On the Generalizations of the Mazur–Ulam Isometric Theorem¹

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This paper contains several generalizations of the Mazur–Ulam isometric theorem in F∗-spaces which are not assumed to be locally bounded. Let X and Y be two real F∗-spaces, and let X be locally pseudoconvex or δ-midpoint bounded. Assume that an operator T maps X onto Y in a δ-locally 1/2-isometric manner for all i ∈ \{0\} ∪ \mathbb{N}. Then T is affine. In addition, we give the sufficient conditions of a mapping between two topological vector spaces being affine. © 2001 Academic Press

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1. PRELIMINARIES

Throughout this paper, we denote by \(\mathbb{N}\), \(\mathbb{R}\), and \(\mathbb{C}\) the sets of positive integers, of reals, and of complex numbers, respectively. Let X be a linear space on \(\mathbb{R}\) or \(\mathbb{C}\). A non-negative-valued function \(\|\cdot\|\) defined on X is called an F-norm if it satisfies the following conditions:

\[(n1) \quad \|x\| = 0 \text{ if and only if } x = 0;\]
\[(n2) \quad \|ax\| = \|x\| \text{ for all } a, |a| = 1;\]
\[(n3) \quad \|x + y\| \leq \|x\| + \|y\|;\]
\[(n4) \quad \|a_n x\| \longrightarrow 0 \text{ provided } a_n \longrightarrow 0;\]
\[(n5) \quad \|a x_n\| \longrightarrow 0 \text{ provided } x_n \longrightarrow 0.\]

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A space $X$ with an $F$-norm is called an $F^*$-space. An $F$-pseudonorm ($\|x\| = 0$ is not necessarily $x = 0$ in (n1)) is called $p$-homogeneous ($p > 0$) if $\|tx\| = |t|^p \|x\|$ for all $x \in X$ and all scalars $t$.

Let $X$ be a TVS (topological vector space). A set $A \subseteq X$ is said to be pseudobounded if, for any neighborhood $U$ of zero, there exists $n \in \mathbb{N}$ such that

$$A \subseteq U + U + \cdots + U.$$  

$n$-times

A TVS is called locally (bounded) pseudobounded if there is a (bounded) pseudobounded neighborhood of zero.

A set $A \subseteq X$ is said to be a starlike set if $tA \subseteq A$ for all $t \in (0, 1)$. The modulus of concavity of a starlike set $A$ is defined by

$$C(A) = \inf \left\{ s > 0 : A + A \subseteq sA \right\}$$

with the convention that $C(\emptyset) = +\infty$. $A$ is called pseudoconvex if $C(A) < +\infty$. A TVS is called locally pseudoconvex if there is a basis of neighborhoods of zero $\{U_n\}$ which are pseudoconvex.

An $F^*$-space $X$ is said to be $\delta$-midpoint bounded if there is $\delta > 0$ such that the sets

$$M = \left\{ x \in X : \|x - x'\| = \|x - x''\| = \left\| \frac{x' - x''}{2} \right\| \right\}$$

are topologically bounded for all $x'$ and $x'' \in X$ whenever $\|x' - x''\| < \delta$.

The author [6] proved that a TVS is locally bounded if and only if it is locally pseudobounded and locally pseudoconvex. Moreover, a locally pseudoconvex space need not be locally bounded; an example of this is $F$-space $(s)$, which is the space of sequences on real or complex number field with the $F$-norm

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n|}{1 + |\xi_n|} \quad (\forall x = \{\xi_n\} \in (s)).$$

In particular, locally convex spaces are locally pseudoconvex.

Let $X$ and $Y$ be two $F^*$-spaces. A mapping $T : X \to Y$ is called isometric if $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in X$. $T$ is said to be $\delta$-locally $t$-isometric ($t > 0$) if there exists $\delta > 0$ such that $\|t(Tx - Ty)\| = \|t(x - y)\|$, whenever $\|x - y\| < \delta$ for any $x, y \in X$.

The question of whether the Mazur–Ulam theorem holds for all metric linear spaces still seems to be open. Mazur and Ulam [3] proved that every surjective isometry $T$ between two normed linear spaces must be affine. Rolewicz [4] proved that if $X$ and $Y$ are real locally bounded $F^*$-spaces, and $T : X \to Y$ is $t$-isometric for all $t > 0$, then $T$ is affine. Ding Guanggui
and Huang Senzhong [1] proved that the Rolewicz theorem holds as long as $T$ is $1/2^n$-isometric ($\forall n \in \{0\} \cup \mathbb{N}$). A different kind of generalization of the Mazur–Ulam theorem was given by Day [2]. He proved that if $X$ and $Y$ are locally convex topological vector spaces and $T: X \to Y$ carries a total family $[p_n]$ of pseudonorms to another such family $[p'_n]$ on $Y$ by the rule $p'_n(Tx - Ty) = p_n(x - y)$, where $T$ is surjective, then $T$ is affine.

In the following we will extend this result to a larger class of $F^*$-spaces which are not assumed to be locally bounded, such as locally pseudoconvex $F^*$-spaces including all locally bounded $F^*$-spaces and all locally convex $F^*$-spaces.

2. THE MAIN RESULT

**Lemmma 1.** Let $X$ be a complex $F^*$-space equipped with an increasing $F$-norm. Suppose that $A \subseteq X$ is topologically bounded. Then for any complex number sequence $\{t_n\}_{n=1}^{\infty}$ with the property that $\{|t_n|\}$ decreasingly tends to 0,

$$\lim_{n \to \infty} D(t_n A) = 0,$$

where $D(A) = \sup_{x, y \in A} \{\|x - y\|\}$ is a diameter of $A$.

**Proof.** Let $x', x'' \in A$. Because $\|\cdot\|$ is increasing and $\{|t_n|\} \searrow 0$, by (n2)

$$\|t_n(x' - x'')\| = \|t_n(x' - x')\| \geq \|t_{n+1}(x' - x')\|$$

$$= \|t_{n+1}(x' - x'')\| \quad (\forall n \in \mathbb{N}),$$

so $D(t_n A) \geq D(t_{n+1} A) (n \in \mathbb{N})$, and hence $\{D(t_n A)\}_{n=1}^{\infty}$ is a decreasing sequence of numbers which has lower bound 0. Therefore, there is $r \geq 0$ such that $\lim_{n \to \infty} D(t_n A) = r$.

We claim $r = 0$. Assume conversely that there is $\varepsilon_0 > 0$ such that

$$D(t_k A) > \varepsilon_0 \quad (\forall k \in \mathbb{N}).$$

By the definition of supremum, there are $x_k$ and $y_k \in A$ such that

$$\|t_n(x_k - y_k)\| > \varepsilon_0.$$

(A is topologically bounded, and so is $A + A$. Moreover, $\{x_k - y_k\}_{k=1}^{\infty} \subseteq A + A$, and we have $\|t_n(x_k - y_k)\| \to 0 (k \to \infty)$ contradicting (L.1.1). This leads to $r = 0$.

**Lemma 2.** Let $X$ be a complex TVS. If a mapping $T: X \to Y$ satisfies the property that there is a $U$ neighborhood of zero in $X$ such that for any $x, y \in X$ with $y - x \in U$,

$$T \left( \frac{x + y}{2} \right) = \frac{T(x) + T(y)}{2},$$

then (L.2.1) holds for all $x, y \in X$. 

Proof. Let \( x, y \in X \). Choose a balanced neighborhood of zero \( V \) such that \( V + V \subseteq U \). We consider the situation where \( y - x \notin U \). We denote \( [u, v] = \{tu + (1 - t)v : 0 \leq t \leq 1\} \) for \( u, v \in X \). By the continuity of number multiplication, there exists \( N \in \mathbb{N} \) such that \( (y - x)/2^N \in V \). Putting \( \{x^{(i)}\}_{i=1}^{2^N-1} \subseteq [x, y] \), such as setting \( x^{(i)} = x + (i/2^N)(y - x) (0 \leq i \leq 2^N) \), we partition \([x, y]\) into \( 2^N \) segments. Then \( x^{(2^N-1)} = (x + y)/2 \) and

\[
x^{(i)} - x^{(i-1)} = \frac{1}{2^N}(y - x) \in V \quad (1 \leq i \leq 2^N),
\]

\[
x^{(i)} = \frac{x^{(i-1)} + x^{(i+1)}}{2} \quad (1 \leq i \leq 2^N - 1).
\]

Hence for every \( 1 \leq i \leq 2^N - 1 \),

\[
x^{(i+1)} - x^{(i-1)} = x^{(i+1)} - x^{(i)} + x^{(i)} - x^{(i-1)} \in V + V \subseteq U.
\]

By assumption (L.2.1), for all \( 1 \leq i \leq 2^N - 1 \), we have

\[
T(x^{(i)}) = T\left(\frac{x^{(i-1)} + x^{(i+1)}}{2}\right) = \frac{T(x^{(i-1)}) + T(x^{(i+1)})}{2}. \tag{L.2.2}
\]

Using (L.2.2), we obtain

\[
T(x^{(i-1)}) + T(x^{(i+1)}) = \frac{T(x^{(i-2)}) + T(x^{(i)}) + T(x^{(i)}) + T(x^{(i+2)})}{2} = \frac{T(x^{(i-2)}) + T(x^{(i+2)})}{2} + T(x^{(i)}) \quad (2 \leq i \leq 2^N - 2). \tag{L.2.3}
\]

From (L.2.2) and (L.2.3), we see that

\[
T(x^{(i)}) = \frac{T(x^{(i-2)}) + T(x^{(i+2)})}{2} \quad (2 \leq i \leq 2^N - 2).
\]

By recurrence, we finally obtain that (L.2.1) holds also when \( y - x \notin U \).

Theorem 1. Let \( X \) and \( Y \) be two complex \( F^*\)-spaces and let \( X \) be \( \delta_1 \)-midpoint bounded (T.1). If an operator \( T : X \to Y \) is \( \delta_2 \)-locally \( 1/2^i \)-isometric for all \( i \in \{0\} \cup \mathbb{N} \) and surjective, then \( T \) is real affine. If \( T \) also satisfies the property that \( T(ix) = iT(x)(i^2 = -1) \) for all \( x \in X \), then \( T \) is affine.

Proof. We will use a method that is similar to that of Mazur and Ulam. Let \( x_1, x_2 \in X \) and \( \delta = \min\{\delta_1, \delta_2/2\} \). First suppose that \( \|x_1 - x_2\| < \delta \). We define

\[
M_1 = \left\{ x \in X : \|x - x_1\| = \|x - x_2\| = \left\| \frac{x_1 - x_2}{2} \right\| \right\}.
\]
We construct by induction
\[ M_n = \left\{ x \in M_{n-1} : \left\| \frac{x-z}{2^{i-1}} \right\| \leq D \left( \frac{M_{n-1}}{2^i} \right) \forall z \in M_{n-1}, \forall i \in \mathbb{N} \right\} \]
\( (n = 2, 3, \ldots) \).

We shall prove by induction that \( M_n \neq \emptyset \) for any \( n \in \mathbb{N} \), and they are such that
\[ \frac{x_1 + x_2}{2} \in M_n, \quad \text{(T.1.1)} \]
\[ \bar{x} = x_1 + x_2 - x \in M_n \quad \text{(whenever } x \in M_n \text{)}. \tag{T.1.2} \]

For \( n = 1 \) this is trivial, since \( \bar{x} - x_1 = x_2 - x \) and \( \bar{x} - x_2 = x_1 - x \).

Suppose that (T.1.1) and (T.1.2) hold for a certain \( k - 1 \). Let \( x \in M_k \) and \( y \in M_{k-1} \). Then, the inductive assumption implies \( \bar{y} \in M_{k-1} \). By definition of \( M_k \),
\[ \left\| \bar{x} - y \right\|_{2^{i-1}} = \left\| \frac{x_1 + x_2 - y}{2^{i-1}} \right\| = \left\| \frac{x - \bar{y}}{2^{i-1}} \right\| \leq D \left( \frac{M_{k-1}}{2^i} \right) \quad (\forall i \in \mathbb{N}), \]
\[ \text{i.e., (T.1.2) holds for } n = k. \]

Since
\[ \left\| \frac{(x_1 + x_2)/2 - y}{2^{i-1}} \right\| = \left\| \frac{x_1 + x_2 - y - \bar{y}}{2^i} \right\| = \left\| \frac{\bar{y} - y}{2^i} \right\| \leq D \left( \frac{M_{k-1}}{2^i} \right) \quad (\forall i \in \mathbb{N}), \]
(T.1.1) holds for \( n = k \). We conclude by induction that (T.1.1) and (T.1.2) hold for any \( n \in \mathbb{N} \).

Next we show that \( \lim_{n \to \infty} D(M_n) = 0 \). Obviously, by \( M_n \subseteq M_{n-1} \), and from the definition of \( M_n \) we have
\[ D \left( \frac{M_n}{2^{i-1}} \right) \leq D \left( \frac{M_{n-1}}{2^i} \right) \quad (\forall i, n \in \mathbb{N}). \]
This implies
\[ D(M_n) \leq D \left( \frac{M_{n-1}}{2} \right) \leq \cdots \leq D \left( \frac{M_1}{2^{n-1}} \right) \quad (\forall n \in \mathbb{N}). \]

We have only to show that \( \lim_{n \to \infty} D(M_1/2^{n-1}) = 0 \).

Let \( \|x\|^* = \sup_{0 < t \leq 1} \|tx\| \). By [4, Theorem 1.2.2], \( \|x\|^* \) is equivalent to the original \( F \)-norm \( \|\cdot\| \), and it is increasing. By Lemma 1 and the assumption of the topological boundedness of \( M_1 \), \( D(M_1/2^{n-1}) \xrightarrow{\|\cdot\|^*} 0 \), so \( D(M_1/2^{n-1}) \xrightarrow{\|\cdot\|} 0 \). We obtain \( \lim_{n \to \infty} D(M_n) = 0 \).
Moreover, the intersection of all sets $M_n$ is the set which consists of one element $(x_1 + x_2)/2$. It is a metric characterization of the center of the points $x_1$ and $x_2$.

We can apply a similar reasoning to the space $Y$ to prove that $(Tx_1 + Tx_2)/2$ is the center of the points $Tx_1$ and $Tx_2$.

We shall show that $T((x_1 + x_2)/2)$ is also the center of $Tx_1$ and $Tx_2$. Suppose that $\tilde{M}$ are subset in $Y$ which are similar to $M_n$. For $n = 1, 2, \ldots$, let

\[
\tilde{M}_1 = \left\{ y \in Y : \| y - Tx_1 \| = \| y - Tx_2 \| = \left\| \frac{Tx_1 - Tx_2}{2} \right\| \right\},
\]

\[
\tilde{M}_n = \left\{ y \in \tilde{M}_{n-1} : \left\| \frac{y - z}{2^{i-1}} \right\| \leq D \left( \frac{M_{n-1}}{2^i} \right), \forall z \in \tilde{M}_{n-1}, \forall i \in \mathbb{N} \right\}.
\]

Since $T$ is $\delta - 1/2^i$-isometric and surjective, $D(\tilde{M}_n/2i) = D(M_n/2^i)$ for all $i \in \{0\} \cup \mathbb{N}$ and for all $n \in \mathbb{N}$. This yields $\lim_{n \to \infty} D(\tilde{M}_n) = \lim_{n \to \infty} D(M_n) = 0$.

We shall prove by induction that

\[
T(M_n) = \tilde{M}_n \quad (\forall n \in \mathbb{N}).
\]

For $n = 1$ this is trivial, since $T$ is a surjection and $\delta$-locally $1/2^i$-isometry ($i = 0, 1$).

Suppose that (T.1.3) holds for a certain $k - 1$. (T.1.4)

Let $y$ be an arbitrary element of $\tilde{M}_k$. Because $T$ is a surjection, there exists an $x \in X$ such that $Tx = y$. For each $z \in M_{k-1}$, we have $Tz \in \tilde{M}_{k-1}$ by (T.1.4). Hence

\[
\left\| \frac{x - z}{2^{i-1}} \right\| = \left\| \frac{Tx - Tz}{2^{i-1}} \right\| = \left\| \frac{y - Tz}{2^{i-1}} \right\| 
\leq D \left( \frac{\tilde{M}_{k-1}}{2^i} \right) = D \left( \frac{M_{k-1}}{2^i} \right) \quad (\forall i \in \mathbb{N}),
\]

and so $x \in M_k$. Thus $\tilde{M}_k \subseteq T(M_k)$.

On the other hand, for any $y \in T(M_k)$, then there is an $x \in M_k$ such that $Tx = y$. By $x \in M_k \subseteq M_{k-1}$ and (T.1.4), we have $y \in \tilde{M}_{k-1}$. Putting any $\tilde{z} \in \tilde{M}_{k-1}$, by (T.1.4), there is a $z \in M_{k-1}$ such that $Tz = \tilde{z}$. It follows that

\[
\left\| \frac{y - \tilde{z}}{2^{i-1}} \right\| = \left\| \frac{x - z}{2^{i-1}} \right\| \leq D \left( \frac{M_{k-1}}{2^i} \right) = D \left( \frac{\tilde{M}_{k-1}}{2^i} \right) \quad (\forall i \in \mathbb{N}).
\]

Then $y \in \tilde{M}_k$, i.e., $T(M_k) \subseteq \tilde{M}_k$. Thus $T(M_n) = \tilde{M}_n$ for all $n \in \mathbb{N}$. It follows from (T.1.3) and $(x_1 + x_2)/2 \in \cap_n M_n$ that $T((x_1 + x_2)/2) \in \cap_n \tilde{M}_n = \cap_n T(M_n)$. Moreover, this leads to

\[
T \left( \frac{x_1 + x_2}{2} \right) = \frac{Tx_1 + Tx_2}{2},
\]

since $\cap_n \tilde{M}_n = \{ (Tx_1 + Tx_2)/2 \}$. 

When \( \|x_1 - x_2\| > \delta \) for \( x_1, x_2 \in X \), by Lemma 2, we obtain that (T.1.5) holds also. Since \( T \) is \( \delta \)-locally isometric, \( T \) is continuous, and hence \( T \) is real affine. 

We can replace condition (T.1) in Theorem 1 with another condition. That is, the following theorem holds.

**Theorem 2.** Let \( X \) be a locally pseudoconvex \( F^* \)-space (T.2). Keeping the other hypotheses of Theorem 1, then \( T \) is affine.

**Proof.** We notice that the condition (T.1) is only used in proving \( \lim_{n \to \infty} D(M_1/2^{n-1}) = 0 \). Therefore, we have only to prove that \( \lim_{n \to \infty} D(M_1/2^{n-1}) = 0 \) holds under the condition (T.2).

Since \( X \) is locally pseudoconvex, by [4, Theorem III.1.3] there is a sequence of \( p_k \)-homogeneous \( F \)-pseudonorms \( \{ \| \cdot \|_k \} \) determining a topology equivalent to the original one.

For each \( k \in \mathbb{N} \), we define \( D_k(A) = \sup_{x,y\in A} \{ \| x - y \|_k \} \), then \( D_k(M_1/2^{n-1}) \leq D_k(M_1)/2^{n-k} \).

Since \( M_1 \subseteq S(x_1, \| (x_1 - x_2)/2 \|) \cap S(x_2, \| (x_1 - x_2)/2 \|) \) (\( S(x_0, r) = \{ x \in X : \| x - x_0 \| = r \} \) for \( r > 0 \)), \( M_1 \) is \( \| \cdot \|_k \)-norm bounded. So \( M_1 \) is also \( \| \cdot \|_k \)-norm bounded for any \( k \in \mathbb{N} \).

Let \( n \to \infty \) to obtain \( D_k(M_1/2^{n-1}) \mid_{l=k} 0 (n \to \infty, k \in \mathbb{N}) \), and thus \( \lim_{n \to \infty} D(M_1/2^{n-1}) = 0 \) by the equivalence of \( \| \cdot \| \) and \( \{ \| \cdot \|_k \} \).

Using the same method, we can find the following corollary.

**Corollary 1.** Let \( X \) and \( Y \) be two real \( F^* \)-spaces. Suppose that there are \( \delta_i > 0 \) and \( 0 < c_i < 1 \) (\( i = 1, 2 \)) such that \( \| x \| \leq c_1 \| x \| \), \( \| z \| \leq c_2 \| y \| \) whenever \( \| x \| < \delta_1 \) and \( \| y \| < \delta_2 \) for any \( x \in X \) and \( y \in Y \) (C.1). If \( T \) maps \( X \) onto \( Y \) in a \( \delta_3 \)-locally isometric manner, then \( T \) is affine.

**Proof.** Let \( \delta = \min \{ \delta_1, \delta_2, \delta_3/2 \}, c = \max \{ c_1, c_2 \} \). For any \( x_1, x_2 \in X \), whenever \( \| x_1 - x_2 \| < \delta \), we define by induction

\[
M_1 = \left\{ x \in X : \| x - x_1 \| = \| x - x_2 \| = \left\| \frac{x_1 - x_2}{2} \right\| \right\};
\]

\[
M_n = \left\{ x \in M_{n-1} : \| x - z \| \leq cD(M_{n-1}), \forall z \in M_{n-1} \right\} \quad (n = 2, 3, \ldots);
\]

\[
\tilde{M}_1 = \left\{ y \in Y : \| y - Tx_1 \| = \| y - Tx_2 \| = \left\| \frac{Tx_1 - Tx_2}{2} \right\| \right\};
\]

\[
\tilde{M}_n = \left\{ y \in \tilde{M}_{n-1} : \| y - z \| \leq cD(\tilde{M}_{n-1}) \quad \forall z \in \tilde{M}_{n-1} \right\} \quad (n = 2, 3, \ldots).
\]

In a way similar to that of the proof of Theorem 1, we can show the result completely.

Corollary 1 is a generalization of Rassias’s result [5, Theorem 2.8].
THEOREM 3. Let $X$ and $Y$ be two TVSs. Let $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ (resp. $\{\|\cdot\|^{\ast}_\lambda\}_{\lambda \in \Lambda}$) be an F-pseudonorm family $X$ (resp. $Y$) which is such that

1. $\|Tx - Ty\|_\lambda = \|x - y\|_\lambda$ for any $x, y \in X$ and $\lambda \in \Lambda$.

2. There is $0 < c_\lambda < 1$ such that $\|\|x\|_\lambda\| \leq c_\lambda \|x\|_\lambda$ and $\|\|x\|_\lambda\| \leq c_\lambda \|y\|_\lambda$ for any $x, y \in X$ and $\lambda \in \Lambda$.

3. $\{\|\cdot\|_\lambda\}^{\ast}$ is total (i.e., $\|y\|_\lambda = 0$ for all $\lambda \in \Lambda$ implies $y = 0$). Then $T$ is affine.

Proof. Let $x_1, x_2 \in X$. For any $\lambda \in \Lambda$, in the same way as in Corollary 1, we can construct $\{M_n^{(\lambda)}\}$ and $\{\tilde{M}_n(\lambda)\}$ which satisfy the property that

$T(M_n^{(\lambda)}) = \tilde{M}_n^{(\lambda)} (\forall n \in \mathbb{N})$, $\lim_{n \to \infty} D(M_n^{(\lambda)}) = \lim_{n \to \infty} D(\tilde{M}_n(\lambda)) = 0$

$(\forall y \in \Lambda)$, $\frac{Tx_1 + Tx_2}{2}, T\left(\frac{x_1 + x_2}{2}\right) \in \bigcap_n \tilde{M}_n(\lambda)$.

Let $M_n = \bigcap_{\lambda \in \Lambda} M_n^{(\lambda)}$, $\tilde{M}_n = \bigcap_{\lambda \in \Lambda} \tilde{M}_n(\lambda) (\forall n \in \mathbb{N})$. By the last three formulas, we have $T(M_n) = \tilde{M}_n(\forall n \in \mathbb{N})$, $\lim_{n \to \infty} D(M_n) = \lim_{n \to \infty} D(\tilde{M}_n) = 0$, and $\left(\frac{Tx_1 + Tx_2}{2}, T\left(\frac{x_1 + x_2}{2}\right)\right) \in \bigcap_n \tilde{M}_n = \bigcap_n T(M_n)$. By the totality of $\{\|\cdot\|_\lambda\}$, $\left(\frac{Tx_1 + Tx_2}{2}, T\left(\frac{x_1 + x_2}{2}\right)\right)$ by assumption (1), $T$ is continuous. Therefore $T$ is affine.

Remark 1. In a way similar to that of the proof Theorem 1, we may generalize immediately the Mazur–Ulam isometric theorem by Lemma 2. Replacing $T$ isometry with $\delta$-local isometry, we have the same conclusion. It shows that every $\delta$-local isometry between two normed spaces is equivalent to an isometry when it is surjective. Furthermore, every $\delta$-local isometry between two $F$-spaces is yet equivalent to an isometry under the conditions of Corollary 1.

Remark 2. Let $X$ and $Y$ be two locally pseudoconvex TVSs and $Y$ with $T_0$ axiom. Suppose that $\{U_\lambda\}_{\lambda \in \Lambda}$ (resp. $\{U^{\ast}_\lambda\}_{\lambda \in \Lambda}$) is a basis of pseudoconvex neighborhoods of 0 in $X$ (resp. $Y$). Without loss of generality, we may assume that they are balanced. Denote $c_\lambda = \max\{C(U_\lambda), C(U^{\ast}_\lambda)\}$ for any $\lambda \in \Lambda$. In the same way as in the proof of [4, Theorem III, 1.3], we can find the systems of F-pseudonorms $\{\|\cdot\|_\lambda\}$ (resp. $\{\|\cdot\|^{\ast}_\lambda\}$) on $X$ (resp. $Y$). For any $\lambda \in \Lambda$, $\|\cdot\|_\lambda$ and $\|\cdot\|^{\ast}_\lambda$ are $p_{\chi}$-homogeneous ($p_{\chi} = (\log 2)/(\log c_\lambda)$). If $T: X \to Y$ is surjective, and $\|Tx - Ty\|_\lambda = \|x - y\|_\lambda$ for any $x, y \in X$ and any $\lambda \in \Lambda$, then $T$ is affine. Note in particular that if $c_\lambda = 2$ for any $\lambda \in \Lambda$, then $X$ and $Y$ are two locally convex TVSs. This shows that we generalize Day’s result.
3. THE ANALYSIS OF CONDITION (C.1)

Throughout this section, let $X$ be an $F^*$-space. Rassias [5] shows that the ratio $\frac{x}{\|x\|}$ plays an important role in the generalizations of the Mazur–Ulam theorem. In Corollary 1, we assume also there exist $\delta > 0$ and $0 < c < 1$ such that $\frac{x}{\|x\|} \leq c\|x\|$ for any $x \in X$ with $\|x\| \leq \delta$ (R.2). In fact, the ratio $c$ satisfies $1/2 \leq c < 1$ for $\frac{x}{\|x\|} \geq 1/2$. Furthermore, we shall see that this condition can imply that $\|\cdot\|$ satisfies a certain kind of approximate homogeneity, and $\|\cdot\|$ is norm if $c = \frac{1}{2}$.

**Proposition 1.** Suppose that $X$ satisfies the condition (R.2). Then

1. $X$ is locally bounded.
2. $\|tx\| \leq \frac{1}{1-c^2} \|x\|$ as $\|x\| < \delta$ for any $t \in B(\mathbb{C}) = \{t \in \mathbb{C} : |t| \leq 1\}$.
3. $\|\|t\|\| - |t\|\|x\|\| \leq \frac{1}{1-c^2} \|x\|$ as $\|x\| < \delta$ for any $t \in \mathbb{C}$ and any $x \in X$.

**Proof.** Let $x \in X$ and $\|x\| < \delta$. By (R.2) and $\|x/2^n\| \leq c^n \|x\|$ for any $n \in \mathbb{N}$. Put any $\{x_n\} \subseteq O(\delta) = \{x \in X : \|x\| < \delta\}$ to obtain $\|x_n/2^n\| \leq c^n \|x_n\| < c^n \delta$. Letting $n \to \infty$, we see, by $0 < c < 1$, that $x_n/2^n \to 0$. Hence $O(\delta)$ is bounded and so $X$ is locally bounded.

Let $r$ be an arbitrary dyadic number in $[0, 1]$,

$$r = \frac{\delta_1}{2} + \frac{\delta_2}{2^2} + \cdots + \frac{\delta_k}{2^k},$$

where $\delta_i$ is equal either to 0 or to 1, $i = 1, 2, \ldots, k$. Then

$$\|rx\| = \left\| \sum_{i=1}^{k} \frac{\delta_i}{2^i} x \right\| \leq \sum_{i=1}^{k} \frac{\|x\|}{2^i} \leq \sum_{i=1}^{k} c^i \|x\| \leq \frac{c}{1-c} \|x\|.$$  

The continuity of number multiplication implies that

$$\|tx\| = \| |t|\|x\| \leq \frac{c}{1-c} \|x\| \quad (\forall t \in [-1, 1]). \quad (P.1.1)$$

Moreover, if $t = 0$, then (2) and (3) are evident. Next we assume that $t \in \mathbb{C}$, $t \neq 0$. In view of (n2), $\|tx\| = \|t\| \|x\| = \|t\| \|x\|$, and it infers from (P.1.1) the assertion (2).

Denote by $[\alpha]$ the integer part of $\alpha$, and by $\{\alpha\}$ the decimal part of $\alpha$ for any $\alpha \geq 0$.

By virtue of (P.1.1), on the one hand,

$$\|tx\| = \|t\| \|x\| = \|(|t|) + \{|t|\})x\| \leq |\{t\}| \|x\| + |\{|t|\}| \|x\|$$

$$\leq |t| \|x\| + \frac{c}{1-c} \|x\| \quad (\forall t \in \mathbb{C}),$$
and on the other hand,

\[ \|tx\| = \| (|t|+1)x - (1-\{t\})x \| \geq \| |t|+1\|x\| + \|1-\{t\}\|x\| \geq |t|\|x\| - \frac{c}{1-c}\|x\\| (\forall t \in \mathbb{C}). \]

Thus, we conclude the assertion (3). ■

**Proposition 2.** If \( \|x\| \leq \frac{1}{2}\|x\| \) for any \( x \in X \) (P2), then \( X \) is a normed space.

**Proof.** We have only to prove that \( \| \cdot \| \) satisfies the absolute homogeneity. Since \( \|x\| \leq \frac{1}{2}\|x\| \) and (P2), \( \|x\| = \frac{1}{2}\|x\| \) for any \( x \in X \). Therefore \( \|2x\| = 2\|\frac{x}{2}\| = 2\|x\| \) for any \( x \in X \). We shall show by induction that

\[ \|nx\| = n\|x\| \quad (\forall n \in \mathbb{N}). \]  

(P2.1)

Obviously, (P2.1) holds for \( n = 1, 2 \). Suppose that (P2.1) holds for \( n \leq k \).

(P2.2)

For \( n = k + 1 \), if \( k \) is an even, then \( \frac{k+2}{2} \leq k \). Hence, by (P2.2)

\[ \|(k+1)x\| \geq \|(k+2)x\| - \|x\| = 2\left\| \frac{k+2}{2}x \right\| - \|x\| = 2 \cdot \frac{k+2}{2}\|x\| - \|x\| = (k+1)\|x\|. \]

If \( k \) is an odd, then \( \frac{k+1}{2} \leq k \). By (P2.2), we have

\[ \|(k+1)x\| = 2\left\| \frac{k+1}{2}x \right\| = 2 \cdot \frac{k+1}{2}\|x\| = (k+1)\|x\|. \]

It is clear that \( \|(k+1)x\| \leq (k+1)\|x\| \). So (P2.1) holds for \( n = k + 1 \). It follows that (P2.1) holds for any \( n \in \mathbb{N} \). Moreover, we obtain that

\[ \left\| \frac{x}{n} \right\| = \frac{1}{n}\left\| \frac{x}{n} \right\| = \frac{1}{n}\|x\| \quad (\forall n \in \mathbb{N}). \]

Hence \( \|\frac{x}{m}\| = \frac{m}{n}\|x\| \) for any \( m \in \{0\} \cup \mathbb{N} \) and \( n \in \mathbb{N} \).

The continuity of number multiplication implies that \( \|tx\| = t\|x\| \) for any \( t \in [0, \infty) \). Thus \( \|tx\| = \| |t|\|x\| = |t|\|x\| \) for any \( t \in \mathbb{C} \). ■

**Corollary 3.** If \( \|x\| \leq \frac{1}{2}\|x\| \) or \( \|kx\| \geq k\|x\| \) for some \( k \in \mathbb{N} \) and any \( x \in X \), then \( X \) is a normed space.
Proof. Note that $\|x\| \leq \frac{1}{k} \|kx\|$ if and only if $\|kx\| \geq k\|x\|$. We may assume that $\|kx\| \geq k\|x\|$. By (n3),

$$\|(k - 1)x\| \geq \|kx\| - \|x\| \geq (k - 1)\|x\|.$$ 

By recurrence, we obtain that $2\|x\| \geq 2\|x\|$. Thus $\|x\| \leq \frac{1}{2} \|x\|$. It follows from Proposition 2 that $\|\cdot\|$ is a norm.

Corollary 4. An $F$-norm $\|\cdot\|$ is convex (i.e., $\|x + y\| \leq 1/2(\|x\| + \|y\|)$ for any $x, y \in X$) if and only if it is a norm.

Proof. The sufficiency is clear and the necessity follows from Proposition 2.

Remark 3. We know that the convexity of norms ensures the convexity of balls. Corollary 4 indicates that the balls are not generally convex in $F^*$-spaces unless they are in normed spaces. Thus it is difficult to study the convexity of balls in $F^*$-spaces with the properties of $F$-norms. We may see this fact from the following examples.

Example 1. Let $X = (\mathbb{R}, \|\cdot\|)$. For any $x \in \mathbb{R}$

$$\|x\| = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| > 1. \end{cases}$$

As in Fig. 1, it is easy to verify that $X$ is an $F$-space which satisfies that $B(\delta) = \{x \in X, \|x\| \leq \delta\}$ is convex for any $\delta > 0$. However, $\|\cdot\|$ is not convex.

Example 2. Let $X = (\mathbb{R}, \|\cdot\|)$. For any $x \in \mathbb{R}$

$$\|x\| = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 2 - |x| & \text{if } 1 < |x| \leq \frac{3}{2} \\ \frac{1}{2} & \text{if } |x| > \frac{3}{2}. \end{cases}$$

As indicated in Fig. 2, $X$ is an $F$-space which satisfies that $B(\delta)$ is convex for any $\delta \in (0, 1/2) \cup [1, +\infty)$ but not for any $\delta \in [1/2, 1)$. 

![Figure 1](image_url)
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REFERENCES