of Propositions 8 and 10). It is a reasonable question to ask if this (being 2-transitive) could be a characterization of the groups that have the property that the independent sets of maximal size in the derangement graph are the cosets of the stabilizer of a point. It is not hard to show that

Table 4
Socle of an almost simple 3-transitive group.

Group	M_{11}	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	$\mathrm{PSL}(2,q)$	$\mathrm{Alt}(n)$
Degree	11	12	12	22	23	24	$q+1$	n

Table 5
Point stabilizer of an affine 3-transitive group.

Group	$\mathrm{SL}(n,2) \leqslant H \leqslant \Gamma\mathrm{L}(n,2)$	$\mathrm{GL}(1,3)$	$\Gamma\mathrm{L}(1,4)$	$\mathrm{Alt}(7)$
Degree	2^n	3	4	16

this is not a characterization, and it is further interesting to see how this property can fail to hold for some 2-transitive groups.

If G is a 2-transitive group, then the permutation character of G is the sum of the trivial character and an irreducible character that we will call ψ. The eigenvalue of the derangement graph Γ_G arising from ψ is $-\frac{d}{\psi(1)}$, where d is the valency of Γ_G. If this eigenvalue is indeed the least eigenvalue of Γ_G, then by Lemma 3, we have that the size of an independent set is no bigger than the size of the stabilizer of a point in G. Thus, if it were true that the eigenvalue arising from the character ψ is the least eigenvalue, then we would have the bound like in Theorem 1. But the characterization of the sets that meet this bound is another question entirely. In fact, there are examples of 2-transitive groups in which there are independent sets of maximal size that are not cosets of the stabilizer of a point.

For example, let $G_{n,q}$ be the 2-transitive group $\mathrm{PGL}(n+1,q)$ in its action on the projective space $\mathbb{P}_q^n$, with $n \geqslant 1$. Since $G_{n,q}$ contains a Singer cycle of length $(q^{n+1}-1)/(q-1)$, the graph $\Gamma_{G_{n,q}}$ has a clique of size $|\mathbb{P}_q^n| = (q^{n+1}-1)/(q-1)$. Thus, any independent set of $\Gamma_{G_{n,q}}$ has size at most $|G_{n,q}|/|\mathbb{P}_q^n|$. Naturally, the stabilizer of a point is an independent set for $\Gamma_{G_{n,q}}$ of size $|G_{n,q}|/|\mathbb{P}_q^n|$ and so it is an independent set of maximal size. But for $G_{n,q}$ with $n \geqslant 2$, the cosets of the stabilizer of a point are not the only independent sets of maximal size. Indeed, it is not hard to see that the stabilizer of a hyperplane of $\mathbb{P}_q^n$ in $G_{n,q}$ is also an independent set of maximal size for $\Gamma_{G_{n,q}}$. Moreover, if $n \geqslant 2$, then the stabilizer of a point and of a hyperplane are not conjugate subgroups of $G_{n,q}$. Therefore, for $n \geqslant 2$, the graph $\Gamma_{G_{n,q}}$ contains at least $2(|G_{n,q}|/|\mathbb{P}_q^n|)^2$ independent sets of maximal size. We make the following conjecture.

Conjecture 2. *Any independent set of maximal size in the derangement graph of* $\mathrm{PGL}(n+1,q)$ *acting on the projective space* $\mathbb{P}_q^n$ *is either the coset of the stabilizer of a point or the coset of the stabilizer of a hyperplane.*

It is not clear that the method in this paper could be used to prove this conjecture. In particular, even if the character table of $\mathrm{PGL}(n+1,q)$ is known [9], it would still be challenging to obtain the minimum eigenvalue of $\Gamma_{G_{n,q}}$ as in Lemma 4.

Further, there exist 2-transitive groups of degree n where the number of independent sets of maximal size is n^{n-1}. Clearly, this means that there are many independent sets of maximal size which are not cosets of the stabilizer of a point. For example, let n be a power of a prime and $\mathbb{F}_n$ the field with n elements. The affine general linear group G on $\mathbb{F}_n$ (i.e. the group generated by the permutations of $\mathbb{F}_n$ of the form $f_{a,b}: \xi \mapsto a\xi + b$, for $a, b \in \mathbb{F}_n$ and $a \neq 0$) is a 2-transitive group. Since G is a Frobenius group with kernel of size n and complement of size $n-1$, we obtain that Γ_G is the disjoint union of $n-1$ complete graphs K_n. In particular, we get that Γ_G has n^{n-1} independent sets of maximal size.

From the above comments it is clear that a result similar to Theorem 1 does not hold for all 2-transitive groups, but perhaps we do better to consider 3-transitive groups. In particular, some information on the 3-transitive groups is listed in Tables 4 and 5, see [2, pp. 195, 197]. Table 4 lists all possible socles of an almost simple 3-transitive group and Table 5 gives all possible point stabilizers of an affine 3-transitive group.

Using GAP [7], it is straight-forward to build the derangement graph for each of the 3-transitive groups of degree $11, 12, 22, 23, 24$ and 16 and to then find all the independent sets of maximal size. Indeed, for every one of these groups, every independent set of maximal size of Γ_G is the coset of the stabilizer of a point. Thus we conclude with the following conjecture.

Conjecture 3. *Let G be a 3-transitive group of degree n. Every independent set S of the derangement graph of G acting on Ω has size at most $|G|/n$. Equality is met if and only if S is the coset of the stabilizer of a point.*

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