Invariant subspaces for tridiagonal operators

Sous-espaces invariants pour des opérateurs tridiagonaux

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Abstract

We consider certain complex sequence spaces $X$ indexed by $\mathbb{N}$ with the canonical basis $(\delta_n)_{n \geq 1}$. Let $T \in \mathcal{L}(X)$ be a tridiagonal operator on $X$. Assume that the associated matrix $(t_{i,j})_{i,j \geq 1}$ has real entries and satisfies the weak symmetry condition that for every integer $n \geq 1$, $t_{n,n+1}t_{n+1,n} \geq 0$. Then $T$ has a non-trivial closed invariant subspace.

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1. Introduction

The aim of this work is to obtain closed non-trivial invariant subspaces for certain classes of operators on Banach spaces. Let $\mathcal{L}(X)$ be the algebra of bounded linear operators...
on a Banach space $X$. We recall that if $T$ belongs to $\mathcal{L}(X)$, a closed subspace $M$ of $X$ is said to be invariant by $T$ if $T(M) \subseteq M$, and the space $M$ is non-trivial if $M \neq \{0\}$ and $M \neq X$.

P. Enflo and C. Read constructed examples of bounded operators on a Banach space which have no non-trivial invariant subspace (see [6] and [8]). But if $X$ is a separable reflexive Banach space, in particular if it is a Hilbert space, it is still unknown whether every bounded operator on $X$ has a non-trivial invariant subspace.

In order to obtain non-trivial invariant subspaces, we use in this work a recent result of A. Atzmon and G. Godefroy [3] which states that every operator on a real Banach space admitting a moment sequence has a non-trivial invariant subspace.

We now give a brief description of our results.

In Section 2, we obtain moment sequences for operators on a real separable Hilbert space $H$ in a purely algebraic way: one replaces the relation of similarity of operators by a weaker one, where the relevant linear mappings need not be continuous. These results are then generalized to a wider class of Banach spaces, called sequence spaces, including for instance the spaces $\ell^p_\omega$, with $1 \leq p < +\infty$, and $c_0^\omega$, where $\omega$ is a positive weight on $\mathbb{N}$.

Section 3 contains the main result of the paper, which runs as follows: assume that $X$ is a Banach space of complex sequences indexed by the set of positive integers $\mathbb{N}$. For every integer $n \geq 1$, let $\delta_n$ denote the sequence $(\delta_{n,k})_{k \geq 1}$, where $\delta_{n,k} = 0$ if $n \neq k$ and $\delta_{n,k} = 1$ if $n = k$. An infinite sequence of complex numbers $(x_n)_{n \geq 1}$ is said to be finitely supported if $x_n = 0$ except for finitely many integers $n$. Suppose that $X$ satisfies the following three conditions:

(a) $X$ contains all finitely supported sequences;
(b) The coordinate functionals on $X$ are continuous;
(c) The space $X$ is self-adjoint, which means that for every sequence $(x_n)_{n \geq 1}$ in $X$, the sequence $(\overline{x_n})_{n \geq 1}$ is also in $X$.

A bounded operator $T$ on $X$ is called tridiagonal if for every integer $n \geq 1$, the vector $T \delta_n$ is a linear combination of $\delta_{n-1}, \delta_n$ and $\delta_{n+1}$, with the convention that $\delta_0 = 0$. Alternatively, $T$ is tridiagonal if it maps the set $E_0^X$ of finitely supported sequences of $X$ into itself and the matrix $(t_{i,j})_{i,j \geq 1}$ of the restriction of $T$ to $E_0^X$ with respect to the basis $(\delta_n)_{n \geq 1}$ is tridiagonal.

**Theorem.** Let $T \in \mathcal{L}(X)$ be a tridiagonal operator on $X$. Assume that the associated matrix $(t_{i,j})_{i,j \geq 1}$ has real entries and satisfies the weak symmetry condition that for every integer $n \geq 1$, $t_{n,n} + t_{n+1,n} \geq 0$. Then $T$ has a non-trivial invariant subspace.

This result appears as Theorem 3.2. It extends a recent theorem of A. Atzmon, who proves in [1] that every bishift on a sequence space has a non-trivial invariant subspace. Section 4 contains some remarks and questions and a result due to A. Atzmon (Theorem 4.2) showing that Theorem 3.2 is in a sense best possible.
2. Moment sequences

We begin by recalling the definition of a moment sequence, which was introduced in [1]:

Definition 2.1. Let $X$ be a real or complex separable Banach space and $T$ a bounded linear operator on $X$. The operator $T$ admits a moment sequence if there exists a non-zero vector $x_0$ in $X$, a non-zero functional $x_0^*$ in $X^*$ and a positive measure $\mu$ on $\mathbb{R}$ such that for every integer $n \geq 0$ we have:

$$\langle T^n x_0, x_0^* \rangle = \int_{\mathbb{R}} t^n \, d\mu(t).$$

Such a pair $(x_0, x_0^*)$ will be called a moment pair for $T$.

Remark 2.2. If $(x_0, x_0^*)$ is a moment pair for $T$, any positive measure $\mu$ satisfying the relation above must actually have compact support. Indeed, it follows from the spectral radius formula that if $\rho(T)$ denotes the spectral radius of $T$, the support of $\mu$ is contained in the interval $[-\rho(T), \rho(T)]$.

A useful fact about moment pairs is the following:

Proposition 2.3. Let $x_0$ be a non-zero vector in $X$, and $x_0^*$ a non-zero functional in $X^*$. The followings are equivalent:

1. $(x_0, x_0^*)$ is a moment pair for $T$;
2. For every polynomial $p$ in $\mathbb{R}[X]$ such that $p \geq 0$ on $\mathbb{R}$, we have $\langle p(T)x_0, x_0^* \rangle \geq 0$;
3. For every polynomial $q$ in $\mathbb{R}[X]$, we have $\langle q(T)x_0, q(T^*)x_0^* \rangle \geq 0$.

Proof. The equivalence between assertions (1) and (2) is well-known (a proof may be found for instance in Koosis’s book [7] on p. 110). The equivalence between (2) and (3) is a straightforward consequence of the fact that every positive polynomial $p$ in $\mathbb{R}[X]$ can be written as a sum $p(X) = q(X)^2 + r(X)^2$ where $q$ and $r$ are polynomials in $\mathbb{R}[X]$. \qed

It is now obvious that self-adjoint operators have moment pairs:

Example 2.4. Let $S$ be a self-adjoint operator on a real or complex separable Hilbert space $H$. Then for every non-zero $x$ in $H$, $(x, x)$ is a moment pair for $S$.

Every operator $T$ on $X$ having a non-trivial invariant subspace clearly has a moment pair: indeed, let $M$ be a non-trivial closed subspace such that $T(M) \subseteq M$. By the Hahn–Banach theorem, there exists a non-zero functional $x_0^*$ in $X^*$ such that for all $x$ in $M$, $(x, x_0^*) = 0$. If $x_0$ is any non-zero vector in $M$, this implies that for every integer $n \geq 0$, $\langle T^n x_0, x_0^* \rangle = 0$. Choosing for $\mu$ the zero measure on $\mathbb{R}$, we see that $(x_0, x_0^*)$ is a moment pair for $T$. Our work relies on the fact that the converse is true on real separable Banach spaces:
Theorem 2.5 [1,3,4]. Let $X$ be a real separable Banach space and $T$ a bounded operator on $X$. If $T$ has a moment pair, then $T$ has a non-trivial invariant subspace.

Thus, in order to obtain non-trivial invariant subspaces, it suffices to look for moment pairs.

Notation. We will be concerned first with linear mappings on a real separable Hilbert space $H$.

1. Let $B = (e_n)_{n \geq 1}$ be any orthonormal basis of $H$. Every vector of $H$ can be written as a sum $x = \sum s_n e_n$ with $\sum |s_n|^2 < +\infty$. A vector $x$ has finite support (or is finitely supported) if $s_n = 0$ except for finitely many $n$’s. We denote by $E_0^H$ (or $E_0$ when no risk of confusion occurs) the space of finitely supported vectors in $H$, and it will be understood that this support is taken with respect to the basis $B$ of $H$.

2. If $L$ is any linear mapping of $E_0$ into $E_0$, $L$ will be identified with its matrix with respect to the basis $(e_n)_{n \geq 1}$, and its coefficients will be denoted by $(l_{i,j})_{i,j \geq 1}$. The adjoint (or transpose) of the matrix $L$ is defined as usual by $L^* = (l_{j,i})_{i,j \geq 1}$. In general, the matrix obtained by transposing $L$ does not even define a linear mapping of $E_0$ into $E_0$.

3. The matrix $L$ of any linear mapping of $E_0$ into $E_0$ is said to be symmetric if $L^* = L$.

4. A moment pair $(x_0, y_0)$ for an operator $T$ on $H$ is said to be finitely supported if the vectors $x_0$ and $y_0$ both have finite support.

Example 2.6. It is worth remarking here that Example 2.4 can be readily extended to the case where $S$ is just an infinite symmetric matrix with $S(E_0) \subseteq E_0$: for every finitely supported non-zero vector $x$ and for every polynomial $p \geq 0$, the quantity $(p(S)x, x)$ is well-defined and $(p(S)x, x) \geq 0$.

The following simple proposition allows us to obtain moment sequences for operators by using purely algebraic devices, notwithstanding any question of continuity of the operators:

Proposition 2.7. Let $T$ be a linear map on $E_0$ such that $T(E_0) \subseteq E_0$. We assume that there exists

- a symmetric matrix $S$ with $S(E_0) \subseteq E_0$, or a matrix $L$ such that $L(E_0) \subseteq E_0$, $L$ is invertible on $E_0$, $L$ admits an adjoint $L^*$ on $E_0$ with $L^*(E_0) \subseteq E_0$, and that either $LT = SL$ on $E_0$ or $TL = LS$ on $E_0$.

Then there exists two non-zero vectors $x$ and $y$ in $E_0$ such that for all positive polynomials $p$ in $\mathbb{R}[X]$, $(p(T)x, y) \geq 0$. 

Remark 2.8. Neither $L$ nor $S$ are supposed to be bounded. But the assumption that $L$ has an adjoint on $E_0$ cannot be dispensed with. It is also worth noticing that $L$ may have an adjoint on $E_0$ without $L^{-1}$ having one. A simple example of this is:

$$L = \begin{pmatrix} 1 & -2 & \cdots & 0 \\ 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad L^{-1} = \begin{pmatrix} 1 & 2 & 4 & 8 & \cdots \\ 1 & 2 & 4 & 8 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots \end{pmatrix}.$$

Proof. Let us first assume that $LT = SL$. Then for all $n \geq 0$, $LT^n = S^n L$. For every positive polynomial $p$ in $\mathbb{R}[X]$ and for every pair $(x, y)$ of non-zero vectors of $E_0$, $(Lp(T)x | y) = (p(S)Lx | y)$. We now take $y = Lx$. Example 2.6 implies that for every positive polynomial $p$ in $\mathbb{R}[X]$ and for every non-zero $x$ in $E_0$, $(Lp(T)x | Lx) \geq 0$. The vectors $p(T)x$ and $Lx$ belong to $E_0$. Since $L$ has an adjoint on $E_0$, this yields that for all $x$ in $E_0 \setminus \{0\}$ and for all positive polynomials $p$, $(p(T)Lx | L^*Lx) \geq 0$. Now $L$ is an isomorphism of $E_0$, so $L^*Lx$ is non-zero, and $(x, L^*Lx)$ is a finitely supported moment pair for $T$ for any $x$ in $E_0 \setminus \{0\}$.

We can now consider a wider class of separable Banach spaces:

Definition 2.9. Let $X$ be a Banach space of real or complex sequences indexed by $\mathbb{N}$. For $n \geq 1$, let $\delta_n$ denote the sequence $(\delta_{n,k})_{k \geq 1}$, where $\delta_{n,k}$ is equal to 0 when $n \neq k$ and to 1 when $n = k$. We also denote by $\delta^*_n$ the $n$th coordinate functional on $X$ defined by $\delta^*_n((x_k)_{k \geq 1}) = x_n$.

The space $X$ will be called a sequence space if it is of one of the following two types:

1. $X$ is a real space which satisfies the following conditions:
   (a) $X$ contains all finitely supported sequences;
   (b) For every positive integer $n$, the coordinate functional $\delta^*_n$ is bounded on $X$.

2. $X$ is a complex space satisfying conditions (a) and (b) and the additional condition
   (c) $X$ is self-adjoint, which means that for every sequence $(x_n)_{n \geq 1}$ in $X$, the sequence $(\overline{x_n})_{n \geq 1}$ is also in $X$.

For instance, $X$ could be one of the spaces $\ell^p_\omega$, $1 \leq p < +\infty$, or $c^0_\omega$, where $\omega$ is a positive weight on $\mathbb{N}$.

We denote by $E_0^X$ the vector subspace of $X$ generated by the sequence $(\delta_n)_{n \geq 1}$, and by $X^*_0$ the vector subspace of $X^*$ generated by the sequence $(\delta^*_n)_{n \geq 1}$. If $T$ is a bounded operator on $L(X)$, a moment pair $(x_0, x^*_0)$ is said to be finitely supported if $x_0$ belongs to $E^X_0$ and $x^*_0$ belongs to $E^*_0$. 
Remark 2.10. Spaces of type (2) were introduced in [1]. In such spaces, it often suffices to work in the space $E$ of real sequences in $X$. Following [1], we note that $E$ is closed in $X$ and that $X = E + iE$ as a topological sum. If $T$ is any bounded operator on $X$ such that $T(E) \subseteq E$, we define $T_0$ to be the restriction of $T$ to $E$. If $M$ is a real non-trivial closed subspace of $E$ which is invariant by $T_0$, then $M + iM$ is a non-trivial complex subspace of $X$ which is invariant by $T$. Thus, in order to obtain invariant subspaces for bounded operators on such spaces, it suffices to obtain moment sequences for their restriction to $E$.

If $X$ is a real sequence space, let $H$ be the usual real Hilbert space $H = \ell_2$, with the canonical basis $(e_n)_{n \geq 1}$. We denote by $(\cdot | \cdot)_H$ the scalar product on $H$. Let $J : E_0^X \rightarrow E_0^H$ be the linear isomorphism which maps $\delta_n$ on $e_n$ for every integer $n \geq 1$. If $T \in \mathcal{L}(X)$ satisfies $T(E_0^X) \subseteq E_0^X$, we define $\widetilde{T}$ on $E_0^H$ by $\widetilde{T} = JTJ^{-1}$. In other words, the matrix of the restriction of $T$ to $E_0^X$ with respect to the basis $(\delta_n)_{n \geq 1}$ is equal to the matrix of the restriction of $\widetilde{T}$ to $E_0^H$ with respect to the basis $(e_n)_{n \geq 1}$.

We now have the following useful proposition:

Proposition 2.11. Let $X$ be a sequence space, and let $T$ belong to $\mathcal{L}(X)$. If $X$ is a complex sequence space, we assume additionally that the space $E$ of real sequences of $X$ is invariant by $T$. If there exists two non-zero vectors $x$ and $y$ of $E_0^H$ such that for all positive polynomials $p$ on $\mathbb{R}$, $(p(\tilde{T})x, y)_H \geq 0$, then $T$ has a finitely supported moment pair. In particular, $T$ has a non-trivial invariant subspace.

Proof. By Remark 2.10, it is possible to assume without loss of generality that $X$ is a real sequence space. Since the vectors $x$ and $y$ are finitely supported, we can consider $J^*$ as a linear mapping between the spaces $E_0^H$ and $E_0^{\mathcal{X}}$, and it is easy to check that for all polynomials $p$ in $\mathbb{R}[X]$, $(p(\tilde{T})x, y)_H = (p(T)J^{-1}x, J^*y)$. Thus $(J^{-1}x, J^*y)$ is a moment pair for $T$ by Proposition 2.7. \qed

3. Tridiagonal operators

In this part, we will be concerned with bounded tridiagonal operators on sequence spaces. We recall the definition of a tridiagonal operator:

Definition 3.1. A bounded operator $T$ on a sequence space $X$ with the canonical basis $(\delta_n)_{n \geq 1}$ is called tridiagonal if for every integer $n \geq 1$, the vector $T\delta_n$ is a linear combination of $\delta_{n-1}$, $\delta_n$ and $\delta_{n+1}$, with the convention that $\delta_0 = 0$. Alternatively, $T$ is tridiagonal if it maps the set $E_0^X$ into itself and the matrix $(t_{i,j})_{i,j \geq 1}$ of its restriction to $E_0^X$ with respect to the basis $(\delta_n)_{n \geq 1}$ is tridiagonal:

$$ T = \begin{pmatrix} t_{1,1} & t_{1,2} & (0) \\ t_{2,1} & t_{2,2} & \ddots \\ (0) & \ddots & \ddots \end{pmatrix}.$$
Theorem 3.2. Let $X$ be a sequence space and let $T$ be a tridiagonal operator on $X$. Assume that the associated matrix $(t_{i,j})_{i,j \geq 1}$ has real entries and satisfies the weak symmetry condition that for every integer $n \geq 1$, $t_{n-1,n+1}t_{n+1,n} \geq 0$. Then $T$ has a finitely supported moment pair. In particular, $T$ has a non-trivial invariant subspace.

Proof. Since the matrix of $T$ has real entries, the space of real sequences of $X$ is invariant by $T$ in the case where $X$ is a complex sequence space. Thus we can suppose to begin with that $X$ is a real sequence space. Let $\tilde{T}$ be the linear map defined on $E^H_0$ as in Proposition 2.11, where $H$ is the real Hilbert space $\ell^2$. We want to prove that there exists two non-zero vectors $x$ and $y$ of $E^H_0$ such that for all positive polynomials $p$ on $\mathbb{R}$, $(p(\tilde{T})x, y)_H \geq 0$. The matrix of $\tilde{T}$ with respect to the canonical basis $(e_n)_{n \geq 1}$ is tridiagonal and satisfies the assumptions of Theorem 3.2. Thus we can suppose that $T$ is an operator on the real Hilbert space $\ell^2$.

There is no loss of generality in assuming that for all $n \geq 1$, $t_{n-1,n+1}t_{n+1,n} > 0$. Indeed, if there exists an $n_0 \geq 1$ such that $t_{n_0-1,n+1}t_{n+1,n_0} = 0$, then $(e_1, e_{n_0+1})$ is clearly a moment pair, and if there exists an $n_0$ such that $t_{n_0-1,n+1}t_{n+1,n_0} = 0$, then in the same way $(e_{n_0+1}, e_1)$ is a moment pair. We now assume that for all $n \geq 1$, $t_{n-1,n+1}t_{n+1,n} > 0$.

A sequence $(\alpha_n)_{n \geq 1}$ may be defined as follows:

$$\alpha_1 = 1, \quad \text{and for all } n \geq 1, \quad \alpha_{n+1} = \sqrt{t_{n,n+1}t_{n+1,n}} \frac{1}{\alpha_n}.$$ 

Let $L$ be the diagonal matrix

$$L = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}.$$ 

Since all the $\alpha_n$’s are non-zero, $L$ is the matrix of an invertible map between $E_0$ and $E_0$. Working in $E_0$, we get $LTL^{-1} = S$, where $S$ is the following tridiagonal matrix:

$$S = \begin{pmatrix} t_{1,1} & \alpha_1 & \alpha_2 & \cdots & 0 \\ \frac{\alpha_2}{\alpha_1} & t_{2,2} & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \frac{\alpha_{n+1}}{\alpha_n} & \alpha_{n+1} & t_{n,n+1} & \cdots \\ 0 & 0 & \cdots & t_{n+1,n} & \alpha_n \end{pmatrix}.$$ 

Then one easily checks that for all $n \geq 1$

$$\frac{\alpha_{n+1}}{\alpha_n} \frac{t_{n,n+1}}{t_{n+1,n}} = \frac{\alpha_n}{\alpha_{n+1}} \frac{t_{n-1,n+1}}{t_{n,n+1}} = \sqrt{t_{n,n+1}t_{n+1,n}}.$$ 

This means that $S$ is a symmetric matrix, and $S(E_0) \subseteq E_0$. Being diagonal, $L$ has an adjoint on $E_0$, and $L^* = L$. Since $LT = SL$, Proposition 2.7 implies that for every $x$ in $E_0 \setminus \{0\}$, the pair $(x, L^2x)$ is a moment pair for $T$. An example of such a pair is $(e_1, e_1)$. This finishes the proof of Theorem 3.2. $\square$
Let $X$ be a sequence space such that $X$ has the following additional property: the transformation $B$ that sends a sequence $(x_n)_{n \geq 1}$ to the sequence $(x_{n-1} + x_{n+1})_{n \geq 1}$ (with $x_0 = 0$) maps $X$ into itself. By the closed graph theorem, $B$ is a bounded operator on $X$.

**Definition 3.3** [1]. $B$ is called the bishift on $X$.

As a special case of Theorem 3.2, we obtain the following result of A. Atzmon:

**Corollary 3.4** [1]. Every bishift on $X$ has a non-trivial invariant subspace.

Bishifts on a Hilbert space were first considered in [5]. If the sequence $\alpha = (\alpha_n)_{n \geq 1}$ is bounded below and above, the matrix of the bishift $B_\alpha$ with respect to the basis $(e_n)_{n \geq 1}$ of $H$ is

$$B_\alpha = \begin{pmatrix} 0 & 1 & (0) \\ \alpha_1 & 0 & \frac{1}{\alpha_2} \\ \alpha_2 & 0 & \ddots \\ (0) & \ddots & \ddots \end{pmatrix}.$$

These bishifts were considered mainly in the special case where $\lim \alpha_n = 1$. $B_\alpha$ is then essentially self-adjoint. It is shown in [5] that if the $\alpha_n$’s satisfy some additional growth conditions, then $B_\alpha$ satisfies:

(a) the spectrum $\sigma(B_\alpha)$ is the whole interval $[-2, 2]$,
(b) for any non-trivial invariant subspace $M$ of $T$, the spectrum $\sigma(B_\alpha|_M)$ of the restriction of $B_\alpha$ to $M$ is $[-2, 2]$,
(c) for any non-trivial invariant subspace $N$ of $T^*$, the spectrum $\sigma(B_\alpha^*|_N)$ of the restriction of $B_\alpha^*$ to $N$ is $[-2, 2]$.

Such operators are said to be completely indecomposable. These examples show that tridiagonal operators can in fact have a very complicated structure in the sense that their invariant subspaces cannot in general be obtained by splitting the spectrum.

4. Concluding remarks and questions

(1) Using essentially the same method, it is possible to prove the following theorem on finite-dimensional perturbations of bishifts:

**Theorem 4.1.** Let $X$ be a sequence space with the canonical basis $(\delta_n)_{n \geq 1}$ such that the bishift $B$ is a bounded operator on $X$. Let $R$ be a finite rank operator having the following property: there exists an integer $r \geq 1$ such that for all $k \in [1, r + 1]$, $R(\delta_k)$ is a linear combination of $\delta_1, \ldots, \delta_{r+1}$ with real coefficients, and for all $k \geq r + 2$, $R(\delta_k) = 0$.
In the other words, the matrix of $B + R$ with respect to the basis $(\delta_n)_n \geq 1$ has the form:

$B + R = \begin{pmatrix}
  t_{1,1} & t_{1,2} & \cdots & t_{1,r+1} \\
  t_{2,1} & t_{2,2} & \vdots & (0) \\
  \vdots & \vdots & \ddots & \vdots \\
  t_{r+1,1} & t_{r+1,2} & \cdots & t_{r+1,r+1} \\
  (0) & \alpha_{r+2} & \frac{1}{\alpha_{r+3}} & \cdots \\
  \alpha_{r+3} & 0 & \ddots & \ddots \\
\end{pmatrix}$

where all the coefficients are real and the sequence $(\alpha_n)_n \geq r+2$ is bounded from below and above.

Then $B + R$ has a finitely supported moment pair, and in particular it has a non-trivial invariant subspace.

The proof of Theorem 4.1 is quite technical and we do not present it here.

(2) If we consider tridiagonal operators on Hilbert spaces, we can wonder if the weak symmetry condition $t_{n,n+1}t_{n+1,n} \geq 0$ is necessary in order to obtain finitely supported moment pairs. It is indeed so. Let us suppose that $T$ is tridiagonal, has zeroes on the main diagonal and satisfies the condition that for all $n \geq 1$, $t_{n,n+1}t_{n+1,n} < 0$. For instance, $T$ could be $S - S^*$, where $S$ is the standard shift on $\ell^2$. The construction of Section 3 allows us to suppose without loss of generality that $T$ is skew-symmetric tridiagonal. The following result is due to A. Atzmon [2], who kindly allowed me to reproduce it here:

**Theorem 4.2.** Let $T$ be a skew-symmetric tridiagonal operator on the real or complex Hilbert space $\ell^2$. If $T$ has no eigenvalue, then every moment pair $(x, y)$ for $T$ satisfies that for all $n \geq 0$, $(T^nx|y) = 0$. Moreover, $x$ and $y$ both have infinitely many non-zero coordinates.

**Proof.** Let us first assume that the underlying space $\ell^2$ is complex. The proof relies on the following fact:

If $T$ is a skew-symmetric injective operator on $\ell^2$, and $(x, y)$ is a moment pair for $T$, then for all $n \geq 0$, $(T^nx|y) = 0$.

Indeed, if $(x, y)$ is such a moment pair, let us write

$$(T^nx|y) = \int_{\mathbb{R}} t^n d\mu(t)$$

for every $n \geq 0$, where $\mu$ is a positive measure on $\mathbb{R}$ with compact support. The operator $S = -iT$ is bounded and self-adjoint. Let $E$ be its spectral measure. It is supported by a
compact set of $\mathbb{R}$. If now $\nu$ is the complex measure defined by $d\nu(t) = dE_{x,y}(-t)$, then one easily checks that for all $n \geq 0$,

$$(S^n x | y) = \int_{\mathbb{R}} (-t)^n d\nu(t) \quad \text{and} \quad (T^n x | y) = \int_{\mathbb{R}} t^n d\mu(t) = i^n \int_{\mathbb{R}} t^n d\nu(t).$$

Let $\mu_1 = \mu - \nu$ and $\mu_2 = i^2(\mu + \nu)$. Both measures annihilate all the monomials $t^{4k}$, $k \geq 0$. Every real-valued continuous even function can be uniformly approximated on a given interval $[-a, a]$ of $\mathbb{R}$ by functions of the form $p(t^4)$, where $p$ is a real polynomial function. Thus, $\mu_1$ and $\mu_2$ annihilate all real-valued continuous even functions with compact support: $\mu_1$ and $\mu_2$ are odd measures. Since $i^2\mu_1$ is odd, the decomposition $2t^2\mu = t^2(\mu - \nu) + t^2(\mu + \nu)$ implies that $i^2\mu$ is an odd measure. It is also positive, so $i^2\mu = 0$. This implies that $t^2\mu$ is a positive multiple of the Dirac measure $\delta_0$ having $\{0\}$ as a support. Plugging this into the relation $(T^n x | y) = \int_{\mathbb{R}} t^n d\mu(t)$, one obtains that for all $n \geq 1$, $(T^n x | y) = 0$.

It remains to prove that $(x | y) = 0$. Let us consider the restriction of $S$ to the closed space $\overline{\text{span}}(S^n x, \ n \geq 0)$. Since $S$ is self-adjoint and injective, the closure of the range of this restriction is equal to $\overline{\text{span}}(S^n x, \ n \geq 0)$. Thus $\overline{\text{span}}(S^n x, \ n \geq 1) = \overline{\text{span}}(S^n x, \ n \geq 0)$ and $x$ belongs to $\overline{\text{span}}(S^n x, \ n \geq 1)$. It is now obvious that $(x | y) = 0$.

If in addition $T$ is tridiagonal and if $(x, y)$ is a moment pair, then it is impossible that both vectors $x$ and $y$ have finite support. Suppose indeed that $(x, y)$ is a finitely supported moment pair. We have just seen that for all $n \geq 0$, $(T^n x | y) = 0$. Now for all $n \geq 1$, $\text{span}(e_1, Te_1, \ldots, T^{n-1} e_1) = \text{span}(e_1, \ldots, e_n)$. There exists a non-zero polynomial $p$ in $\mathbb{R}[X]$ such that $p(T)e_1 = x$. So for all $n \geq 0$, $(T^n p(T)e_1 | y) = 0$, and since $T$ is skew-symmetric, $(T^n p(T)e_1 | p(-T)y) = 0$. Since $e_1$ is cyclic for $T$, $p(-T)y = 0$, so there exists a non-zero polynomial $q$ such that $q(T)e_1 = 0$: this contradicts the fact that $(e_1, e_2, \ldots)$ is an independent set of $E_0$.

If we moreover suppose that $T$ has no eigenvalue, a minor modification of the above proof shows that if $(x, y)$ is a moment pair for $T$, then neither $x$ nor $y$ can have finite support. This finishes the proof in the case where $\ell_2$ is complex.

If the space $\ell_2$ is real, the operator $T$ can be complexified in the obvious way. We thus get that for every moment pair $(x, y)$, where $x$ and $y$ belong to the real Hilbert space $\ell_2$, and for all $n \geq 0$, $(T^n x | y) = 0$. The proof is now just the same. □

(3) It is also natural to ask whether the result of Section 3 can be extended to matrices having more than three non-zero diagonals. For instance:

**Question.** Let $T$ be a matrix with only five non-zero diagonals, which are the main diagonal and the two upper and lower diagonals. If $T$ has positive entries, is it true that $T$ has a finitely supported moment pair?

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References