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Second order initial value problems of Lane–Emden type

Ravi P. Agarwal[a,*], Donal O'Regan[b]

[a] Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901-6975, USA
[b] Department of Mathematics, National University of Ireland, Galway, Ireland

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Abstract

This work discusses the existence of positive solutions to the initial value problem $(py')' \pm pqg(y) = 0$, $t \in [0, T)$, with $y(0) = 1$ and $\lim_{t\to 0^+} p(t)y'(t) = 0$. Our analysis is based on the Schauder–Tychonoff theorem.

Keywords: Initial value problems; Positive solutions; Lane–Emden equation; Schauder–Tychonoff theorem

1. Introduction

Consider a spherical cloud of gas and denote its total pressure at a distance r from the center by $p(r)$. The total pressure is due to the usual gas pressure and a contribution from radiation,

$$p = \frac{1}{3}aT^4 + \frac{RT}{v}$$

where a, T, R and v are respectively the radiation constant, the absolute temperature, the gas constant, and the volume. Pressure and density $\rho = v^{-1}$ vary with r and $p = K\rho^\gamma$ where γ and K are constants. Let m be the mass within a sphere of radius r and G the constant of gravitation. The equilibrium equations for the configuration are

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} \quad \text{and} \quad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho.$$

Eliminating m yields

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) + 4\pi G\rho = 0.$$

Now let $\gamma = 1 + \mu^{-1}$ and set $\rho = \lambda\phi^\mu$, so

$$p = K\rho^{1+\mu^{-1}} = K\lambda^{1+\mu^{-1}}\phi^{\mu+1}.$$

* Corresponding author. Tel.: +1 321 674 7202.
E-mail address: agarwal@fit.edu (R.P. Agarwal).

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Thus

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)+k^2\,\phi^\mu=0$$

with

$$k^2=\frac{4\pi G\,\lambda^{1-\mu^{-1}}}{(\mu+1)\,K}.$$

Now with $x=k\,r$ we have

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}+\frac{2}{x}\frac{\mathrm{d}\phi}{\mathrm{d}r}+\phi^\mu=0.$$

If we let $\lambda=\rho_0$, the density at $r=0$, then we may take $\phi=1$ at $x=0$. By symmetry the other condition is $\frac{\mathrm{d}\phi}{\mathrm{d}x}=0$ when $x=0$. A solution of the differential equation satisfying these initial conditions is called a Lane–Emden function of index $\mu=(\gamma-1)^{-1}$.

The differential equation

$$y''+\frac{2}{t}\,y'+g(y)=0, \tag{1.1}$$

was first studied by Emden [2] when he examined the thermal behavior of spherical clouds of gas acting on gravitational equilibrium and subject to the laws of thermodynamics. The usual interest is in the case $g(y)=y^n$, $n\geq 1$, which was treated by Chandrasekhar in his study of stellar structure. The natural initial conditions for (1.1) are

$$y(0)=1,\qquad y'(0)=0. \tag{1.2}$$

It is known that (1.1) and (1.2) can be solved exactly if $n=1$ with the solution

$$y(t)=\frac{\sin t}{t}$$

and if $n=5$ with the solution

$$y(t)=\left(1+\frac{1}{3}t^2\right)^{-\frac{1}{2}};$$

we refer the reader to [1, pp. 106–108]. It is also of interest to note that the Emden differential equation $y''-t^a\,y^b=0$ arises in various astrophysical problems, including the study of the density of stars. Of course one is interested only in positive solutions in the above models.

In this work we examine the following initial value problems:

$$\begin{cases}(py')'\pm pqg(y)=0, & t\in[0,T)\\ y(0)=1, & \lim\limits_{t\to 0^+}p(t)y'(t)=0\end{cases} \tag{1.3}$$

and

$$\begin{cases}(py')'\pm pqg(y)=0, & t\in[0,T)\\ y(0)=1, & y'(0)=0\end{cases} \tag{1.4}$$

and we obtain very general existence results (existence of positive solutions) for both (1.3) and (1.4); here $0<T\leq\infty$ with $p\geq 0$, $q\geq 0$ and $g:[0,\infty)\to[0,\infty)$.

For notational purposes in this work if $u\in C[0,T)$ then for every $m\in\{1,2,\ldots\}=N^+$ we define the seminorms $\rho_m(u)$ by

$$\rho_m(u)=\sup_{t\in[0,t_m]}|u(t)|$$

where $t_m\uparrow T$. Note that $C[0,T)$ is a locally convex linear topological space. The topology on $C[0,T)$, induced by the seminorms $\{\rho_m\}_{m\in N^+}$, is the topology of uniform convergence on every compact interval of $[0,T)$. From the

Arzela–Ascoli theorem a set $\Omega \subseteq C[0,T)$ is compact if and only if it is uniformly bounded and equicontinuous on each compact interval of $[0,T)$. Also in Section 2 we use $L^1_w[0,a]$ (w is a weight function) which is the space of functions u with $\int_0^a w(t)|u(t)|\,\mathrm{d}t < \infty$.

2. Second order initial value problems

In this section we examine both

$$\begin{cases} (py')' + pqg(y) = 0, & t \in [0,T) \\ y(0) = a > 0, \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \end{cases} \tag{2.1}$$

and

$$\begin{cases} (py')' + pqg(y) = 0, & t \in [0,T) \\ y(0) = a > 0, \\ y'(0) = 0 \end{cases} \tag{2.2}$$

where $0 < T \le \infty$ with $p \ge 0$, $q \ge 0$ and $g : [0,\infty) \to [0,\infty)$. We establish three existence results for both (2.1) and (2.2).

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$p \in C[0,T) \cap C^1(0,T) \quad \textit{with } p > 0 \text{ on } (0,T) \tag{2.3}$$

$$q \in L^1_p[0,t^\star] \quad \textit{for any } t^\star \in (0,T) \textit{ with } q > 0 \textit{ on } (0,T) \tag{2.4}$$

$$\int_0^{t^\star} \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s < \infty \quad \textit{for any } t^\star \in (0,T) \tag{2.5}$$

and

$$\begin{cases} g : [0,\infty) \to [0,\infty) & \textit{is continuous and nondecreasing} \\ \textit{on } [0,\infty) & \textit{with } g(u) > 0 \textit{ for } u > 0. \end{cases} \tag{2.6}$$

Let $H(z) = \int_z^a \frac{\mathrm{d}x}{g(x)}$ *for* $0 < z \le a$ *(note* H *is decreasing on* $(0,a)$*) and assume*

$$\int_0^{t^\star} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)\,\mathrm{d}x\mathrm{d}s < a \quad \textit{for any } t^\star \in (0,T); \tag{2.7}$$

here

$$\tau(x) = g\left(H^{-1}\left(\int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)\,\mathrm{d}z\mathrm{d}w\right)\right).$$

Then (2.1) *has a solution* $y \in C[0,T)$ *with* $py' \in C[0,T)$, $(py')' \in L^1_{pq}[0,T)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T)$.

If in addition either

$$p(0) \ne 0 \tag{2.8a}$$

or

$$p(0) = 0 \quad \textit{and} \quad \lim_{t\to 0^+} \frac{p(t)q(t)}{p'(t)} = 0 \tag{2.8b}$$

holds, then y *is a solution of* (2.2).

Proof. We first consider the problem

$$\begin{cases} (py')' + pqg^\star(y) = 0, & t \in [0,T) \\ y(0) = a > 0, \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \end{cases} \tag{2.9}$$

where

$$g^{\star}(y) = \begin{cases} g(a), & y \geq a \\ g(y), & 0 \leq y \leq a \\ g(0), & y < 0. \end{cases}$$

A solution to (2.9) is a fixed point of the operator $N : C[0, T) \to C[0, T)$ defined by

$$Ny(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g^{\star}(y(x))\,\mathrm{d}x\mathrm{d}s. \tag{2.10}$$

Let

$$b(s) = a + M \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s$$

where

$$M = \max\{g^{\star}(y) : y \in \mathbf{R}\} = \max\{g(u) : 0 \leq u \leq a\}.$$

Next let

$$K = \{y \in C[0, T) : |y(t)| \leq b(t) \text{ for } t \in [0, T)\}.$$

Clearly K is a closed, convex, bounded subset of $C[0, T)$. Also $N : K \to K$ since if $y \in K$ then

$$\begin{aligned} |Ny(t)| &\leq a + \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g^{\star}(y(s))\mathrm{d}x\mathrm{d}s \\ &\leq a + \max\{g^{\star}(v) : v \in \mathbf{R}\} \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s \\ &= b(t). \end{aligned}$$

Next we show that $N : K \to K$ is continuous and compact. To see continuity let $y_n \to y$ in $C[0, T)$, i.e. $\rho_m(y_n) \to \rho_m(y)$ for each $m \in N^+ = \{1, 2, \ldots\}$. The Lebesgue dominated convergence theorem guarantees that $Ny_n \to Ny$ uniformly on $[0, t_m]$ for each t_m so $N : C[0, T) \to C[0, T)$ is continuous. To show that $N : C[0, T) \to C[0, T)$ is completely continuous let $A \subseteq C[0, T)$ be bounded. The above analysis guarantees that $N(A)$ is uniformly bounded and equicontinuous on $[0, t_m]$ for each t_m since

$$|Ny(t) - Ny(s)| \leq M \left| \int_s^t \frac{1}{p(x)} \int_0^x p(z)\,q(z)\,\mathrm{d}z\mathrm{d}x \right| \quad \text{for } t, s \in [0, t_m].$$

Hence $N : C[0, T) \to C[0, T)$ is completely continuous.

The Schauder–Tychonoff theorem guarantees that N has a fixed point $y \in K$, i.e. y is a solution of (2.9). We now claim

$$0 < y(t) \leq a \quad \text{for } t \in [0, T). \tag{2.11}$$

If (2.11) is true then y is a solution of (2.1) and also note if (2.8a) or (2.8b) occur then y is a solution of (2.2) since

$$\begin{aligned} \lim_{t\to 0^+} \frac{y(t) - y(0)}{t} &= \lim_{t\to 0^+} \frac{-\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g^{\star}(y(s))\,\mathrm{d}x\mathrm{d}s}{t} \\ &= \begin{cases} 0 & \text{if } p(0) \neq 0 \\ g(a) \lim\limits_{t\to 0^+} \dfrac{p(t)q(t)}{p'(t)} = 0 & \text{if } p(0) = 0. \end{cases} \end{aligned}$$

It remains to show (2.11). Now $(py')' \leq 0$ on $(0, T)$ so $y' \leq 0$ on $(0, T)$ and as a result $y(t) \leq a$ for $t \in [0, T)$. It remains to show

$$y(t) > 0 \quad \text{for } t \in [0, T). \tag{2.12}$$

If (2.12) is false then there exists a $\eta \in (0, T)$ with $y > 0$ on $[0, \eta)$ and $y(\eta) = 0$. Now for $t \in (0, \eta)$ we have

$$y'(t) = -\frac{1}{p(t)} \int_0^t p(s)\, q(s)\, g(y(s))\, \mathrm{d}s$$

and since $y' \leq 0$ on $(0, T)$ (so $y(s) \geq y(t)$ for $s \in [0, t]$) and g is nondecreasing we have

$$y'(t) \leq -\frac{1}{p(t)}\, g(y(t)) \int_0^t p(s)\, q(s)\, \mathrm{d}s.$$

As a result we have for $t \in (0, \eta)$ that

$$\frac{y'(t)}{g(y(t))} \leq -\frac{1}{p(t)} \int_0^t p(s)\, q(s)\, \mathrm{d}s.$$

Now integrate from 0 to t ($t \in (0, \eta)$) to obtain

$$-H(y(t)) \leq -\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s$$

and so

$$H(y(t)) \geq \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s.$$

Now since H^{-1} is decreasing we have

$$y(t) \leq H^{-1}\left(\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s\right) \quad \text{for } t \in [0, \eta]. \tag{2.13}$$

Finally notice

$$y(\eta) = a - \int_0^\eta \frac{1}{p(s)} \int_0^s p(x)q(x)g(y(x))\, \mathrm{d}x\mathrm{d}s \tag{2.14}$$

together with (2.13) and g nondecreasing yields

$$y(\eta) \geq a - \int_0^\eta \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)\, \mathrm{d}x\mathrm{d}s > 0$$

using (2.7); here

$$\tau(x) = g\left(H^{-1}\left(\int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)\, \mathrm{d}z\mathrm{d}w\right)\right).$$

This is a contradiction, so (2.12) is true. □

Example. If $g(y) = y^\alpha$, $\alpha > 1$ and $a = 1$ then (2.7) reduces to

$$\int_0^{t^\star} \frac{1}{p(s)} \int_0^s p(x)q(x) \frac{1}{\left(1 + (\alpha - 1) \int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)\, \mathrm{d}z\mathrm{d}w\right)^{\frac{\alpha}{\alpha-1}}}\, \mathrm{d}x\mathrm{d}s < 1$$

for any $t^\star \in (0, T)$, since $H(z) = \frac{1}{\alpha-1}\left[\frac{1}{z^{\alpha-1}} - 1\right]$ so $H^{-1}(z) = \frac{1}{[1+(\alpha-1)\, z]^{\frac{1}{\alpha-1}}}$.

Theorem 2.2. *Suppose (2.3)–(2.6) hold. In addition assume*

$$p' \geq 0 \quad \text{on } (0, T) \tag{2.15}$$

and

$$\exists M > 0 \quad \text{with } q(t) \leq M \text{ for } t \in [0, T). \tag{2.16}$$

Let $Q(z) = \int_z^a g(x)\,\mathrm{d}x$ *for* $0 < z \le a$ *(note* Q *is deceasing on* $(0, a)$*) and assume*

$$t^\star < \frac{1}{\sqrt{2M}} \int_0^a \frac{\mathrm{d}x}{\sqrt{Q(x)}} \quad \text{for any } t^\star \in (0, T). \tag{2.17}$$

Then (2.1) *has a solution* $y \in C[0, T)$ *with* $0 < y(t) \le a$ *for* $t \in [0, T)$*. If in addition* (2.8a) *or* (2.8b) *holds then* y *is a solution of* (2.2).

Proof. Essentially the same reasoning as in Theorem 2.1 guarantees that (2.9) has a solution y with $y(t) \le a$ for $t \in [0, T)$. It remains to show (2.12). If (2.12) is false then there exists a $\eta \in (0, T)$ with $y > 0$ on $[0, \eta)$ and $y(\eta) = 0$. Now

$$y'\,y'' + \frac{p'}{p}[y']^2 = -q g(y)\, y' \quad \text{on } (0, \eta) \tag{2.18}$$

together with (2.15) and (2.16) and $y' \le 0$ on $(0, T)$ guarantees that

$$y'y'' \le M g(y)[-y'] \quad \text{on } (0, \eta).$$

Now integrate from 0 to t ($t \in (0, \eta)$) to obtain

$$[y'(t)]^2 \le 2M\, Q(y(t))$$

so

$$\frac{-\,y'(t)}{\sqrt{Q(y(t))}} \le \sqrt{2M} \quad \text{for } t \in (0, \eta). \tag{2.19}$$

Now integrate (2.19) from 0 to η to obtain

$$\int_0^a \frac{\mathrm{d}x}{\sqrt{Q(x)}} \le \eta\sqrt{2M} < \int_0^a \frac{\mathrm{d}x}{\sqrt{Q(x)}},$$

using (2.17). This is a contradiction, so (2.12) is true. □

Example. If $g(y) = y^\alpha$, $\alpha > 0$ and $a = q = 1$ then (2.17) reduces to

$$t^\star < \sqrt{\frac{\alpha+1}{2}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^{\alpha+1}}} \quad \text{for any } t^\star \in (0, T)$$

since $Q(z) = \frac{1}{\alpha+1}\left[1 - z^{\alpha+1}\right]$.

Remark 2.1. One can relax assumption (2.16) to $q \in L^\alpha[0, t^\star)$ for any $t^\star \in [0, T)$ (here $1 < \alpha < \infty$) provided (2.17) is appropriately adjusted. The idea here is to multiply the differential equation by $[-\,y']^\beta$ (here $1 < \beta < \infty$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) so (2.18) becomes

$$-[-y']^\beta y'' + \frac{p'}{p}[-y']^{\beta+1} = q g(y)[-y']^\beta \quad \text{on } (0, \eta).$$

Now apply Hölder's inequality (the details are left to the reader).

Theorem 2.3. *Suppose* (2.3)–(2.6) *hold. In addition assume there exists a function* $f : [0, \infty) \to [0, \infty)$ *such that*

$$\text{for any } 0 < y \le a \quad \text{we have } g(y) \le f(y) \tag{2.20}$$

and

$$\begin{cases} \text{for any } \epsilon, 0 < \epsilon < T, & \text{then any solution } y \text{ of} \\ (py')' + pq f(y) \ge 0 & \text{on } (0, \epsilon),\ y(0) = a, \\ \lim_{t \to 0^+} p(t) y'(t) = 0 & \text{satisfies } y > 0 \text{ on } [0, \epsilon]. \end{cases} \tag{2.21}$$

Then (2.1) *has a solution* $y \in C[0, T)$ *with* $0 < y(t) \le a$ *for* $t \in [0, T)$*. If in addition* (2.8a) *or* (2.8b) *holds then* y *is a solution of* (2.2).

Proof. Essentially the same reasoning as in Theorem 2.1 guarantees that (2.9) has a solution y with $y(t) \le a$ for $t \in [0, T)$. It remains to show (2.12). If (2.12) is false then there exists a $\eta \in (0, T)$ with $y > 0$ on $[0, \eta)$ and $y(\eta) = 0$. Now if $t \in (0, \eta)$ then (2.20) implies

$$0 = (py')' + pqg^\star(y) = (py')' + pqg(y) \le (py')' + pqf(y),$$

so (2.21) implies $y(\eta) > 0$, a contradiction. □

Notice it is also possible to use the ideas in this section to discuss both

$$\begin{cases} (py')' - pqg(y) = 0, & t \in [0, T) \\ y(0) = a > 0, \\ \lim\limits_{t\to 0^+} p(t)y'(t) = 0 \end{cases} \tag{2.22}$$

and

$$\begin{cases} (py')' - pqg(y) = 0, & t \in [0, T) \\ y(0) = a > 0, \\ y'(0) = 0 \end{cases} \tag{2.23}$$

where $0 < T \le \infty$ with $p \ge 0$, $q \ge 0$ and $g : [0, \infty) \to [0, \infty)$.

Theorem 2.4. *Suppose (2.3)–(2.6) hold. Then (2.22) has a solution $y \in C[0, T)$ with $y(t) \ge a$ for $t \in [0, T)$. If in addition (2.8a) or (2.8b) holds then y is a solution of (2.23).*

Proof. Consider (2.9) with

$$g^\star(y) = \begin{cases} g(\beta(t)), & y \ge \beta(t) \\ g(y), & a \le y \le \beta(t) \\ g(a), & y \le a \end{cases}$$

with

$$\beta(t) = R^{-1}\left(\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s\right)$$

and $R(z) = \int_a^z \frac{\mathrm{d}x}{g(x)}$ for $z > a$ (note that R is increasing on (a, ∞)). Essentially the same reasoning as in Theorem 2.1 guarantees that (2.9) has a solution y. Notice also that $(py')' \ge 0$ on $(0, T)$ so $y' \ge 0$ on $(0, T)$ and as a result $y(t) \ge a$ for $t \in [0, T)$. It remains to show

$$y(t) \le \beta(t) \quad \text{for } t \in [0, T). \tag{2.24}$$

Notice that, since g is nondecreasing and $y \ge a$, $g^\star(y) \le g(y)$. Now for $t \in (0, T)$ we have

$$\begin{aligned} y'(t) &= \frac{1}{p(t)} \int_0^t p(s)\,q(s)\,g^\star(y(s))\,\mathrm{d}s \\ &\le \frac{1}{p(t)} \int_0^t p(s)\,q(s)\,g(y(s))\,\mathrm{d}s \\ &\le g(y(t))\frac{1}{p(t)} \int_0^t p(s)\,q(s)\,\mathrm{d}s \end{aligned}$$

since $y' \ge 0$ on $(0, T)$. As a result

$$\frac{y'(t)}{g(y(t))} \le \frac{1}{p(t)} \int_0^t p(s)\,q(s)\,\mathrm{d}s \quad \text{for } t \in (0, T),$$

so integration from 0 to t yields

$$y(t) \le R^{-1}\left(\int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathrm{d}x\mathrm{d}s\right) = \beta(t)$$

for $t \in [0, T)$. As a result (2.24) holds. □

Remark 2.2. One could also consider the case $a = 0$ in (2.22) and (2.23); the details are left to the reader.

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