A sampling theorem for non-bandlimited signals using generalized Sinc functions

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\section*{Abstract}
A ladder shaped filter of two real parameters $a_1, a_2 \in (-1, 1)$ is introduced in this note. The impulse response of the corresponding Linear Time Invariant (LTI) system is a generalized Sinc function of two parameters. Consequently a generalized Shannon-type sampling theorem is established for a class of non-bandlimited signals with special spectrum properties associated with a ladder shaped filter of two parameters. Finally, a mathematical characterization for the class of non-bandlimited signals satisfying the generalized sampling theorem is offered. These signals are restrictions to the real line of certain analytic functions in stripped domains symmetric about the real axis in the complex plane. For these signals, their spectra in higher frequency bands are measured by the spectrum of their base bands.

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\section{Introduction}
A LTI system is completely determined by its impulse response $h$. Recall that the impulse response of a LTI system is the output signal of the LTI system when the input is the Dirac Delta function. The Fourier transform $\hat{h}$ of $h$ is called its transfer function. Given a LTI system with impulse response $h$, for an input signal $f_{\text{in}}$, the output signal $f_{\text{out}}$ is the convolution of $h$ with $f_{\text{in}}$, that is,

$$f_{\text{out}}(t) = \int_{-\infty}^{\infty} f_{\text{in}}(t-x)h(x)dx, \quad t \in \mathbb{R},$$

which has an equivalent representation in frequency domain of

$$\hat{f}_{\text{out}}(\omega) = \sqrt{2\pi} \hat{f}_{\text{in}}(\omega) \hat{h}(\omega), \quad \omega \in \mathbb{R},$$

where, the Fourier transform of a signal $f$ is defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \quad \omega \in \mathbb{R}.$$
higher frequency band is ignored. The impulse response $h$ is
\begin{equation}
    h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(-\frac{t}{2}) \cdot \hat{f}(\omega) e^{j\omega t} d\omega = \frac{\sqrt{2\pi}}{2} \text{Sinc} \left( \frac{\pi t}{2} \right),
\end{equation}
where the classic Sinc function $\text{Sinc}(t) = \frac{\sin t}{t}$.

It is shown in [1,2] that the signal $\cos \theta_b(t)$, $t \in \mathbb{R}$ with the phase $\theta_b(t)$ defined by
\begin{equation}
    \theta_b(t) := t + 2 \arctan \left( \frac{|b| \sin(t - t_0)}{1 - |b| \cos(t - t_0)} \right), \quad t \in \mathbb{R}
\end{equation}
for a complex number $b = |b|e^{i\theta}$ with $|b| < 1$, satisfies
\[ \mathcal{H} \cos \theta_b(t) = \sin \hat{\theta}_b(t), \quad t \in \mathbb{R}, \]
where, $\mathcal{H}$ is the Hilbert transform defined by
\[ \mathcal{H} f(t) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} dx, \quad t \in \mathbb{R}. \]
That is to say, the complex signal $e^{\theta_b(t)}$ is an analytic signal with strictly increasing nonlinear instantaneous phase $\theta_b(t)$. There are a large number of literatures discussing the notions of analytic signals and instantaneous frequencies (see [3–12]). The instantaneous frequency $\frac{d\theta_b(t)}{dt}$ of the signal $\cos \theta_b(t)$ or $e^{\theta_b(t)}$ is just the periodic Poisson kernel
\begin{equation}
    p_b(t) := \frac{1 - |b|^2}{1 - 2|b| \cos(t - t_0) + |b|^2}, \quad \text{with} \quad b = |b|e^{i\theta}.
\end{equation}
The spectrum properties of this signal are studied in [13]. Some further results about the time-frequency properties of the analytic signal $e^{\theta_b(t)}$ and other generalizations appeared in [14–17]. This signal when $b$ is a real parameter $a$ with $|a| < 1$, was related to a kind of ladder shaped filter of one parameter [18]. The impulse response of the corresponding LTI system is a so-called generalized Sinc function
\begin{equation}
    \text{Sinc}_a(t) := p_a(t) \text{Sinc}(t) = \frac{\sin \theta_a(t)}{t}, \quad t \in \mathbb{R},
\end{equation}
where,
\begin{equation}
    \sin \theta_a(t) = \frac{(1 - a^2) \sin t}{1 - 2a \cos t + a^2}.
\end{equation}
The basic idea can be described as follows. First, the frequency space is partitioned into different frequency intervals
\begin{equation}
    \tilde{I}_n = \left[ -\frac{\pi}{2}(n + 1), -\frac{\pi}{2}n \right] \cup \left[ \frac{\pi}{2}n, \frac{\pi}{2}(n + 1) \right], \quad n = 0, 1, 2, \ldots.
\end{equation}
By the Riemann–Lebesgue lemma, for a signal with finite energy, when $n$ is very big, the spectrum $|\hat{f}(\omega)|$ in $\tilde{I}_n$ is very small. This suggests that we consider a filtering process: for an input signal $f_{\text{in}}$, output signal $f_{\text{out}}$ keeps the frequency information of $f_{\text{in}}$ with decaying weights for different frequency bands $\tilde{I}_n$. That is
\[ \hat{f}_{\text{out}}(\omega) = a^n(1 + a)\hat{f}_{\text{in}}(\omega), \quad \omega \in \tilde{I}_n, \quad n = 0, 1, 2, \ldots \]
with the real number $a$ satisfying $|a| < 1$. Consequently, the transfer function of the corresponding LTI system is
\begin{equation}
    \hat{h}_a(\omega) = a^n(1 + a), \quad \omega \in \tilde{I}_n, \quad n = 0, 1, 2, \ldots.
\end{equation}
It is easy to see that $\hat{h}_a(\omega)$ is a piecewise constant function like a ladder. Applying the inverse Fourier transform to both sides of the above equation offers us the impulse response in the time domain:
\begin{equation}
    h_a(t) = \frac{\sqrt{2\pi}}{2} \frac{\sin \theta_a(t)}{t} = \frac{\sqrt{2\pi}}{2} \text{Sinc}_a \left( \frac{\pi t}{2} \right)
\end{equation}
which is just a constant multiple of a dilation of the generalized Sinc function defined in (1.7).

In this note, we introduce a generalized ladder shaped filter of two parameters. Subsequently a generalized Sinc function of two parameters is defined, and then a corresponding Shannon-type sampling theorem for a class of non-bandlimited signals is established. Our starting point is based on the interesting but non-intuitive formula
\begin{equation}
    \sin (\theta_{a_1}(t) + \theta_{a_2}(t)) = \frac{1 - a_1 a_2}{a_1 - a_2} (\sin \theta_{a_1}(t) - \sin \theta_{a_2}(t)),
\end{equation}
where $a_1, a_2 \in (0, 1)$.
where, \(|a_i| < 1, i = 1, 2, \) and \(a_1 \neq a_2\). This observation inspires us to consider a kind of ladder shaped filter of two parameters whose impulse response is discovered to be the generalized Sinc function of two parameters

\[
\text{Sinc}_{a_1,a_2}(t) := \frac{\sin(\theta_{a_1}(t) + \theta_{a_2}(t))}{t}, \quad t \in \mathbb{R},
\]

which essentially equals

\[
\text{Sinc}_{a_1,a_2}(t) = \frac{1 - a_1a_2}{a_1 - a_2} (\text{Sinc}_{a_1}(t) - \text{Sinc}_{a_2}(t))
= \frac{1 - a_1a_2}{a_1 - a_2} (p_{a_1}(t) - p_{a_2}(t)) \text{Sinc}(t),
\]

by Eqs. (1.7) and (1.12).

This paper is organized as follows: In Section 2, a ladder shaped filter of two parameters is introduced. The impulse response of such an LTI system is shown to be a generalized Sinc function of two parameters. Some studies on the ladder shape filter and the associated generalized Sinc function are also presented. In Section 3, a Shannon-type sampling theorem for certain type of non-bandlimited signals is provided. The frequency spectrum of this type of signal is a constant multiple of the product of a periodic interpolating function with a generalized ladder shaped filter of two parameters. In Section 4, we establish that signals with the previously stated frequency spectrum are restrictions to the real axis of the difference of two analytic functions in a strip symmetric about the real axis in the complex plane.

2. A ladder shaped filter of two parameters

In this section, we introduce a ladder shaped filter of two parameters. We then associate the impulse response of the corresponding LTI system with a generalized Sinc function of two parameters. Some studies of the ladder shaped filter and the generalized Sinc function are also given here.

For a pair of real numbers \((a_1, a_2)\) with \(|a_1| < 1, |a_2| < 1\) and \(a_1 \neq a_2\), we consider the following filtering process: given an input signal \(f_{\text{in}}\), the output signal \(f_{\text{out}}\) keeps the weighted frequency information of \(f_{\text{in}}\) with the weight \((1 - a_1a_2)b_n\), where, we define

\[
b_n = \frac{a_1^2(1 + a_1) - a_2^2(1 + a_2)}{a_1 - a_2}
\]
in the frequency band \(l_n, n = 0, 1, \ldots\) with

\[
l_n = [-n(1 + \Omega), -n\Omega] \cup [n\Omega, (n + 1)\Omega], \quad \Omega > 0.
\]

That is to say,

\[
\hat{f}_{\text{out}}(\omega) = (1 - a_1a_2)b_n \hat{f}_{\text{in}}(\omega), \quad \omega \in l_n.
\]

The transfer function of this system is denoted by \(H_{a_1,a_2,\Omega}\) with

\[
H_{a_1,a_2,\Omega}(\omega) := (1 - a_1a_2)b_n, \quad \omega \in l_n.
\]

The condition \(|a_i| < 1, i = 1, 2\) ensures \(\lim_{|\omega| \to +\infty} |H_{a_1,a_2,\Omega}(\omega)| = 0\). We call \(H_{a_1,a_2,\Omega}\) a ladder shaped filter of two parameters.

Recalling Eq. (1.10), we deduce that

\[
H_{a_1,a_2,\Omega}(\omega) = \frac{1 - a_1a_2}{a_1 - a_2} (H_{a_1,\Omega}(\omega) - H_{a_2,\Omega}(\omega)), \quad \omega \in \mathbb{R}
\]

with

\[
H_{a,\Omega}(\omega) = \hat{h}_a \left( \frac{\pi}{2\Omega} \omega \right).
\]

That is to say, a ladder shaped filter of two parameters is a scaled difference of two ladder shaped filters of one parameter. Note we also can express \(H_{a_1,a_2,\Omega}\) as a series, i.e., for \(\omega \in \mathbb{R},\)

\[
H_{a_1,a_2,\Omega}(\omega) = \sum_{n=0}^{\infty} \frac{1 - a_1a_2}{a_1 - a_2} \left( a_1^n(1 + a_1) - a_2^n(1 + a_2) \right) \left( \chi_{[(n, (n+1))]}(|\omega|) \right).
\]

Observe that

- For \(|a_1| < 1 \) and \(|a_2| < 1\), the ladder shaped filter \(H\) is symmetric about the two parameters in the sense that

\[
H_{a_1,a_2,\Omega} = H_{a_2,a_1,\Omega};
\]
If \( a_1 = -a_2 = a \), with \(|a| < 1\), and \( a \neq 0\),

\[
H_{a,-a,\Omega}(\omega) = (1 + a^2) \left( a^2 \right)^{\frac{n}{2}} H_{a^2,\Omega} \left( \frac{1}{2} \omega \right), \quad \omega \in I_n.
\]

Therefore in this case, the ladder shaped filter of two parameters \( a \) and \(-a\) degenerates to a ladder shaped filter of one parameter \( a^2 \) (see Eq. (1.10)). But note that each piece of the ladder shaped filter has the support of length \( 2\Omega \). An example of such ladder shaped filter \( H_{0.5,-0.5,\pi/2} \) can be seen from Fig. 1.

Simple calculations can show that the ladder shaped filters \( H_{a_1,a_2,\Omega} \) may be classified into two groups:

- If one of the parameters with bigger absolute value is positive, then \( H_{a_1,a_2,\Omega}(\omega) > 0 \), for \( \omega \in \mathbb{R} \); Moreover, except possibly the first several terms in the series representation of \( H_{a_1,a_2,\Omega} \) (see Eq. (2.5)); all other terms of \( H_{a_1,a_2,\Omega} \) decrease monotonically to 0 as \( l \to \infty \) (see Fig. 2).

- If one of the parameters with bigger absolute value is negative, then \( H_{a_1,a_2,\Omega} \) is essentially an alternating series, except possibly the first several terms. The magnitude \( |H_{a_1,a_2,\Omega}| \) decreases monotonically to 0 as \( l \to \infty \) after the first several terms (see Fig. 3).

From Figs. 1–3, it is easily seen that through different combinations of \( a_1 \) and \( a_2 \), a ladder shaped filter of two parameters has the flexibility of catering to various types of signals with different frequency properties. It is obviously more applicable than a ladder shaped filter of one parameter.
If we denote the impulse response of the above system by \( g \), then we have
\[
\hat{g}(\omega) = \frac{1 - a_1 a_2}{a_1 - a_2} \left( \hat{h}_{a_1} \left( \frac{\pi}{2\Omega} \omega \right) - \hat{h}_{a_2} \left( \frac{\pi}{2\Omega} \omega \right) \right), \quad \omega \in \mathbb{R}.
\]
Applying the inverse Fourier transform to both sides of the above equation leads to
\[
g(t) = \frac{2\Omega}{\pi} \frac{1 - a_1 a_2}{a_1 - a_2} \left( h_{a_1} \left( \frac{2\Omega}{\pi} t \right) - h_{a_2} \left( \frac{2\Omega}{\pi} t \right) \right), \quad t \in \mathbb{R}. \tag{2.6}
\]
We now relate the impulse response \( g \) to a generalized Sinc function of two parameters defined in (1.13). To that end, we first establish the following lemma.

**Lemma 2.1.** For any pair of real numbers \( (a_1, a_2) \) with \(|a_1| < 1, |a_2| < 1 \) and \( a_1 \neq a_2 \), the Eq. (1.12) holds true.

**Proof.** By recalling the two formulae
\[
\sin \theta_a(t) = p_a(t) \sin t = \frac{(1 - a^2) \sin t}{1 - 2a \cos t + a^2}, \quad t \in \mathbb{R}
\]
and
\[
\cos \theta_a(t) = \frac{(1 + a^2) \cos t - 2a}{1 - 2a \cos t + a^2}, \quad t \in \mathbb{R},
\]
we deduce that
\[
\sin(\theta_{a_1}(t) + \theta_{a_2}(t)) = \sin \theta_{a_1}(t) \cos \theta_{a_2}(t) + \sin \theta_{a_2}(t) \cos \theta_{a_1}(t)
\]
\[
= \frac{(1 - a_1^2) \sin t}{1 - 2a_1 \cos t + a_1^2} \cdot \frac{(1 + a_2^2) \cos t - 2a_2}{1 - 2a_2 \cos t + a_2^2} + \frac{(1 - a_2^2) \sin t}{1 - 2a_2 \cos t + a_2^2} \cdot \frac{(1 + a_1^2) \cos t - 2a_1}{1 - 2a_1 \cos t + a_1^2}
\]
\[
= \frac{\sin t}{1 - 2a_1 \cos t + a_1^2} \left[ (1 + a_2^2) \cos t - 2a_2 \right] + \frac{\sin t}{1 - 2a_2 \cos t + a_2^2} \left[ (1 + a_1^2) \cos t - 2a_1 \right]
\]
\[
= \frac{2(1 - a_1 a_2) \sin t}{(1 - 2a_1 \cos t + a_1^2)(1 - 2a_2 \cos t + a_2^2)} \left[ (1 + a_1 a_2) \cos t - (a_1 + a_2) \right].
\]
We therefore obtain that
\[
\sin(\theta_{a_1}(t) + \theta_{a_2}(t)) = \frac{2 \sin t(1 - a_1 a_2)}{a_1 - a_2} \frac{(a_1 - a_2)(a_1 + a_2) \cos t + (a_2^2 - a_1^2)}{(1 - 2a_1 \cos t + a_1^2)(1 - 2a_2 \cos t + a_2^2)}.
\]
Combining this with the equations \( \sin \theta_a(t) = p_a(t) \sin t \) and
\[
p_a(t) = \frac{2[(a_1 - a_2)(1 + a_1 a_2) \cos t + (a_2^2 - a_1^2)]}{(1 - 2a_1 \cos t + a_1^2)(1 - 2a_2 \cos t + a_2^2)}.
\]
This completes the proof. ■

Consequently we have three representations of $g$

$$g(t) = \sqrt{\frac{2}{\pi}} \frac{1 - a_1 a_2}{a_1 - a_2} \left( \text{Sinc}_{a_1} (\Omega t) - \text{Sinc}_{a_2} (\Omega t) \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(\theta_{a_1}(t) + \theta_{a_2}(t))}{\Omega t}$$

$$= \sqrt{\frac{2}{\pi}} \Omega \text{Sinc}_{a_1, a_2} (\Omega t).$$

We call the function $\text{Sinc}_{a_1, a_2}$ a generalized Sinc function of two parameters $a_1$ and $a_2$. We offer the following observations about the function $\text{Sinc}_{a_1, a_2}$:

- For $|a_1| < 1$ and $|a_2| < 1$, the generalized Sinc function of two parameters is symmetric about the two parameters in the sense that
  $$\text{Sinc}_{a_1, a_2} = \text{Sinc}_{a_2, a_1}.$$
- If $a_1 = -a_2 = a$, with $|a| < 1$, and $a \neq 0$,
  $$\text{Sinc}_{a, -a, 0}(t) = 2\text{Sinc}_{a^2} (2t).$$

Therefore in this case, the generalized Sinc of two parameters $a$ and $-a$ is 2 times a generalized Sinc of one parameter $a^2$. But note that it is compressed horizontally by a factor of 2. An example of such generalized Sinc $\text{Sinc}_{0.5, -0.5}$ can be seen in Fig. 4.

One may classify the generalized Sinc function of two parameters $\text{Sinc}_{a_1, a_2}$ into the following two groups:

- When one of the two parameters is positive. Examples of such generalized Sinc functions compared to the classic Sinc are depicted in Fig. 5.
- When both two parameters are negative. Examples of such generalized Sinc functions compared to the classic Sinc are depicted in Fig. 6.
3. A sampling theorem for non-bandlimited signals

We now turn to establish a sampling theorem for a family of non-bandlimited signals. The famous Whittaker–Kotelnikov–Shannon sampling theorem states that for any bandlimited signal $f$ with $\text{supt} \hat{f} \subset [-\Omega, \Omega]$ for an arbitrarily given positive number $\Omega$, the signal $f$ can be reconstructed from its sampling sequence $\{ f(n\pi/\Omega) : n \in \mathbb{Z} \}$ by Nyquist frequency $\frac{\Omega}{2}$, that is,

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n \pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}, \quad t \in \mathbb{R}.$$ 

Thus, we can say any bandlimited function $f$ with $\text{supt} \hat{f} \subset [-\Omega, \Omega]$ is related to a $2\Omega$-periodic function

$$M_{f,\Omega}(\omega) := \frac{\sqrt{2\pi}}{2\Omega} \sum_{n \in \mathbb{Z}} f\left(\frac{n \pi}{\Omega}\right) e^{-i\pi n \omega}.$$ 

In fact, the function $M_{f,\Omega}$ is well-defined for any function with a suitable decay rate such that the sequence $\{ f(n\pi/\Omega) : n \in \mathbb{Z} \}$ belongs to $L^2(\mathbb{Z})$. It is easy to see that, for a bandlimited signal $f$, the compactly supported function $\hat{f}$ is a pulse in $[-\Omega, \Omega]$ of the $2\Omega$-periodic signal $M_{f,\Omega}$. This suggests us to consider the space

$$B_{\Omega} := \{ f \in L^2(\mathbb{R}) : \hat{f}(\omega) = M_{f,\Omega}(\omega) \chi_{[-\Omega,\Omega]}(\omega) \}.$$ 

By the Whittaker–Kotelnikov–Shannon sampling theorem, we know that $f$ is a bandlimited signal with $\text{supt} \hat{f} \subset [-\Omega, \Omega]$ if and only if $f \in B_{\Omega}$. Next we will extend the space $B_{\Omega}$ of bandlimited signals to a space of non-bandlimited signals. Denote
by \( G_\Omega \), the nonempty space of signals satisfying

\[
G_\Omega = \{ f \in L^2(\mathbb{R}) : \hat{f}(\omega) = M_{\Omega}(\omega)H_{\Omega,1,2,\Omega}(\omega), \ \omega \in \mathbb{R} \}
\]

with the ladder shaped filter \( H_{\Omega,1,2,\Omega} \) defined in Eq. \((2.2)\).

When both \( a_1 \) and \( a_2 \) equal 0, the space \( G_\Omega \) is just the space \( \mathbb{B}_\Omega \) of bandlimited signals. It is time to establish the sampling theorem based on the generalized Sinc function \( \text{Sinc}_{a_1, a_2} \).

**Theorem 3.1.** A signal \( f \) belongs to \( G_\Omega \) if and only if for \( t \in \mathbb{R} \),

\[
f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n \pi}{\Omega}\right) \sin \left(\frac{\theta_0(\Omega t - n\pi)}{\Omega t - n\pi}\right).
\]

**Proof.** We first show the necessity. From the definition of \( G_\Omega \), we know that, any signal \( f \in G_\Omega \) has the following representation in the frequency domain

\[
\hat{f}(\omega) = M_{\Omega}(\omega)H_{\Omega,1,2,\Omega}(\omega).
\]

Thus, we have

\[
\hat{f}(\omega) = M_{\Omega}(\omega)H_{\Omega,1,2,\Omega}(\omega) = \sum_{l=0}^{\infty} \frac{1 - a_1 a_2}{a_1 - a_2} \left[ (1 + a_1) a_1^l - (1 + a_2) a_2^l \right] M_{\Omega}(\omega) \left( \chi_{\{l \in \mathbb{Z}, \mid \omega \mid \}}(\mid \omega \mid) \right) \]

The above equation with the equation

\[
\sum_{l=0}^{\infty} (1 + a_1) a_1^l \chi_{\{l \in \mathbb{Z}, \mid \omega \mid \}}(\mid \omega \mid) = (1 - a_1^2) \sum_{l=0}^{\infty} a_1^{-1} \chi_{\{-l \in \mathbb{Z}, \mid \omega \mid \}}(\mid \omega \mid), \quad j = 1, 2
\]

leads to

\[
\hat{f}(\omega) = \frac{\sqrt{2\pi}}{2\Omega} \sum_{n \in \mathbb{Z}} \left( \frac{n \pi}{\Omega} \right) e^{-i \frac{\pi}{\Omega} a_1 a_2} \left[ (1 - a_1^2) \sum_{l=0}^{\infty} a_1^{-1} \chi_{\{-l \in \mathbb{Z}, \mid \omega \mid \}}(\mid \omega \mid) \right].
\]

Applying the inverse Fourier transform to both sides of the above equation and using

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i \frac{\pi}{\Omega} a_1 a_2} \chi_{\{-l \in \mathbb{Z}, \mid \omega \mid \}}(\mid \omega \mid) e^{i \omega t} d\omega = \frac{2\Omega}{\sqrt{2\pi}} \sin \left( \frac{\Omega(t - \frac{\pi}{2} n)}{\Omega(t - \frac{\pi}{2} n)} \right)
\]

yields that

\[
f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n \pi}{\Omega}\right) \frac{1 - a_1 a_2}{a_1 - a_2} \sum_{l=0}^{\infty} ((1 - a_1^2) a_1^{-1} - (1 - a_2^2) a_2^{-1}) \sin \left( \frac{\Omega(t - \frac{\pi}{2} n)}{\Omega(t - \frac{\pi}{2} n)} \right).
\]

By noting that

\[
\sum_{l=1}^{\infty} \sin lt = \frac{a \sin t}{1 - 2a \cos t + a^2}, \quad |a| < 1,
\]

we observe that

\[
\sin \theta_a(t) = (1 - a^2) \sum_{l=1}^{\infty} a^{l-1} \sin lt.
\]

Therefore we deduce that

\[
f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n \pi}{\Omega}\right) \frac{1 - a_1 a_2}{a_1 - a_2} \left[ \frac{\sin \theta_{a_1}(\Omega(t - \frac{n \pi}{\Omega}))}{\Omega(t - \frac{n \pi}{\Omega})} - \frac{\sin \theta_{a_2}(\Omega(t - \frac{n \pi}{\Omega}))}{\Omega(t - \frac{n \pi}{\Omega})} \right].
\]

Combining this with the definition of the generalized Sinc function of two parameters and Eq. \((1.14)\) gives us that

\[
f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n \pi}{\Omega}\right) \text{Sinc}_{a_1, a_2} \left( \Omega\left( t - n\frac{\pi}{\Omega} \right) \right).
\]
That is to say,
\[
  f(t) = \sum_{n \in \mathbb{Z}} f\left(\pi n \frac{\pi}{T}\right) \sin\left(\theta_1(\Omega t - n\pi) + \theta_2(\Omega t - n\pi)\right) \Omega t - n\pi.
\] (3.5)

Reversing the process of the proof of necessity, one can establish the sufficiency. Consequently, we have completed the proof of the theorem.

4. Characterizing the space \(G_{\Omega}\)

In this section, we will show that any function in the space \(G_{\Omega}\) is the restriction to the real axis of the difference of two analytic functions in a strip symmetric about the real axis in the complex plane. To verify this result, we establish the following lemma, which asserts that any function \(g\) having the following frequency representation
\[
  \hat{g}(\omega) = M_{g,\Omega}(\omega)H_{g,\Omega}(\omega)
\] (4.1)
with \(M_{g,\Omega}(\omega)\) and \(H_{g,\Omega}(\omega)\) defined in (3.1) and (2.4), respectively, may be analytically extended to a strip symmetric about the real axis.

**Lemma 4.1.** Suppose that a function \(g\) is defined by (4.1) with \(|a| < 1\). Then the function \(g\) may be analytically extended to the strip
\[
  \{z = x + iy \mid \frac{\ln|a|}{\Omega} < y < -\frac{\ln|a|}{\Omega}, -\infty < x < \infty\},
\]
and, inside the strip, the extended function satisfies the estimate
\[
  |g(z)| \leq \frac{C_{g,\Omega}}{1 - e^{2(i\arg z + \ln|z|)}},
\]
where \(C_{g,\Omega}\) is a constant depending on \(a\) and \(\Omega\).

**Proof.** Consider a possible complex number \(z\) that makes the following two integrals \(g^- (z)\) and \(g^+ (z)\) well defined:
\[
  g^- (z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\omega z} M_{g,\Omega}(\omega) H_{g,\Omega}(\omega) d\omega,
\]
\[
  g^+ (z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{i\omega z} M_{g,\Omega}(\omega) H_{g,\Omega}(\omega) d\omega.
\]
Since \(M_{g,\Omega}\) is \(2\Omega\)-periodic, by the definition of \(H_{g,\Omega}\) (see (2.4)), we have for \(\omega \in [0, 2\Omega)\):
\[
  M_{g,\Omega}(2\Omega + \omega)H_{g,\Omega}(2\Omega + \omega) = e^{2i\omega}M_{g,\Omega}(\omega)H_{g,\Omega}(\omega).
\]
Write \(\ln a = \ln|a| + i \arg a\), where, \(\arg a\) is the principle argument of \(a\). Noting that \(\arg a = 0\) or \(\arg a = \pi\), we thus have
\[
  g^- (z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\omega z} \sum_{n=-\infty}^{\infty} a^{2n|a|} e^{2i\omega n\Omega} M_{g,\Omega}(\omega) H_{g,\Omega}(\omega) d\omega
\]
\[
  = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\omega z} \sum_{n=1}^{\infty} a^{2(n-1)|a|} e^{2i(n-1)\omega|a|} e^{-i(2n-1)\omega|a|^2} M_{g,\Omega}(\omega) H_{g,\Omega}(\omega) d\omega
\]
\[
  = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\omega z} \sum_{n=1}^{\infty} a^{2(n-1)|a|} e^{2i(1-n)\omega|a|^2} M_{g,\Omega}(\omega) H_{g,\Omega}(\omega) d\omega
\]
\[
  = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\omega z} S_{g,\Omega}^z M_{g,\Omega}(\omega) H_{g,\Omega}(\omega) d\omega,
\]
where \(S_{g,\Omega}^z\) represents the geometric series in the integral, namely,
\[
  S_{g,\Omega}^z = \sum_{n=1}^{\infty} e^{2(1-n)(\ln|a|+i\arg a - i\Omega/2)}.
\]

Note \(e^{i\arg a} = \pm 1\), so for \(z = x + iy\),
\[
  e^{2(i\arg z + \ln|z|)} \leq e^{2(i\arg z + \ln|z|)}.
\]

One can then see that for \(y < -\frac{\ln|a|}{\Omega}\), the geometric series is absolutely convergent to
\[
  S_{g,\Omega}^z = \frac{1}{1 - e^{2(i\arg z + \ln|z|)}},
\]
The function $S_{\alpha, \Omega}$ is bounded by $\frac{1}{1-e^{-2(\ln|a| + iy)}}$. Therefore $g^-$ is well defined in the half-plane $y < -\frac{\ln|a|}{\Omega}$ with

$$|g^-(z)| \leq \frac{C_{\alpha, \Omega}}{1-e^{2(\ln|a| + iy)}},$$

where the constant

$$C_{\alpha, \Omega} = \frac{e^{-2\ln|a|}}{\sqrt{2\pi}} \int_{-2\Omega}^{0} |M_{\alpha, \Omega}(\omega)H_{\alpha, \Omega}(\omega)|d\omega.$$

The function $g^-$ is obviously continuous in the half plane $y < -\frac{\ln|a|}{\Omega}$. The theorems of Fubini and Cauchy show that $\int_{\gamma} g^-(z)dz = 0$ for every closed path $\gamma$ in the half plane. By Morera’s theorem, $g^-$ is analytic in the half plane.

Similarly,

$$|g^+(z)| \leq \frac{C_{\alpha, \Omega}}{1-e^{2(\ln|a| + iy)}} \quad \text{for } y > \frac{\ln|a|}{\Omega},$$

with the same constant $C_{\alpha, \Omega}$, and the function $g^+$ is analytic and bounded in the half plane

$$\left\{ |t + iy| - \infty < t < \infty, y > \frac{\ln|a|}{\Omega} \right\}.$$

It follows that in the strip

$$\frac{\ln|a|}{\Omega} < y < -\frac{\ln|a|}{\Omega},$$

the function $g(z) = g^+(z) + g^-(z)$ is analytic and satisfies the estimate

$$|g(z)| \leq \frac{C_{\alpha, \Omega}}{1-e^{2(\ln|a| + iy)}}.$$

This completes the proof. ■

We are ready to characterize the space $G_{\alpha, \Omega}$.

**Theorem 4.2.** If a function $f$ belongs to $G_{\alpha, \Omega}$, then $f$ may be analytically extended to the strip

$$[z] = \gamma < \text{Im} z < \gamma, -\infty < \text{Re} z < \infty,$$

with $\gamma = \min\{ -\frac{\ln|a|}{\Omega}, -\frac{\ln|a|}{\Omega} \}$ and, inside the strip, the extended function satisfies the estimate

$$|f(z)| \leq \frac{C_{\alpha, \Omega}}{1-e^{2(\ln|a| + |\text{Im} z|)}},$$

where $C_{\alpha, \Omega}$ is a constant depending on $\Omega$ and the vector $a = (a_1, a_2)$.

**Proof.** We consider a possible complex number $z$ that makes the following two integrals $f^+(z)$ and $f^-(z)$ both well defined:

$$f^+(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{iz\Omega}M_{\alpha, \Omega}(\omega)H_{\alpha, \Omega}(\omega)d\omega$$

and

$$f^-(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{iz\Omega}M_{\alpha, \Omega}(\omega)H_{\alpha, \Omega}(\omega)d\omega.$$

By recalling the relation of two parameter filters to one parameter filters (see (2.3)), we have

$$f^+(z) = \frac{1-a_1a_2}{a_1-a_2} (f_1^+(z) - f_2^+(z))$$

and

$$f^-(-z) = \frac{1-a_1a_2}{a_1-a_2} (f_1^-(z) - f_2^-(z)),$$

where the complex variable functions $f_j^+, f_j^-, j = 1, 2$ are defined by

$$f_j^+(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{iz\Omega}M_{\alpha, \Omega}(\omega)H_{\alpha, \Omega}(\omega)d\omega, \quad j = 1, 2$$

and

$$f_j^-(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{iz\Omega}M_{\alpha, \Omega}(\omega)H_{\alpha, \Omega}(\omega)d\omega, \quad j = 1, 2.$$


The proof of this theorem is completed. ■

To close this paper, we remark that for a non-bandlimited signal \( f \in \mathbb{C}_\Omega \), its high frequency spectrum is just a scaled copy of its spectrum in the base band (see the Eq. (3.3)). A Shannon-type sampling theorem holds true for such signals (see the Theorem 3.1). These signals are restrictions to the real line of certain analytic functions in stripped domains symmetric about the real axis in the complex plane (see the Theorem 4.2).

A Fourier transform for discrete data essentially is only suitable for bandlimited signals. For a signal of short duration (certainly non-bandlimited), its Fourier transform becomes periodic and hence the information at the boundary is misleading. Yet our research presented in the paper shows that for certain non-bandlimited signals characterized by the presented generalized sampling theorem, we can obtain frequency information at any band of the original signals. We feel this is very important wherever one deals with transient signals, such as in radar signal processing.

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