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Constructing Pointed Hopf Algebras by Ore Extensions

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We present a general construction producing pointed co-Frobenius Hopf algebras and give some classification results for the examples obtained.

0. INTRODUCTION AND PRELIMINARIES

In recent years a serious effort has been made to understand and classify Hopf algebras over an algebraically closed field of characteristic zero. Nevertheless, the classification of finite dimensional Hopf algebras has been completed only for some small dimensions and for prime dimensions ([24]).

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A nice survey of the present state of the classification of finite dimensional Hopf algebras with emphasis on the semisimple case is given in [16]. The main purpose of this paper is to present a general construction producing pointed Hopf algebras.

Many current papers have defined interesting pointed Hopf algebras by generators and relations. In [18] two families of such Hopf algebras are constructed. The first, denoted by $H_{n,q,N,\nu}$, includes the classical four dimensional Sweedler Hopf algebra and the Taft Hopf algebra of dimension n^2. The second is a family of finite dimensional pointed unimodular ribbon Hopf algebras, denoted by $U_{(N,\nu,\omega)}$, generating invariants of knots and 3-manifolds. A special object in the second class is $U_q(sl(2))'$, where q is a root of unity (see [20]). These examples were further studied in [9]. In [22] an example of a unimodular Hopf algebra, whose antipode is not inner, is constructed. An example of an infinite dimensional co-Frobenius Hopf algebra which is not the tensor product of a cosemisimple Hopf algebra and a finite dimensional Hopf algebra was given in [23]. This example was generalized in [3], where a large class of such examples was constructed using Ore extensions of a group algebra. In [4] examples constructed by Ore extensions were used to classify pointed Hopf algebras of dimension p^n, p prime, with the coradical the group algebra of an abelian group of order p^{n-1}. Pointed Hopf algebras of dimension p^3 were classified in [7], [2], and [21]. Hopf algebras of dimension 2^n with coradical kC_2 were described in [8].

Our general construction produces pointed co-Frobenius Hopf algebras which are generated by grouplikes and (g, h)-primitives. Briefly, we start with a group algebra (the coradical), add indeterminates (the (g, h)-primitives) by repeated Ore extensions and then factor by a Hopf ideal. The idea is very simple, but it sheds some light on many complicated examples, providing a natural algebraic framework for their construction. In particular, all the examples mentioned above may be obtained by this construction, as well as other infinite dimensional co-Frobenius Hopf algebras, and finite dimensional quantum groups. We remark that a very different approach to constructing bialgebras may be found in [14] where an example of a noncommutative noncocommutative bialgebra of dimension 5 over a field of characteristic 2 is given.

In many cases we can determine explicitly when two Hopf algebras constructed as described above are isomorphic. This leads to a proof that infinite families of non-isomorphic Hopf algebras of the same finite dimension exist, generalizing the counter-example in [5] to Kaplansky's tenth conjecture [11]. This conjecture that there exist only finitely many types of Hopf algebras of a given finite dimension over an algebraically closed field has only recently been refuted independently, and with different approaches, in [5, 2, 10].

The paper is organized as follows. In Sect. 1 we develop the general construction, i.e., we start with a group algebra $A = kC$, where C is a finitely generated abelian group, then we form a sequence of Ore extensions which we endow with a Hopf algebra structure. The Hopf algebra obtained after t steps, denoted by A_t, is pointed. By computing the injective envelopes of the simple subcomodules, we see that A_t is not co-Frobenius. In order to produce a pointed co-Frobenius Hopf algebra, we factor A_t by a Hopf ideal. Section 2 contains some classification results; in many cases an isomorphism between two such Hopf algebras essentially reduces to an automorphism of the group C. The classification of our "Ore extension Hopf algebras" is complete in the case where each Ore extension has zero derivation. If there are non-zero derivations, we still get a classification result which is sufficient to produce infinitely many types of Hopf algebras with the same dimension. In Sect. 3, we compute the duals of some finite dimensional Ore extension Hopf algebras defined with zero derivations; these duals are also Ore extension Hopf algebras with zero derivations. We remark that duals of Ore extension Hopf algebras with non-zero derivations may not even be pointed. The special case of C cyclic is discussed in Sect. 4. We determine when Ore extension Hopf algebras can be constructed starting with kC and how many. The result depends on the arithmetic properties of the order of C. Since $\mathrm{Aut}(C)$ is easy to describe when C is cyclic, the classification of the Hopf algebras that we obtain is more precise. In Sect. 5 we consider a special class of examples of the general construction, namely those constructed using a non-zero derivation with each Ore extension after the first. These, also, can be classified. A large class of quantum groups arises here. In Sect. 6 we list all pointed Hopf algebras of dimension p^3 produced by the preceding constructions, and count how many non-isomorphic Hopf algebras occur. By [7], these are indeed all pointed Hopf algebras of dimension p^3.

Throughout, k will be an algebraically closed field of characteristic 0 although, in fact, we only need that k contain enough roots of unity. The set of non-zero elements of k is denoted by k^*. All maps, $\otimes$, etc., are k-linear. We use $\mathbf{N}$ to denote the non-negative integers and $\mathbf{Z}^+$ for the positive integers.

In order to compute comultiplication on products and powers of (g, h)-primitives, we will require q-binomial coefficients, $\binom{n}{l}_q$, $q \in k^*$. Note that this is a formal notation; $\binom{n}{l}_q$ is a polynomial in q. For n, l integers with $0 \leq l \leq n$, the q-binomial coefficients are defined by $\binom{n}{l}_q = (n)_q!/(l)_q!(n-l)_q!$. If l is a positive integer, $(l)_q = 1 + q + \cdots + q^{l-1}$, and $(l)_q! = (l)_q(l-1)_q \cdots (1)_q$. By definition, $(0)_q! = 1$. For more detail, we refer the reader to [12, Chap. 4].

Suppose a and b are elements of a k-algebra and $ba = qab$. Then the expansion of $(a+b)^n$ is described by the following.

LEMMA 0.1. *For $q \neq 0$, $ba = qab$,*

(i) $(a+b)^n = \sum_{l=0}^{n} \binom{n}{l}_q a^{n-l} b^l$.

(ii) $(a+b)^n = a^n + b^n$ *if q is a primitive* nth *root of unity.*

REMARK 0.2. Note that in Lemma 0.1 (ii), it is essential that q be a primitive nth root. For example, if $q = -1$ and $n = 4$, $ba = -ab$, $(a+b)^4 = (a^2+b^2)^2 = a^4 + 2a^2b^2 + b^4$. The coefficients of $a^{n-i}b^i$, $0 < i < n$, in the expansion of $(a+b)^n$ are all 0 if and only if q is a primitive nth root.

We follow the standard notation in [15]. For H a Hopf algebra, $G(H)$ will denote the group of grouplike elements and $H_0, H_1, H_2, \ldots$ will denote the coradical filtration of H. H is called pointed if $H_0 = kG(H)$. If g and h are group-like elements of a Hopf algebra H, then x is called a (g,h)-primitive of H if

$$\triangle(x) = x \otimes g + h \otimes x;$$

$P_{g,h}$ denotes the k-vector space of (g,h)-primitives of H. Then $P_{g,h} = k(g-h) \oplus P'_{g,h}$ for some vector space $P'_{g,h}$. If H is a pointed Hopf algebra then by the Taft–Wilson Theorem ([15, Theorem 5.4.1]), $H_1 = H_0 \oplus P'$, where H_0 is the coradical, H_1 the next term of the coradical filtration and $P' = \oplus_{g,h\in G} P'_{g,h}$. If H is finite dimensional, then $P_{1,1} = 0$. This implies that if H is also pointed of dimension > 1, then $G(H)$ is not trivial. A Hopf algebra is co-Frobenius if it has a left (and a right) integral in H^*.

1. THE ORE EXTENSION CONSTRUCTION

Recall (for example, from [19, 1.6.16]) that for a k-algebra A, an algebra endomorphism φ of A, and a φ-derivation δ of A (i.e., a linear map $\delta\colon A \to A$ such that $\delta(ab) = \delta(a)b + \varphi(a)\delta(b)$ for all $a, b \in A$), the Ore extension $A[X, \varphi, \delta]$ is $A[X]$ as an abelian group, with multiplication induced by $Xa = \delta(a) + \varphi(a)X$ for all $a \in A$. The following is an obvious extension of the universal property for polynomial rings.

LEMMA 1.1. *Let $A[X, \varphi, \delta]$ be an Ore extension of A and $i\colon A \to A[X, \varphi, \delta]$ the inclusion morphism. Then for any algebra B, any algebra morphism $f\colon A \to B$ and every element $b \in B$ such that $bf(a) = f(\delta(a)) + f(\varphi(a))b$ for all $a \in A$, there exists a unique algebra morphism $\overline{f}\colon A[X, \varphi, \delta] \to B$ such that $\overline{f}(X) = b$ and the following diagram* *is commutative*:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & A[X, \varphi, \delta] \arrow[d, "\overline{f}"] \\ & B \end{tikzcd}$$

In this section, we construct pointed Hopf algebras by starting with the coradical, forming Ore extensions, and then factoring out a Hopf ideal.

Let $A = kC$ be the group algebra of a finitely generated abelian group C with the usual Hopf algebra structure, and let C^* be the character group of C. Let $c_1 \in C$ and $c_1^* \in C^*$.

Let φ_1 be an algebra automorphism of A defined by $\varphi_1(g) = \langle c_1^*, g\rangle g$ for all $g \in C$. Consider the Ore extension $A_1 = A[X_1, \varphi_1, \delta_1]$, where $\delta_1 = 0$. Apply Lemma 1.1 first with $B = A_1 \otimes A_1$, $f = (i \otimes i) \cdot \triangle_A$, $b = c_1 \otimes X_1 + X_1 \otimes 1$ and then with $B = k$, $f = \epsilon_A$, $b = 0$, to define algebra homomorphisms $\triangle\colon A_1 \to A_1 \otimes A_1$ and $\epsilon : A_1 \to k$ by

$$\triangle(X_1) = c_1 \otimes X_1 + X_1 \otimes 1 \qquad \text{and } \epsilon(X_1) = 0. \tag{1.2}$$

It is easily checked that Δ and ϵ define a bialgebra structure on A_1. The antipode S of A extends to an antipode on A_1 by $S(X_1) = -c_1^{-1}X_1$.

Next, let $c_2^* \in C^*$, $\gamma_{12} \in k^*$, and let $\varphi_2 \in \operatorname{Aut}(A_1)$ be defined by

$$\varphi_2(g) = \langle c_2^*, g\rangle g \qquad \text{for } g \in C, \qquad \varphi_2(X_1) = \gamma_{12}X_1.$$

We seek a φ_2-derivation δ_2 of A_1, such that δ_2 is zero on kC and $\delta_2(X_1) \in kC$. (The assumption that δ_2 is zero on kC will be at least partially justified by Proposition 1.20.) We want the Ore extension $A_2 = A_1[X_2, \varphi_2, \delta_2]$ to have a Hopf algebra structure with X_2 a $(1, c_2)$-primitive for some $c_2 \in C$, i.e., $\triangle(X_2) = c_2 \otimes X_2 + X_2 \otimes 1$. Then

$$X_2X_1 = \delta_2(X_1) + \gamma_{12}X_1X_2. \tag{1.3}$$

Applying $\triangle$ to both sides of (1.3), we see that

$$\gamma_{12} = \langle c_1^*, c_2\rangle^{-1} = \langle c_2^*, c_1\rangle \qquad \text{and}$$

$$\triangle(\delta_2(X_1)) = c_1c_2 \otimes \delta_2(X_1) + \delta_2(X_1) \otimes 1.$$

Thus $\delta_2(X_1)$ is a $(1, c_1c_2)$-primitive in kC and so we must have

$$\delta_2(X_1) = b_{12}(c_1c_2 - 1)$$

for some scalar b_{12}. If $c_1c_2 - 1 = 0$, then we define b_{12} to be 0. If $b_{12} = 0$, then δ_2 is clearly a φ_2-derivation. Suppose that $\delta_2 \neq 0$. In this case it remains to check that δ_2 is a φ_2-derivation of A_1. In order that δ_2 be well defined we must have, for all $g \in C$,

$$\delta_2(gX_1) = \varphi_2(g)\delta_2(X_1) = \delta_2(\langle c_1^*, g\rangle^{-1}X_1g) = \langle c_1^*, g\rangle^{-1}\delta_2(X_1)g.$$

Thus $\varphi_2(g) = \langle c_2^*, g\rangle g = \langle c_1^*, g\rangle^{-1}g$ and therefore $c_1^*c_2^* = 1$ and $\gamma_{12} = \langle c_1^*, c_2\rangle^{-1} = \langle c_2^*, c_2\rangle = \langle c_2^*, c_1\rangle = \langle c_1^*, c_1\rangle^{-1}$.

Now we compute

$$\begin{aligned}\delta_2(X_1^2) &= \delta_2(X_1)X_1 + \varphi_2(X_1)\delta_2(X_1)\\ &= b_{12}(1 + \langle c_1^*, c_1\rangle)c_1c_2X_1 - b_{12}(1 + \langle c_1^*, c_1\rangle^{-1})X_1,\end{aligned}$$

and, by induction, we see that for every positive integer r, we have

$$\delta_2(X_1^r) = b_{12}\left(\sum_{k=0}^{r-1}\langle c_1^*, c_1\rangle^k\right)c_1c_2X_1^{r-1} - b_{12}\left(\sum_{k=0}^{r-1}\langle c_1^*, c_1\rangle^{-k}\right)X_1^{r-1}. \tag{1.4}$$

A straightforward (tedious) computation now ensures that for $g, g' \in C$,

$$\delta_2(gX_1^rg'X_1^p) = \delta_2(gX_1^r)g'X_1^p + \varphi_2(gX_1^r)\delta_2(g'X_1^p)$$

and our definition of $A_2 = A_1[X_2, \varphi_2, \delta_2]$ is complete.

Summarizing, A_2 is a Hopf algebra with generators $g \in C$, X_1, X_2, such that the elements of C are commuting grouplikes, X_j is a $(1, c_j)$-primitive and the following relations hold

$$gX_j = \langle c_j^*, g\rangle^{-1}X_jg \text{ and } X_2X_1 - \gamma_{12}X_1X_2 = b_{12}(c_1c_2 - 1),$$
$$\text{where } \gamma_{12} = \langle c_1^*, c_2\rangle^{-1} = \langle c_2^*, c_1\rangle,$$

and, if $\delta_2(X_1) \neq 0$,

$$c_1^*c_2^* = 1 \text{ and } \gamma_{12} = \langle c_1^*, c_1\rangle^{-1} = \langle c_1^*, c_2\rangle^{-1} = \langle c_2^*, c_1\rangle = \langle c_2^*, c_2\rangle.$$

We continue forming Ore extensions. Define an algebra automorphism φ_j of A_{j-1} by $\varphi_j(g) = \langle c_j^*, g\rangle g$ where $c_j^* \in C^*$, and $\varphi_j(X_i) = \langle c_j^*, c_i\rangle X_i$ where $c_i \in C$, and X_i is a $(1, c_i)$-primitive. The derivation δ_j of A_{j-1} is 0 on kC and $\delta_j(X_i) = b_{ij}(c_ic_j - 1)$. If $c_ic_j = 1$, we define $b_{ij} = 0$. We write X^p for $X_1^{p_1}\cdots X_t^{p_t}$ where $p \in \mathbf{N}^t$. After t steps, we have a Hopf algebra A_t.

Definition 1.5. A_t is the Hopf algebra generated by $g \in C$ and X_j, $j = 1, \ldots, t$ where

(i) the elements of C are commuting group-likes;

(ii) the X_j are $(1, c_j)$-primitives;

(iii) $X_j g = \langle c_j^*, g\rangle g X_j$;
(iv) $X_j X_k = \langle c_j^*, c_k\rangle X_k X_j + b_{kj}(c_k c_j - 1)$ for $1 \le k < j \le t$;
(v) $\langle c_k^*, c_j\rangle\langle c_j^*, c_k\rangle = 1$ for $j \ne k$;
(vi) If $b_{ij} \ne 0$ then $c_i^* c_j^* = 1$.

The antipode of A_t is given by $S(g) = g^{-1}$ and $S(X_j) = -c_j^{-1} X_j$.

Note that $S^2(X_j) = c_j^{-1} X_j c_j =< c_j^*, c_j > X_j$ so that if $t = 1$, S^2 is inner. The relations show that A_t has basis $\{gX^p | g \in C,\ p \in \mathbf{N}^t\}$. Since for $q_j = \langle c_j^*, c_j\rangle$,

$$(X_j \otimes 1)(c_j \otimes X_j) = q_j(c_j \otimes X_j)(X_j \otimes 1),$$

then, for $k \in \mathbf{Z}^+$, $\triangle(X_j^k) = \triangle(X_j)^k = (c_j \otimes X_j + X_j \otimes 1)^k$, and expansion of this power follows the rules in Lemma 0.1. For $g \in C,\ p = (p_1, \ldots, p_t) \in \mathbf{N}^t$,

$$\triangle(gX_1^{p_1} \cdots X_t^{p_t}) = \triangle(gX^p) = \sum_d \alpha_d g c_1^{d_1} c_2^{d_2} \cdots c_t^{d_t} X^{p-d} \otimes gX^d, \qquad (1.6)$$

where $d = (d_1, \ldots, d_t) \in \mathbf{Z}^t$, the jth entry d_j in the t-tuple d ranges from 0 to p_j, and the α_d are scalars resulting from the q-binomial expansion described in Lemma 0.1 and the commutation relations. In particular, for $1 \le j \le t$, $n \in \mathbf{Z}^+$,

$$\Delta(X_j^n) = \sum_{k=0}^{n} \binom{n}{k}_{q_j} c_j^k X_j^{n-k} \otimes X_j^k. \qquad (1.7)$$

PROPOSITION 1.8. *The Hopf algebra A_t has the following properties*:

(i) *The $(n+1)$th term, $(A_t)_n$, in the coradical filtration of A_t is generated by gX^p, $g \in C$, $p \in \mathbf{N}^t$, $p_1 + \cdots + p_t \le n$. In particular, A_t is pointed with coradical kC.*

(ii) *For $g \in C$, the injective envelope of kg in the category of right A_t-comodules is the k-space $\mathcal{E}_g$ spanned by all $gc_1^{-p_1} c_2^{-p_2} \cdots c_t^{-p_t} X_1^{p_1} \cdots X_t^{p_t} = gc_1^{-p_1} \cdots c_t^{-p_t} X^p$, $p = (p_1, \ldots, p_t) \in \mathbf{N}^t$.*

(iii) *A_t is not a co-Frobenius Hopf algebra.*

Proof. (i) An induction argument using Eq. (1.6) shows that for all n,

$$\langle gX^p | g \in C,\ p \in \mathbf{N}^t,\ p_1 + \cdots + p_t \le n\rangle \subseteq \wedge^{(n+1)} kC.$$

Thus, $\wedge^{(\infty)} kC = A_t$ and by [1, 2.3.9], Corad $(A_t) \subseteq kC$. Since kC is a cosemisimple coalgebra, it is exactly the coradical of A_t.

(ii) Again by Eq. (1.6), $\mathscr{E}_g$ is a right A_t-subcomodule of A_t and kg is essential in $\mathscr{E}_g$. On the other hand, $A_t = \oplus_{(g)}\mathscr{E}_g$, $g \in C$. Thus the $\mathscr{E}_g$'s are injective.

(iii) This follows directly from [13, Theorem 10] and the fact that the $\mathscr{E}_g$'s are infinite dimensional. ∎

In order to obtain a co-Frobenius Hopf algebra, we factor A_t by a Hopf ideal.

LEMMA 1.9. *Let $n_1, n_2, \ldots, n_t \geq 2$ and $a = (a_1, \ldots, a_t) \in \{0, 1\}^t$. The ideal $J(a)$ of A_t generated by*

$$(X_1^{n_1} - a_1(c_1^{n_1} - 1), \ldots, X_t^{n_t} - a_t(c_t^{n_t} - 1))$$

is a Hopf ideal if and only if $q_j = \langle c_j^*, c_j \rangle$ is a primitive n_jth root of unity for $1 \leq j \leq t$.

Proof. Since $c_j^{n_j} - 1$ is a $(1, c_j^{n_j})$-primitive, it follows that $X_j^{n_j} - a_j(c_j^{n_j} - 1)$ is a $(1, c_j^{n_j})$-primitive if and only if $X_j^{n_j}$ is. By (1.7) and Remark 0.2, this occurs if and only if $\binom{n_j}{k}_{q_j} = 0$ for every $0 < k < n_j$, i.e., if and only if q_j is a primitive n_jth root of unity. Moreover, since $S(X_j) = -c_j^{-1}X_j$, induction on n shows that

$$S(X_j^n) = (-1)^n q_j^{-n(n-1)/2} c_j^{-n} X_j^n.$$

Now, since $q_j^{n_j} = 1$, checking the cases n_j even and n_j odd, we see that $(-1)^{n_j} q_j^{-n_j(n_j-1)/2} = -1$ and hence

$$S(X_j^{n_j} - a_j(c_j^{n_j} - 1)) = -c_j^{-n_j}(X_j^{n_j} - a_j(c_j^{n_j} - 1))$$

for $1 \leq j \leq t$, so that the ideal $J(a)$ is invariant under the antipode S, and is thus a Hopf ideal. ∎

By Lemma 1.9, $H = A_t/J(a)$ is a Hopf algebra. However, the coradical may be affected by taking this quotient. Since we want H to be a pointed Hopf algebra with coradical kC, some additional restrictions are required. We denote by x_i the image of X_i in H and write x^p for $x_1^{p_1} \cdots x_t^{p_t}$, $p = (p_1, \ldots, p_t) \in \mathbf{N}^t$.

PROPOSITION 1.10. *Assume $J(a)$ as in Lemma 1.9 is a Hopf ideal. Then $J(a) \cap kC = 0$ if and only if for each i either $a_i = 0$ or $(c_i^*)^{n_i} = 1$. If this is the case then $\{gx^p | g \in C, p \in \mathbf{N}^t, 0 \leq p_j \leq n_j - 1\}$ is a basis of $A_t/J(a)$.*

Proof. By Lemma 1.9, we know that $J(a)$ is a Hopf ideal if and only if $q_i = \langle c_i^*, c_i\rangle$ is a primitive n_ith root of unity for $1 \le i \le t$. Now suppose that $J(a) \cap kC = 0$. Since

$$(X_i^{n_i} - a_i(c_i^{n_i} - 1))g = \langle c_i^*, g\rangle^{n_i} g(X_i^{n_i} - \langle c_i^*, g\rangle^{-n_i} a_i(c_i^{n_i} - 1))$$

is in $J(a)$ for every $g \in C$, it follows that $X_i^{n_i} - \langle c_i^*, g\rangle^{-n_i} a_i(c_i^{n_i} - 1)$ is in $J(a)$. But then for every $g \in C$, both $a_i(1 - \langle c_i^*, g\rangle^{-n_i})(c_i^{n_i} - 1)$ and $(1 - \langle c_i^*, g\rangle^{-n_i})X_i^{n_i}$ are in $J(a)$. If $a_i \neq 0$, which by our convention implies that $c_i^{n_i} - 1 \neq 0$, then we must have $\langle c_i^*, g\rangle^{n_i} = 1$ for all g, and thus $c_i^{*n_i} = 1$.

Conversely, assume that $c_i^{*n_i} = 1$ whenever $a_i \neq 0$. By Definition 1.5 (iii), $X_i^{n_i} g = \langle c_i^*, g\rangle^{n_i} g X_i^{n_i}$. In particular, $X_i^{n_i} g = g X_i^{n_i}$ if $a_i \neq 0$. Also, if $i < j$ then by (1.4),

$$X_j X_i^{n_i} = \varphi_j(X_i^{n_i})X_j + \delta_j(X_i^{n_i})$$

$$= \langle c_j^*, c_i\rangle^{n_i} X_i^{n_i} X_j + b_{ij}\left(\sum_{k=0}^{n_i-1} \langle c_i^*, c_i\rangle^k\right) c_i c_j X_i^{n_i-1} - b_{ij}\left(\sum_{k=0}^{n_i-1} \langle c_i^*, c_i\rangle^{-k}\right) X_i^{n_i-1}.$$

So, if $b_{ij} = 0$, then $X_j X_i^{n_i} = \langle c_j^*, c_i\rangle^{n_i} X_i^{n_i} X_j$, where $\langle c_j^*, c_i\rangle^{n_i} = \langle c_i^*, c_j\rangle^{-n_i} = 1$ if $a_i \neq 0$. If $b_{ij} \neq 0$ then $c_i^* c_j^* = 1$, hence $\langle c_i^*, c_i\rangle$ is a primitive n_ith root of unity, so that $X_j X_i^{n_i} = X_i^{n_i} X_j$. A similar argument works for $i > j$. Thus, $X_i^{n_i}$ is a central element of A_t if $a_i \neq 0$. It follows that

$$X_j(X_i^{n_i} - a_i(c_i^{n_i} - 1)) = \langle c_j^*, c_i\rangle^{n_i}(X_i^{n_i} - a_i(c_i^{n_i} - 1))X_j,$$

so that $J(a)$ is equal to the left ideal generated by $\{X_j^{n_j} - a_j(c_i^{n_j} - 1) | 1 \le j \le t\}$, and A_t is a free left module with basis $\{X^p | 0 \le p_j \le n_j - 1\}$ over the subalgebra B generated by C and $X_1^{n_1}, \ldots, X_t^{n_t}$. We now show that no non-zero linear combination of elements of the form gX^p, $p \in \mathbf{N}^t$, $0 \le p_j \le n_j - 1$ lies in $J(a)$. Otherwise there exist $f_j \in A_t$, not all zero, such that

$$\sum_{1 \le j \le t} (X_j^{n_j} - a_j(c_j^{n_j} - 1))f_j = \sum \alpha_{g,p} g X^p,$$

where in the second sum $g \in C$, $p \in \mathbf{N}^t$, $0 \le p_j \le n_j - 1$. Since A_t is a free left B-module with basis $\{X^p | 0 \le p_j \le n_j - 1\}$, each f_j can be expressed in terms of this basis, and we find that $\sum_{1 \le j \le t}(X_j^{n_j} - a_j(c_j^{n_j} - 1))F_j \in kC - \{0\}$ for some $F_j \in B$. Now, B is isomorphic to the algebra R obtained from kC by a sequence of Ore extensions with zero derivations in the indeterminates $Y_i = X_i^{n_i}$, so that $Y_i g = \langle c_i^{*n_i}, g\rangle g Y_i$ and $Y_j Y_i = \langle c_j^{*n_j}, c_i^{n_i}\rangle Y_i Y_j$. Thus, we have

$$\sum_{1 \le j \le t} (Y_j - a_j(c_j^{n_j} - 1))G_j \in kC - \{0\}$$

for some $G_j \in R$. It follows from Lemma 1.1 by induction on the number of indeterminates that there exists a kC-algebra homomorphism $\theta: R \to kC$ such that $\theta(Y_j) = c_j^{n_j} - 1$ if $a_j \neq 0$ and $\theta(Y_j) = 0$ otherwise. Then $\theta(\sum_{1\leq j\leq t}(Y_j - a_j(c_j^{n_j} - 1))G_j) = 0$, a contradiction. ∎

From now on, we assume that $n_j \geq 2$, $q_j = \langle c_j^*, c_j \rangle$ is a primitive n_jth root of 1, and $c_j^{*n_j} = 1$ whenever $a_j \neq 0$, and we study the new Hopf algebra $H = A_t/J(a)$. We have shown that the following defines a Hopf algebra structure on H.

DEFINITION 1.11. Let $t \geq 1$, C a finitely generated abelian group, $n \in \mathbf{N}^t$, $c = (c_j) \in C^t$, $c^* = (c_j^*) \in C^{*t}$, $a \in \{0,1\}^t$, $b = (b_{ij})_{1\leq i<j\leq t}$ as above. Define $H = A_t/J(a) = H(C, n, c, c^*, a, b)$ to be the Hopf algebra generated by the commuting grouplike elements $g \in C$, and the $(1, c_j)$-primitives x_j, $1 \leq j \leq t$, where, as well,

(i) $x_j g = \langle c_j^*, g \rangle g x_j$;

(ii) $x_j^{n_j} = a_j(c_j^{n_j} - 1)$;

(iii) $x_k x_j = \langle c_k^*, c_j \rangle x_j x_k + b_{jk}(c_j c_k - 1)$ for $1 \leq j < k \leq t$;

(iv) $\langle c_j^*, c_k \rangle \langle c_k^*, c_j \rangle = 1$ for $j \neq k$; $\langle c_j^*, c_j \rangle$ is a primitive n_jth root of unity;

(v) $a_j = 0$ whenever $c_j^{n_j} = 1$; if $a_j \neq 0$, $c_j^{*n_j} = 1$;

(vi) $b_{ij} = 0$ if $c_i c_j = 1$; if $b_{ij} \neq 0$, $c_i^* c_j^* = 1$.

REMARK 1.12. (i) If $a_i = 0$ for all i, we write $a = 0$. Similarly if $b_{ij} = 0$ for all $i < j$, we write $b = 0$. If $t = 1$ so that no non-zero derivation occurs, we also write $b = 0$.

(ii) If $a = 0$ and $b = 0$, then we write $H = H(C, n, c, c^*)$ instead of $H(C, n, c, c^*, 0, 0)$.

(iii) If in Definition 1.11, the a_i's were arbitrary elements of k, then a simple change of variables would reduce to the case where the a_i's are 0 or 1.

REMARK 1.13. In order to construct $H(C, n, c, c^*, a, b)$, it suffices to have c^* and c such that $\langle c_i^*, c_i \rangle$ is a root of unity not equal to 1, and $\langle c_i^*, c_j \rangle \langle c_j^*, c_i \rangle = 1$ for $i \neq j$. Then n_i is the order of $\langle c_i^*, c_i \rangle$, and we choose a and b such that $a_i = 0$ whenever $c_i^{n_i} = 1$, $a_i = 0$ whenever $c_i^{*n_i} \neq 1$, $b_{ij} = 0$ whenever $c_i c_j = 1$, and $b_{ij} = 0$ whenever $c_j^* c_i^* \neq 1$. The remaining a_i's and b_{ij}'s are arbitrary. ∎

By Proposition 1.10, $\{g x^p | g \in C,\ p \in \mathbf{N}^t,\ 0 \leq p_j \leq n_j - 1\}$ is a basis for H. As in Eq. (1.6), comultiplication on a general basis element is given by

$$\triangle(gx^p) = \sum_d \alpha_d g c_1^{d_1} c_2^{d_2} \cdots c_t^{d_t} x^{p-d} \otimes gx^d, \tag{1.14}$$

where $d = (d_1, \ldots, d_t) \in \mathbf{Z}^t$ with $0 \le d_j \le p_j$. Here the scalars α_d are non-zero products of q_j-binomial coefficients and powers of $\langle c_j^*, c_i \rangle$.

In particular, for $k \in \mathbf{Z}^+$,

$$\triangle(x_j^k) = \sum_{0 \le d \le k} \binom{k}{d}_{q_j} c_j^d x_j^{k-d} \otimes x_j^d. \tag{1.15}$$

PROPOSITION 1.16. *$H = H(C, n, c, c^*, a, b)$ has the following properties.*

(i) *H is pointed and the $(r+1)$th term in the coradical filtration of H is $H_r = \langle gx^p | g \in C,\ p \in \mathbf{N}^t,\ p_1 + \cdots + p_t \le r \rangle$. $H = H_n$ where $n = n_1 + \cdots + n_t - t$ so that the coradical filtration has $n_1 + \cdots + n_t - t + 1$ terms.*

(ii) *The elements gx_j form a k-basis for P', where $H_1 = H_0 \oplus P'$ as in Section 0. Thus $P_{1,g} = k(g-1)$ unless $g = c_j$ for some $1 \le j \le t$. In particular, if C is finite, the k-dimension of P' is mt where m is the order of C.*

Proof. The proof of (i) is similar to the proof of Proposition 1.8. The second part follows from the fact that the α_d are non-zero. Statement (ii) follows from the coradical filtration. ∎

Unlike A_t, the Hopf algebra H is co-Frobenius. We compute the left and right integrals in H^* explicitly. For $g \in C$, and $w = (w_1, \ldots, w_t) \in \mathbf{Z}^t$, let $E_{g,w} \in H^*$ be the map taking gx^w to 1 and all other basis elements to 0.

PROPOSITION 1.17. *The Hopf algebra $H = H(C, n, c, c^*, a, b)$ is co-Frobenius. The space of left integrals in H^* is $kE_{l,n-1}$, where $l = c_1^{1-n_1} c_2^{1-n_2} \cdots c_t^{1-n_t} = \prod_{j=1}^t c_j^{-(n_j-1)}$, and where $n-1$ is the t-tuple $(n_1 - 1, \ldots, n_t - 1)$. The space of right integrals for H is $kE_{1,n-1}$ where 1 denotes the identity in C.*

Proof. We show that $E_{l,n-1}$ is a left integral by evaluating $h^*E_{l,n-1}$ for $h^* \in H^*$. This is non-zero only on elements $z \otimes lx^{n-1}$ and such an element can only occur as a summand in $\triangle(\prod_{j=1}^t (c_j^{-1}x_j)^{n_j-1}) = \triangle(\gamma l x^{n-1})$ where $\gamma \in k^*$. Now $h^*E_{l,n-1}(lx^{n-1}) = h^*(1)E_{l,n-1}(lx^{n-1})$.

Similarly $x^{n-1} \otimes z$ only occurs in $\triangle(x^{n-1})$. Since $\triangle(x^{n-1}) = x^{n-1} \otimes 1 + \cdots$, thus $E_{1,n-1}h^* = E_{1,n-1}h^*(1)$. ∎

COROLLARY 1.18. *H is unimodular, i.e., the spaces of left and right integrals in H^* coincide, if and only if $l = 1$.* ∎

If G is a group and $g \in G^t$, we write g^{-1} to denote the t-tuple $(g_1^{-1}, \ldots, g_t^{-1})$.

EXAMPLE 1.19. (i) If $H = H(C, n, c, c^*, a, b)$ then H^{op} and H^{cop} are also of this type. Indeed, $H^{\mathrm{op}} \cong H(C, n, c, c^{*-1}, a, b')$, where $b'_{ij} = -\langle c_j^*, c_i\rangle b_{ij}$ for $i < j$.

Also $H^{\mathrm{cop}} \cong H(C, n, c^{-1}, c^*, a, b'')$; the isomorphism is given by the map f taking g to g and x_j to $z_j = -c_j^{-1}x_j$. Then z_j is a $(1, c_j^{-1})$-primitive and, using the fact that $(-1)^{n_j} q_j^{-n_j(n_j-1)/2} = -1$ where q_j is a primitive n_jth root of 1, we see that its n_jth power is either 0 or $c_j^{-n_j} - 1$. The last parameter, b'', is given by $b''_{ij} = - < c_j^*, c_i > b_{ij}$ for $i < j$.

(ii) In particular if $H = H(C, n, c, c^*)$ then

$$H^{\mathrm{op}} \cong H(C, n, c, c^{*-1}) \qquad \text{and } H^{\mathrm{cop}} \cong H(C, n, c^{-1}, c^*).$$

(iii) The Hopf algebras $H_{n,q,N,\nu}$ and $H_{(N,\nu,\omega)}$ defined in [18, 5.1] are of this type. In particular for N, n positive integers with $n|N, 1 \leq \nu < N$, q a primitive nth root of 1 and $r = |q^\nu| = \frac{n}{(n,\nu)}$, the Hopf algebra $H_{n,q,N,\nu}$ is, in our notation, $H(C_N, r = \frac{n}{(n,\nu)}, c^\nu, c^*)^{\mathrm{cop}} = H(C_N, r, c^{-\nu}, c^*)$, where $t = 1$, $C_N = \langle c\rangle$ is cyclic of order N and $\langle c^*, c\rangle = q$. The Hopf algebras $H_{(N,\nu,\omega)}$ are the $H_{n,q,N,\nu}$ which are self dual; if ω is a primitive Nth root of 1, then $q = \omega^d$ and, as Corollary 4.6 will show, we may take $d = \nu$. In our notation, $H_{(N,\nu,\omega)} = H(C_N, r = \frac{n}{(n,\nu)} = N/(N, \nu^2), c^\nu, c^{*\nu})$, where $\langle c^*, c\rangle = \omega$. The Taft Hopf algebras of dimension n^2, including Sweedler's four dimensional example, are of this form.

(iv) The Hopf algebras defined in [22] to show that for a unimodular Hopf algebra, the square of the antipode need not be inner, are also Ore extension Hopf algebras. $C = \langle g_1\rangle \times \cdots \times \langle g_s\rangle$ where $\langle g_i\rangle$ has order m_i, and $m_i = n_i$. Also the gcd of $m_1, \ldots, m_t$ is greater than 1 and for l a divisor of gcd $(m_1, \ldots, m_t)$, ω is a primitive lth root of 1. For each i, η_i is a primitive m_ith root of 1. Let $c_j = (1, \ldots, 1, g_j^{-1}, 1, \ldots, 1) \in C$ and $c_j^* \in C^*$ be defined by

$$\langle c_j^*, (1, \ldots, 1, g_i, 1, \ldots, 1)\rangle = \begin{cases} \omega^{-1} & \text{if } i < j \\ \eta_j & \text{if } i = j \\ \omega & \text{if } i > j. \end{cases}$$

Then the Hopf algebra B defined in [22] is $H(C, m, c^*, c)$.

(v) The infinite dimensional non-unimodular co-Frobenius Hopf algebra defined in [23, 5.6] is also an Ore extension Hopf algebra. Here $C = \langle a\rangle$ is cyclic of infinite order and there is one indeterminate b with $\triangle(b) = a \otimes b + b \otimes a^{-1}, b^n = 0$. Also λ is a primitive $(2n)$th root of 1 and $ba = \lambda^{-1}ab$. It is straightforward to check that the Hopf algebra A generated by a and b is isomorphic to $H(C, n, a^2, a^*)$ where $\langle a^*, a\rangle = \sqrt{\lambda}$.

(vi) Let $C = C_1 \times \cdots \times C_s$ be an abelian group of order p^{n-1}, with $C_i = \langle g_i \rangle$ of order m_i, $\lambda_i \in k$ an m_ith root of 1, and $c = \prod_{i=1}^s g_i^{r_i}$ where the r_i are such that $\lambda = \prod_{i=1}^s \lambda_i^{r_i}$ is a primitive pth root of 1. Let $c^* \in C^*$ be defined by $\langle c^*, g_i \rangle = \lambda_i^{-1}$. If $c^p \neq 1$, then $H(C, p, c, c^*, 1, 0)$ is the Hopf algebra with generators $g_1, \ldots, g_s, x$, subject to relations

$$g_i^{m_i} = 1, \qquad xg_i = \lambda_i^{-1} g_i x, \qquad x^p = c^p - 1$$

$$\triangle(g_i) = g_i \otimes g_i, \qquad \triangle(x) = c \otimes x + x \otimes 1, \qquad \epsilon(g_i) = 1, \qquad \epsilon(x) = 0.$$

This Hopf algebra was useful in [4] for the classification of pointed Hopf algebras of dimension p^n with abelian coradical of dimension p^{n-1}.

(vii) Let $C = C_2 = \langle c \rangle$, the cyclic group of order 2, $\langle c_j^*, c \rangle = -1$, and $c_j = c$ for all $1 \leq j \leq t$. Then $H(C, n, c, c^*)$ is the Hopf algebra with generators $c, x_1, \ldots, x_t$ subject to relations

$$c^2 = 1, \qquad x_i^2 = 0, \qquad x_i c = -c x_i, \qquad x_j x_i = -x_i x_j,$$

$$\Delta(c) = c \otimes c, \qquad \Delta(x_i) = c \otimes x_i + x_i \otimes 1.$$

It is proved in [8] that this is the only type of Hopf algebra of dimension 2^{t+1} with coradical kC_2.

(viii) For $t = 2$, the Hopf algebra $U_{(N,\nu,\omega)}$ constructed in [18, 5.2] and studied in [18] and [9] is exactly $H(C_N = \langle g \rangle, r, c = (g^\nu, g^\nu), c^*, 0, b)^{\mathrm{cop}}$, where $b_{12} = 1$. The character c_i^* is defined by $\langle c_1^*, g \rangle = q$, $\langle c_2^*, g \rangle = q^{-1}$. Here $\nu \in \mathbf{Z}$, $1 \leq \nu < N$ with ν^2 not divisible by N. For $\omega \in k$ a primitive Nth root of 1, $q = \omega^\nu$ and r is the order of $q^\nu = \omega^{\nu^2}$.

We end this section by showing that our assumption that the derivations are zero on kC is not unreasonable. Assume that φ is an algebra automorphism of kC of the form $\varphi(g) = \langle c^*, g \rangle g$ for $g \in C$, as usual. Give the Ore extension $(kC)[X, \varphi]$ a Hopf algebra structure such that X is a $(1, c)$-primitive as in the beginning of this section.

PROPOSITION 1.20. *Assume that $\langle c^*, g \rangle \neq 1$ if $g \in C$ has infinite order. If δ is a φ-derivation of kC such that the Ore extension $(kC)[Y, \varphi, \delta]$ has a Hopf algebra structure extending that of kC with Y a $(1, c)$-primitive, then there is a Hopf algebra isomorphism $(kC)[Y, \varphi, \delta] \simeq (kC)[X, \varphi]$.*

Proof. Let $U = \{g \in C | \langle c^*, g \rangle \neq 1\}$ and $V = \{g \in C | \langle c^*, g \rangle = 1\}$. Thus, if $g \in V$ then by our assumption g has finite order. In this case, $\varphi(g^n) = g^n$ for all n, and induction on $n \geq 1$ shows that $\delta(g^n) = ng^{n-1}\delta(g)$. Then $\delta(1) = mg^{-1}\delta(g)$, where m is the order of g, and $\delta(1) = 0$ imply that $\delta(g) = 0$.

Now let $g \in U$. Applying Δ to the relation $Yg = \langle c^*, g \rangle gY + \delta(g)$, we find that $\Delta(\delta(g)) = cg \otimes \delta(g) + \delta(g) \otimes g$. Thus $\delta(g)$ is a (g, cg)-primitive, and so $\delta(g) = \alpha_g g(c-1)$ for some scalar α_g.

Therefore, for any two elements g and h of U

$$\delta(gh) = \delta(g)h + \varphi(g)\delta(h) = \alpha_g g(c-1)h + \langle c^*, g\rangle g\alpha_h h(c-1)$$
$$= (\alpha_g + \alpha_h\langle c^*, g\rangle)(c-1)gh,$$

and similarly

$$\delta(hg) = (\alpha_h + \alpha_g\langle c^*, h\rangle)(c-1)gh.$$

Since C is abelian $\alpha_g + \alpha_h\langle c^*, g\rangle = \alpha_h + \alpha_g\langle c^*, h\rangle$, or $\alpha_g/(1-\langle c^*, g\rangle) = \alpha_h/(1-\langle c^*, h\rangle)$. Denote by γ the common value of the $\alpha_g/(1-\langle c^*, g\rangle)$ for $g \in U$. We have $\alpha_g - \gamma + \langle c^*, g\rangle\gamma = 0$.

Let $Z = Y - \gamma(c-1)$. For any $g \in U$ we have that

$$Zg = Yg - \gamma(c-1)g = \langle c^*, g\rangle gY + \alpha_g g(c-1) - \gamma(c-1)g$$
$$= \langle c^*, g\rangle gZ + \langle c^*, g\rangle \gamma g(c-1) + \alpha_g g(c-1) - \gamma g(c-1)$$
$$= \langle c^*, g\rangle gZ + (\alpha_g - \gamma + \langle c^*, g\rangle\gamma)g(c-1) = \langle c^*, g\rangle gZ.$$

Obviously, $Zg = gZ$ if $g \in V$, and thus we have proved that $(kC)[Y, \varphi, \delta] \cong (kC)[Z, \varphi]$ as algebras. Since Z is clearly a $(1, c)$-primitive, this is also a coalgebra morphism and the proof is complete.

2. CLASSIFICATION RESULTS

We first classify Hopf algebras of the form $H(C, n, c, c^*, a, 0)$, i.e., they are constructed as in Sect. 1 by using Ore extensions with zero derivations. Suppose $H = H(C, n, c, c^*, a, 0) \simeq H' = H(C', n', c', c^{*'}, a', 0)$ and write g, x_i (g', x_i') for the generators of H (H', respectively). Let f be a Hopf algebra isomorphism from H to H'. Since the coradicals must be isomorphic, we may assume that $C = C'$, and the Hopf algebra isomorphism induces an automorphism of C. Also by Proposition 1.16, $t = t'$. If π is a permutation of $\{1, \ldots, t\}$ and $v \in \mathbf{Z}^t$, we write $\pi(v)$ to denote $(v_{\pi(1)}, \ldots, v_{\pi(t)})$.

THEOREM 2.1. *Let $H = H(C, n, c, c^*, a, 0)$ and $H' = H(C', n', c', c^{*'}, a', 0)$ be Hopf algebras as described above. Then $H \cong H'$ if and only if $C = C'$, $t = t'$ and there is an automorphism f of C and a permutation π of $\{1, \ldots, t\}$ such that for $1 \le i \le t$*

$$n_i = n'_{\pi(i)},\ f(c_i) = c'_{\pi(i)},\ c_i^* = c^{*'}_{\pi(i)} \circ f, \text{ and } a_i = a'_{\pi(i)}.$$

Proof. Let $I = \{i | 1 \le i \le t,\ c_i = c_1, c_i^* = c_1^*\}$ and let

$$\tilde{J} = \{j | 1 \le j \le t,\ c'_j = f(c_1)\} \supseteq J = \{j | 1 \le j \le t,\ j \in \tilde{J}, c_j^{*'} \circ f = c_1^*\}.$$

Note that since $\langle c_i^*, c_i\rangle$ is a primitive n_ith root of 1 and for $i \in I$, $\langle c_i^*, c_i\rangle = \langle c_1^*, c_1\rangle$, then $n_i = n_1$ for $i \in I$. Similarly, since for $j \in J$, $\langle c_j^{*'}, c_j'\rangle = \langle c_j^{*'}, f(c_1)\rangle = \langle c_1^*, c_1\rangle$, $n_j' = n_1$ for $j \in J$. Let L be the Hopf subalgebra of H generated by C and $\{x_i | i \in I\}$ and L' the Hopf subalgebra of H' generated by C and $\{x_j' | j \in J\}$.

Since x_1 is a $(1, c_1)$-primitive, $f(x_1)$ is a $(1, f(c_1)$-primitive and so

$$f(x_1) = \alpha_0(f(c_1) - 1) + \sum_{i=1}^{r} \alpha_i x_{j_i}' \qquad \text{with } \alpha_i \in k, j_i \in \tilde{J}.$$

Then, since $gx_1 = \langle c_1^*, g\rangle^{-1} x_1 g$ for all $g \in C$, we see that $\alpha_0 = 0$, and

$$\sum_{i=1}^{r} f(g)\alpha_i x_{j_i}' = \sum_{i=1}^{r} \alpha_i \langle c_1^*, g\rangle^{-1} x_{j_i}' f(g) = \sum_{i=1}^{r} \alpha_i \langle c_1^*, g\rangle^{-1} \langle c_{j_i}^{*'}, f(g)\rangle f(g) x_{j_i}'$$

and thus $\alpha_i = 0$ for any i for which $c_1^* \neq c_{j_i}^{*'} \circ f$. Thus $f(L) \subseteq L'$. The same argument using f^{-1} shows that $f^{-1}(L') \subseteq L$ and so $f(L) = L'$.

If $L \neq H$, we repeat the argument for M, the Hopf subalgebra of H generated by C and the set $\{x_i : c_i = c_k, c_i^* = c_k^*\}$ where x_k is the first element in the list $x_2, \ldots, x_t$ which is not in L. Continuing in this way, we see that there exists a permutation σ such that

$$n_i = n_{\sigma(i)}', f(c_i) = c_{\sigma(i)}, c_i^* = c_{\sigma(i)}^{*'} \circ f.$$

It remains to find π such that $a_i = a_{\pi(i)}'$. First suppose $n_1 > 2$. Then $I = \{1\}$. For if $k \in I$, $k \neq 1$, then $\langle c_1^*, c_1\rangle = \langle c_k^*, c_1\rangle = \langle c_1^*, c_k\rangle^{-1} = \langle c_1^*, c_1\rangle^{-1}$ and $\langle c_1^*, c_1\rangle^2 = 1$, a contradiction. Similarly $J = \{\sigma(1)\}$. Hence $f(x_1) = \alpha x_{\sigma(1)}'$ for some non-zero scalar α, and the relation $x_1^{n_1} = a_1(c_1^{n_1} - 1)$ implies $\alpha^{n_1} {x'}_{\sigma(1)}^{n_{\sigma(1)}'} = a_1(c_{\sigma(1)}^{\prime n_1} - 1)$, so that $a_{\sigma(1)}' = a_1$.

Next suppose $n_1 = 2$. Let $I_1 = \{i \in I | a_i = 1\}$ and $J_1 = \{j \in J | a_{j'} = 1\}$. For any $i \in I$, there exist $\alpha_{ij} \in k$ such that $f(x_i) = \sum_{j \in J} \alpha_{ij} x_j'$. As above, for all $i \in I$, $\langle c_i^*, c_i\rangle = -1$ (for all $j \in J$, $\langle c_j^{*'}, c_j'\rangle = -1$) and thus the x_i (respectively, the x_j') anticommute. If $i \in I_1$, f applied to $x_i^2 = c_1^2 - 1$ yields $\sum_{j \in J_1} \alpha_{ij}^2 = 1$. On the other hand, comparing $f(x_i x_k)$ and $f(x_k x_i)$ for $i, k \in I_1$, $i \neq k$, we see that

$$\sum_{j \in J_1} \alpha_{ij}\alpha_{kj} = -\sum_{j \in J_1} \alpha_{kj}\alpha_{ij}$$

and thus $\sum_{j \in J_1} \alpha_{ij}\alpha_{kj} = 0$.

This implies that the vectors $B_i \in k^{J_1}$, defined by $B_i = (\alpha_{ij})_{j \in J_1}$ for $i \in I_1$, form an orthonormal set in k^{J_1} under the ordinary dot product. Thus the space k^{J_1} contains at least $|I_1|$ independent vectors and so $|J_1| \geq |I_1|$. The reverse inequality is proved similarly. Now define π to be a refinement of the permutation σ such that for $i \in I_1$, $\pi(i) \in J_1$ and then $a_i = a'_{\pi(i)}$ for all $i \in I$.

Conversely, let f be an automorphism of C and let π be a permutation of $\{1, 2, \ldots, t\}$ such that for all $1 \leq i \leq t$,

$$n_i = n'_{\pi(i)}, f(c_i) = c'_{\pi(i)}, c_i^* = c^{*'}_{\pi(i)} \circ f, \text{ and } a_i = a'_{\pi(i)}.$$

Extend f to a Hopf algebra isomorphism from H to H' by $f(x_i) = x'_{\pi(i)}$. If we note that

$$\langle c^{*'}_{\pi(i)}, c'_{\pi(j)} \rangle = \langle c^{*'}_{\pi(i)}, f(c_j) \rangle = \langle c_i^*, c_j \rangle,$$

the rest of the verification that f induces a Hopf algebra isomorphism is straightforward. ∎

Note that in the proof above, it was shown that if $n_k > 2$, then $|I| = |J| = 1$ where $I = \{i | 1 \leq i \leq t,\ c_i = c_k, c_i^* = c_k^*\}$ and $J = \{j | 1 \leq j \leq t,\ c'_j = f(c_k), c^{*'}_j \circ f = c_k^*\}$. Thus we can also classify Hopf algebras of the form $H(C, n, c, c^*, a, b)$ if all $n_i > 2$. We revisit the case where some $n_i = 2$ in Sect. 5.

THEOREM 2.2. *Let $H = H(C, n, c, c^*, a, b)$ and $H' = H(C', n', c', c^{*'}, a', b')$ be such that all n_i and $n'_i > 2$. Then $H \cong H'$ if and only if $C = C'$, $t = t'$ and there is an automorphism f of C, non-zero scalars $(\alpha_i)_{1 \leq i \leq t}$, and a permutation π of $\{1, \ldots, t\}$ such that*

$$n_i = n'_{\pi(i)}, f(c_i) = c'_{\pi(i)}, c_i^* = c^{*'}_{\pi(i)} \circ f, \text{ and } a_i = a'_{\pi(i)},$$

$\alpha_i^{n_i} = 1$ for any i such that $a_i = 1$, and for any $1 \leq i < j \leq t$,

$$b_{ij} = \alpha_i \alpha_j b'_{\pi(i)\pi(j)} \text{ if } \pi(i) < \pi(j)$$
$$\text{and } < c_i^*, c_j > b_{ij} = -\alpha_i \alpha_j b'_{\pi(j)\pi(i)} \text{ if } \pi(j) < \pi(i).$$

Proof. The argument is similar to that in Theorem 2.1. An application of the isomorphism f to the equation $x_j x_i = \langle c_j^*, c_i \rangle x_i x_j + b_{ij}(c_i c_j - 1)$, $i < j$, yields the relationship between b and b'. ∎

The following corollary answers in the negative to Kaplansky's tenth conjecture on Hopf algebras [11].

COROLLARY 2.3. *Suppose that $C, c \in C^t, c^* \in C^{*t}$, are such that $\langle c_j^*, c_l \rangle = \langle c_l^*, c_j \rangle^{-1}$ if $l \neq j$, $\langle c_i^*, c_i \rangle$ is a primitive root of unity of order $n_i > 2$, and there exist $i < j$ such that $c_i^{*n_i} = c_j^{*n_j} = 1$, $c_i^{n_i} \neq 1, c_j^{n_j} \neq 1$, $c_i c_j \neq 1$, and $c_i^* c_j^* = 1$. Then for any a with $a_i = a_j = 1$ and satisfying the conditions of Remark 1.13, there exist infinitely many non-isomorphic Hopf algebras of the form $H(C, n, c, c^*, a, b)$.*

Proof. Let b and b' be such that $H = H(C, n, c, c^*, a, b)$ and $H' = H(C, n, c, c^*, a, b')$ are well defined. By Remark 1.13, infinitely many such b and b' exist. If $f: H \to H'$ is a Hopf algebra isomorphism, then the permutation π in Theorem 2.2 is the identity and thus $b_{ij} = \alpha_i \alpha_j b'_{ij}$ for some n_ith and n_jth roots of unity α_i and α_j. Since there exist only finitely many such roots, and k is infinite, the result follows. ∎

EXAMPLE 2.4. To find a concrete example of a class consisting of infinitely many types of Hopf algebras of the same finite dimension, we need some data (C, c, c^*) as in Corollary 2.3, with C finite. The simplest such data are the following.

(i) Let p be an odd prime, and ρ a primitive p-th root of 1. Take $C = C_{p^2} = \langle g \rangle$, the cyclic group of order p^2, $t = 2$, $c = (g, g)$, $c^* = (g^*, g^{*-1})$ where $\langle g^*, g \rangle = \rho$ and $a = (1, 1)$. Then $n_1 = n_2 = p$ and by Corollary 2.3, $H(C, n, c, c^*, a, b) \cong H(C, n, c, c^*, a, b')$ if and only if $b_{12} = \gamma b'_{12}$ for γ a primitive pth root of 1. Thus there are infinitely many types of Hopf algebras of dimension p^4. This is the example from [5].

(ii) Let $C = C_{pq} = \langle g \rangle$, the cyclic group of order pq where p is an odd prime, $q > 1$, and $t = 2$, $c = (g, g)$, $c^* = (g^*, g^{*-1})$ where $\langle g^*, g \rangle = \rho$, ρ a primitive pth root of 1. Let $a_1 = a_2 = 1$. Then again $n_1 = n_2 = p$, and as in (i), there are infinitely many types of Hopf algebras $H(C, n, c, c^*, a, b)$ of dimension $p^3 q$.

We end this section by demonstrating that in case some of the n_i's are equal to 2, a classification result like Theorem 2.2 does not hold. Although clearly the partial data (C, n, c, c^*) and $(C', n', c', c^{*'})$ are related as in the proof of Theorem 2.1, $a \in \{0, 1\}^t$ and $a' \in \{0, 1\}^t$ may contain different numbers of 0's and 1's. To see this we cite the following examples from [6].

EXAMPLE 2.5. (i) Let $C = C_4 = \langle g \rangle, t = 2, n = (2, 2), c = (g, g), c^* = (g^*, g^*)$ where $\langle g^*, g \rangle = -1, b_{12} = 1, a = (1, 1), a' = (0, 1)$. Then there exists a Hopf algebra isomorphism $f : H(C, n, c, c^*, a, b) \to H(C, n, c, c^*, a', b)$ defined by $f(g) = g, f(x_1) = -(\beta^2 + \beta)x'_1 + \beta x'_2, f(x_2) = x'_2$, where $\beta \in k$ is a primitive cube root of -1.

(ii) Let $C = C_4 = \langle g \rangle, t = 2, n = (2, 2), c = (g, g), c^* = (g^*, g^*)$ where $\langle g^*, g \rangle = -1, a = (1, 1)$ and $b_{12} = 2, a' = (0, 1), b'_{12} = 0$. Then the map f from $H(C, n, c, c^*, a', b')$ to $H(C, n, c, c^*, a, b)$ defined by $f(g) = g, f(x_1) = x_2, f(x_2) = x_1 - x_2$, is a Hopf algebra isomorphism. Note that one of the Hopf algebras is an extension with non-trivial derivation while the other is an extension with trivial derivation.

3. DUALS

In this section, we study the duals of the Hopf algebras $H(C, n, c^*, c)$ for C finite. Suppose $C = C_1 \times C_2 \times \cdots \times C_s =< g_1 > \times \cdots \times \langle g_s \rangle$ where C_i is cyclic of order m_i. For $i = 1, \ldots, s$, let $\zeta_i \in k^*$ be a primitive m_ith root of 1. The dual $C^* = \langle g_1^* \rangle \times \cdots \times \langle g_s^* \rangle$, where $\langle g_i^*, g_i \rangle = \zeta_i$ and $\langle g_i^*, g_j \rangle = 1$ for $i \neq j$ is then isomorphic to C. We identify C and C^{**} using the natural isomorphism $C \cong C^{**}$ where $\langle g^{**}, g^* \rangle = \langle g^*, g \rangle$.

Now we show that for C finite, the dual of a Hopf algebra $H = H(C, n, c, c^*)$ constructed via t Ore extensions with zero derivation and with all indeterminates nilpotent, is again an "Ore extension Hopf algebra," and that there is a very natural relationship between H and H^*.

THEOREM 3.1. $H(C, n, c, c^*)^* \cong H(C^*, n, c^*, c)$.

Proof. First we determine the grouplikes in H^*. Let $g_i^* \in H^*$ be the algebra map defined by $g_i^*(g_j) = \langle g_i^*, g_j \rangle$ and $g_i^*(x_j) = 0$ for all i, j. Since the g_i^* are algebra maps from H to k, H^* contains a group of grouplikes generated by the g_i^*, and so isomorphic to C^*.

Now, let $y_j \in H^*$ be defined by $y_j(gx_j) = \langle c_j^{*-1}, g \rangle$, and $y_j(gx^w) = 0$ for $x^w \neq x_j$.

We determine the nilpotency degree of y_j. Clearly y_j^r is non-zero only on basis elements gx_j^r. Note that by (1.15) and the fact that $q_j = \langle c_j^*, c_j \rangle$,

$$\begin{aligned} y_j^2(gx_j^2) &= (y_j \otimes y_j)[\binom{2}{1}_{q_j} gc_j x_j \otimes gx_j] \\ &= \binom{2}{1}_{q_j} \langle c_j^*, g^2 c_j \rangle^{-1} \\ &= (1 + q_j) q_j^{-1} \langle c_j^*, g^2 \rangle^{-1} \\ &= (1 + q_j^{-1}) \langle c_j^*, g^2 \rangle^{-1}. \end{aligned}$$

By induction, using the fact that $\binom{r}{1}_{q_j} = (1 + q_j + \cdots + q_j^{r-1})$, we see that for $\eta_j = q_j^{-1}$,

$$y_j^r(gx_j^r) = (1 + \eta_j) \cdots (1 + \eta_j + \cdots + \eta_j^{r-1}) \langle c_j^*, g^r \rangle^{-1}.$$

Since q_j, and thus η_j, is a primitive n_j-th root of 1, this expression is 0 if and only if $r = n_j$. Thus the nilpotency degree of y_j is n_j.

Let $g^* \in H^*$ be an element of the group of grouplikes generated by the g_i^* above. We check how the y_j multiply with g^* and with each other. Clearly, both $y_j g^*$ and $g^* y_j$ are non-zero only on basis elements gx_j. We compute

$$g^* y_j(gx_j) = g^*(gc_j) y_j(gx_j) = \langle g^*, g \rangle \langle g^*, c_j \rangle \langle c_j^*, g \rangle^{-1}$$

and

$$y_j g^*(gx_j) = y_j(gx_j)g^*(g) = \langle c_j^*, g\rangle^{-1}\langle g^*, g\rangle$$

so that

$$g^* y_j = \langle g^*, c_j\rangle y_j g^*, \text{ or } y_j g^* = \langle c_j^{**-1}, g^*\rangle g^* y_j.$$

Let $j < k$. Then $y_j y_k$ and $y_k y_j$ are both non-zero only on basis elements $gx_k x_j = \langle c_k^*, c_j\rangle gx_j x_k$. We compute

$$y_k y_j(gx_k x_j) = y_k(gx_k)y_j(gx_j) = \langle c_k^{*-1}, g\rangle\langle c_j^{*-1}, g\rangle$$

and

$$y_j y_k(\langle c_k^*, c_j\rangle gx_j x_k) = \langle c_k^*, c_j\rangle y_j(gx_j)y_k(gx_k) = \langle c_k^*, c_j\rangle\langle c_j^{*-1}, g\rangle\langle c_k^{*-1}, g\rangle.$$

Therefore for $j < k$,

$$y_k y_j = \langle c_k^{*-1}, c_j\rangle y_j y_k = \langle c_j^{*-1}, c_k^{-1}\rangle y_j y_k = \langle c_k^{**-1}, c_j^{*-1}\rangle y_j y_k.$$

Finally we confirm that the elements y_j are (ϵ_H, c_j^{*-1})-primitives and then we will be done. The maps $c_j^{*-1} \otimes y_j + y_j \otimes \epsilon_H$ and $m^*(y_j)$ are both only non-zero on elements of $H \otimes H$ which are sums of elements of the form $g \otimes lx_j$ or $gx_j \otimes l$. We check

$$(c_j^{*-1} \otimes y_j + y_j \otimes \epsilon_H)(g \otimes lx_j) = (c_j^{*-1} \otimes y_j)(g \otimes lx_j) = \langle c_j^{*-1}, g\rangle\langle c_j^{*-1}, l\rangle,$$

and

$$m^*(y_j)(g \otimes lx_j) = y_j(glx_j) = \langle c_j^{*-1}, gl\rangle.$$

Similarly,

$$(c_j^{*-1} \otimes y_j + y_j \otimes \epsilon_H)(gx_j \otimes l) = y_j(gx_j) = \langle c_j^{*-1}, g\rangle,$$

and

$$y_j(gx_j l) = y_j(\langle c_j^*, l\rangle glx_j) = \langle c_j^*, l\rangle\langle c_j^{*-1}, gl\rangle = \langle c_j^{*-1}, g\rangle.$$

Thus the Hopf subalgebra of H^* generated by the g_i^*, y_j is isomorphic to $H(C^*, n, c^{*-1}, c^{-1})$ and by a dimension argument it is all of H^*. Now we only need note that for any $H = H(C, n, c, c^*)$, the group automorphism of C which maps every element to its inverse induces a Hopf algebra isomorphism from H to $H(C, n, c^{-1}, c^{*-1})$, and the statement is proved. ∎

COROLLARY 3.2. *Let $H = H(C, n, c, c^*)$ where C is a finite abelian group. Then $H \cong H^*$ if and only if there is an isomorphism $f: C \to C^*$ and a permutation $\pi \in S_t$ such that for all $1 \le j \le t$,*

$$n_{\pi(j)} = n_j,\ f(c_j) = c_{\pi(j)}^*, \qquad \langle f(c_j), g\rangle = \langle f(g), c_{\pi^2(j)}\rangle \qquad \text{for all } g \in C.$$

If we work with a general $H = H(C, n, c, c^*, a, b)$ with C finite, then the dual H^* is not necessarily an "Ore extension Hopf algebra." In [18, Proposition 11], Radford points out that the duals of the Hopf algebras $U_{(N,\nu,\omega)}$ of Example 1.19(viii) may have trivial group of grouplikes. Even if $t = 1$, the dual may not be pointed. In [17], Radford shows that in the dual, the Hopf algebra of dimension pn^2 generated by g and x with λ a pth root of 1 and

$$g^{np} = 1, \quad gx = \lambda xg, \quad x^n = g^n - 1, \quad \Delta(g) = g \otimes g,$$
$$\Delta(x) = g \otimes x + x \otimes 1,$$

the coradical is not a Hopf subalgebra.

4. ORE EXTENSIONS OVER A CYCLIC GROUP

In this section $C = \langle g \rangle$ will be a cyclic group, either of order m, or infinite cyclic. We first determine for which values of the parameters t and m, finite dimensional Hopf algebras $H = H(C_m, n, c, c^*, a, b)$ exist. By Remark 1.13, for a given t, in order to construct H, we need $c \in C_m^t$, $c^* \in (C_m^*)^t$ such that $\langle c_i^*, c_i \rangle$ is a root of unity different from 1, and $\langle c_i^*, c_j \rangle \langle c_j^*, c_i \rangle = 1$ for $i \neq j$. Let ζ be a primitive mth root of unity, and then $g^* \in C_m^*$ defined by $\langle g^*, g \rangle = \zeta$ generates C_m^*. Thus we may write $c_i = g^{u_i}$ and $c_i^* = g^{*d_i}$. To find suitable c and c^*, we require $u, d \in \mathbf{Z}^t$ with $u_i, d_i \in \mathbf{Z}$ mod m such that,

$$(d_i u_j + d_j u_i) \equiv 0 \text{ if } i \neq j \text{ and } d_i u_i \not\equiv 0. \tag{4.1}$$

Then H will be the Hopf algebra with basis $g^i x^p$, $p \in \mathbf{Z}^t$, $0 \leq p_i \leq n_i$, and $0 \leq i \leq m-1$, and such that

$$x_i^{n_i} = a_i(g^{n_i u_i} - 1), \qquad x_i g^j = \zeta^{d_i j} g^j x_i, \qquad \triangle(x_i) = g^{u_i} \otimes x_i + x_i \otimes 1$$
$$x_j x_i = \zeta^{d_j u_i} x_i x_j + b_{ij}(g^{u_i + u_j} - 1) \qquad \text{for } 1 \leq i < j \leq t.$$

PROPOSITION 4.2. *Let m be a positive integer.*

(i) *If m is even, then the system* (4.1) *has solutions for any t.*

(ii) *If m is odd, then the system* (4.1) *has solutions if and only if $t \leq 2s$, where s is the number of distinct primes dividing m.*

Proof. (i) If $m = 2r$ then $d_i = r, u_i = 1$, $1 \leq i \leq t$, is a solution of (4.1).

(ii) We first prove by induction on s that the system has solutions for $t = 2s$ and thus for any $t \le 2s$. If $s = 1$ then $d_1 = u_1 = 1 = u_2$, $d_2 = -1$ is a solution of (4.1). Now suppose the assertion holds for $s-1$ and let $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ with the p_i prime. Then $m' = m/p_s^{\alpha_s}$ has $s-1$ distinct prime divisors, so by the induction hypothesis there exist d_i', u_i' for $1 \le i \le 2s-2$, such that $(d_i'u_j' + d_j'u_i') \equiv 0 \bmod m'$ for $1 \le i \ne j \le 2s-2$ and $d_i'u_i' \not\equiv 0 \bmod m'$ for $1 \le i \le 2s-2$. Now a solution of the system for $t = 2s$ is given by $d_i = p_s^{\alpha_s} d_i'$, $u_i = p_s^{\alpha_s} u_i'$ for $1 \le i \le 2s-2$ and $d_{2s} = d_{2s-1} = u_{2s-1} = m'$, $u_{2s} = -m'$.

Next we show that for $m = p^\alpha$ and $t = 3$ the system has no solutions. Suppose $d, u \in \mathbf{Z}^3$ is a solution, and suppose $d_i = d_i' p^{\alpha_i}$, $u_i = u_i' p^{\beta_i}$ where $(d_i', p) = (u_i', p) = 1$ for $1 \le i \le 3$. For $i \ne j$, p^α divides $d_i u_j + d_j u_i = p^{\alpha_i + \beta_j} d_i' u_j' + p^{\alpha_j + \beta_i} d_j' u_i'$, and so $\alpha_i + \beta_j = \alpha_j + \beta_i$. Since p^α does not divide $d_i u_i$ for any i, then $\alpha_i + \beta_i < \alpha$, so $\alpha_i + \beta_j + \alpha_j + \beta_i < 2\alpha$ for all i, j. Thus $d_i' u_j' \equiv -d_j' u_i' \bmod p$ for all $i \ne j$. Multiplying these three congruences, we obtain $d_1' d_2' d_3' u_1' u_2' u_3' \equiv 0 \bmod p$, a contradiction.

Now suppose that $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $2s + 1 \le t$. If the system had a solution d, u, then for every i there would exist j_i, $1 \le j_i \le s$, such that $p_{j_i}^{\alpha_{j_i}}$ does not divide $d_i u_i$. By the Pigeon Hole Principle we find i_1, i_2, i_3 such that $j_{i_1} = j_{i_2} = j_{i_3}$; denote this integer by j. Then $p_j^{\alpha_j}$ does not divide any $d_k u_k$, but divides $d_k u_r + d_r u_k$ for all distinct $r, k \in \{i_1, i_2, i_3\}$, and this contradicts what we proved in the case $m = p^\alpha$. ∎

COROLLARY 4.3. (i) *If m is even, then Hopf algebras of the form $H(C_m, n, c, c^*, a, b)$ exist for every t.*

(ii) *If m is odd, then $H(C_m, n, c, c^*, a, b)$ exist for any $t \le 2s$, where s is the number of distinct prime factors of m.*

Now let $C = \langle g \rangle$ be an infinite cyclic group.

COROLLARY 4.4. *Hopf algebras $H(C, n, c, c^*, a, b)$ exist for all t.*

Proof. Let t be a positive integer and choose m such that $t \le 2s$ where s is the number of distinct prime divisors of m. Then by Proposition 4.2, there exist $d_i, u_i, 1 \le i \le t$ solutions for the system (4.1). Now let $c_i = g^{u_i}$ and $c_i^* = g^{*d_i}$ for $\langle g^*, g \rangle = \zeta$, a primitive mth root of 1, as before. ∎

The classification results presented in Sect. 2 depend upon knowledge of the automorphism group of C. In case C is cyclic, $\mathrm{Aut}(C)$ is well known, and Theorem 2.1 specializes to the following.

PROPOSITION 4.5. *If $C = \langle g \rangle$ is cyclic, then $H(C, n, c, c^*, a, 0) \cong H(C', n', c', c^{*'}, a', 0)$ if and only if $C = C'$, $t = t'$ and there is an automorphism f of C mapping g to g^h and a permutation π of $\{1, \dots, t\}$ such* *that*

$$n_i = n'_{\pi(i)},\, c_i^h = c'_{\pi(i)} \text{ i.e., } hu_i \equiv u_{\pi(i)},\, c_i^* = (c^{*'}_{\pi(i)})^h, \text{ i.e., } d_i \equiv hd_{\pi(i)}, \quad \textit{and } a_i = a'_{\pi(i)}.$$

If C is cyclic of order m, then $(h, m) = 1$; if C is infinite cyclic, then $h = 1$ or $h = -1$.

If C is cyclic, then it is easy to see when $H(C_m, n, c, c^*)$ is isomorphic to its dual, its opposite or co-opposite Hopf algebra.

COROLLARY 4.6. *Let $C = C_m = \langle g\rangle$, finite, and $H = H(C_m, n, c, c^*)$ where $c_i = g^{u_i}$, $c_i^* = (g^*)^{d_i}$ and $\langle g^*, g\rangle = \zeta$, a fixed primitive* mth *root of* 1.

(i) *$H \cong H^*$ if and only if there exist h, π as in Proposition* 4.5 *such that for all $1 \le j \le t$,*

$$n_{\pi(j)} = n_j, \qquad hu_j \equiv d_{\pi(j)} \bmod m, \qquad u_{\pi^2(j)} \equiv u_j \bmod m.$$

(ii) *$H \cong H^{\mathrm{cop}}$ if and only if there exist h, π such that for all $1 \le j \le t$,*

$$n_{\pi(j)} = n_j,\; hu_j \equiv -u_{\pi(j)} \bmod m,\; d_j \equiv hd_{\pi(j)} \bmod m.$$

(iii) *$H \cong H^{\mathrm{op}}$ if and only if there exist h, π such that for all $1 \le j \le t$,*

$$n_{\pi(j)} = n_j,\; hu_j \equiv u_{\pi(j)} \bmod m,\; d_j \equiv -hd_{\pi(j)} \bmod m.$$

Note that for $t = 1$, if $H \cong H^*$ and ζ, the fixed primitive mth root of 1 is replaced by ζ^h, then $u_i = d_i$ in the parametrization of H, i.e., $H = H(C_m, n, (g^{u_1}, \ldots, g^{u_t}), (g^{*u_1}, \ldots, g^{*u_t}))$. For k algebraically closed, Proposition 8 of [18] follows immediately. Now for such $H \cong H^*$, parts (c), (d), (e) of [18, Theorem 4] follow from the theorem above. Similarly Lemma 1.1.2 of [9] follows easily from the above discussion. For with $t = 1$, $H = H(C_m, n, g^u, g^{*d}) \cong H(C_m, n, g^u, g^{*d'})^{*\mathrm{cop}} \cong H(C_m, n, g^{-d'}, g^{*u})$ if and only if there exists h such that ζ^h is also a primitive mth root of unity and $(\zeta^h)^d = (\zeta^h)^{-d'}$.

5. ORE EXTENSIONS WITH NON-ZERO DERIVATIONS

In this section we study Hopf algebras of the form $H(C, n, c^*, c, 0, 1)$, where $b = 1$ means that $b_{ij} = 1$ for all $i < j$. Thus, the skew-primitives x_i are all nilpotent and for $i \neq k$, $x_i x_k - \langle c_i^*, c_k\rangle x_k x_i$ is a non-zero element of kC. It is easy to see that if $a = 0$ and all b_{ij} are non-zero, then a change of variables ensures that all b_{ij} equal 1. This class produces many interesting examples.

The following two definitions are particular cases of Definition 1.11.

DEFINITION 5.1. For $t = 2$, let $n \geq 2$, $c = (c_1, c_2) \in C^2$, $g^* \in C^*$ with $\langle g^*, c_1\rangle = \langle g^*, c_2\rangle$ a primitive nth root of unity, and $c_1c_2 \neq 1$. Denote the pair (n, n) by (n), and, if $c_1 = c_2 = g$, denote (c_1, c_2) by (g). Then $H(C, (n), (c_1, c_2), (g^*, g^{*-1}), 0, 1)$ denotes the Hopf algebra generated by the commuting grouplike elements $g \in C$, and the $(1, c_j)$-primitives x_j, $j = 1, 2$, with multiplication relations

$$x_j^n = 0,\ x_1g = \langle g^*, g\rangle gx_1, \qquad x_2g = \langle g^{*-1}, g\rangle gx_2,$$
$$x_2x_1 - \langle g^{*-1}, c_1\rangle x_1x_2 = c_1c_2 - 1.$$

DEFINITION 5.2. Let $t > 2$ and let $c \in C^t$, $g^* \in C^*$ such that $< g^*, c_i >= -1$ for all i and $c_ic_j \neq 1$ if $i \neq j$. We denote the t-tuple $(2, \ldots, 2)$ by (2), and the t-tuple $(g^*, \ldots, g^*)$ by (g^*). Then $H(C, (2), (c_1, \ldots, c_t), (g^*), 0, 1)$ is the Hopf algebra generated by the commuting grouplike elements $g \in C$, and the $(1, c_j)$-primitives x_j, with relations

$$x_i^2 = 0, \qquad x_ig = \langle g^*, g\rangle gx_i, \qquad x_kx_j + x_jx_k = c_kc_j - 1 \qquad \text{for } k \neq j.$$

REMARK 5.3. Note that the Hopf algebras in this section have a non-zero derivation at each step of the Ore extension construction after the first. The notation $H(C, n, c, c^*)$ of earlier sections indicates that the derivations are all zero.

In each of the examples below, the coradical is kC for a cyclic group C.

EXAMPLE 5.4. (i) Let $C_m = \langle g\rangle$ be cyclic of finite order $m \geq 2$, let n be an integer ≥ 2, and let $c_1 = g^{u_1}$, $c_2 = g^{u_2}$, $g^* \in C^*$ be such that $\langle g^*, g\rangle = \lambda$ where $\lambda^m = 1$, $u_1 + u_2 \not\equiv 0 \bmod m$, and $\lambda^{u_1} = \lambda^{u_2}$, a primitive nth root of 1. Then $H = H(C_m, (n), c, (g^*, g^{*-1}), 0, 1)$ is a Hopf algebra of dimension mn^2, with coradical kC_m and generators g, x_1, x_2 such that g is grouplike of order m, x_i is a $(1, g^{u_i})$-primitive, and

$$x_1^n = x_2^n = 0, \qquad x_1g = \lambda gx_1, \qquad x_2g = \lambda^{-1}gx_2,$$
$$x_2x_1 - \lambda^{-u_1}x_1x_2 = g^{u_1+u_2} - 1.$$

The Hopf algebra $U_{(N,\nu,\omega)}$ (see [18] or Ex. 1.19(viii)) is just $H(C_N, (r), (g^\nu), (g^*, g^{*-1}), 0, 1)^{cop}$.

(ii) Let $m \geq 2, t > 2$ be integers, m even, and let $C = C_m = \langle g\rangle$. Let $u_1, \ldots, u_t$ be odd integers such that $u_i + u_j \not\equiv 0 \bmod m$ if $i \neq j$ and let $c_i = g^{u_i}$, $c_i^* = g^*$ where $\langle g^*, g\rangle = -1$. Then the Hopf algebra $H(C_m, (2), c, (g^*), 0, 1)$ has dimension 2^tm and has generators $g, x_1, \ldots, x_t$ such that g is grouplike, x_i is a $(1, g^{u_i})$-primitive, and

$$g^m = 1, \qquad x_i^2 = 0, \qquad x_ig = -gx_i, \qquad x_jx_i + x_ix_j = g^{u_i+u_j} - 1.$$

(iii) Suppose $C = \langle g \rangle$ is infinite cyclic, and $n \geq 2$. Let u_1, u_2 be integers such that $u_1 + u_2 \neq 0$, and let $\lambda \in k$ such that $\lambda^{u_1} = \lambda^{u_2}$ is a primitive nth root of 1. Let $g^* \in C^*$ with $\langle g^*, g \rangle = \lambda$. Then there is an infinite dimensional pointed co-Frobenius Hopf algebra $H(C, (n), (g^{u_1}, g^{u_2}), (g^*, g^{*-1}), 0, 1)$ with generators g, x_1, x_2 such that g is grouplike of infinite order, x_i is a $(1, g^{u_i})$-primitive, and

$$x_1^n = x_2^n = 0, \qquad x_1 g = \lambda g x_1, \qquad x_2 g = \lambda^{-1} g x_2,$$
$$x_2 x_1 - \lambda^{-u_1} x_1 x_2 = g^{u_1+u_2} - 1.$$

(iv) Let $C = \langle g \rangle$ be infinite cyclic, $t > 2$ and let $u_1, \ldots, u_t$ be odd integers such that $u_i + u_j \neq 0$ for $i \neq j$. Then there is an infinite dimensional co-Frobenius pointed Hopf algebra $H(C, (2), c, (g^*), 0, 1)$, where $c_i = g^{u_i}$ and $\langle g^*, g \rangle = -1$. The generators are $g, x_1, \ldots, x_t$ such that g is group-like of infinite order, x_i is a $(1, g^{u_i})$-primitive, and

$$x_i^2 = 0, \qquad x_i g = -g x_i, \qquad x_j x_i + x_i x_j = g^{u_i+u_j} - 1.$$

By an argument similar to the proof of Theorem 2.1, we can classify the Hopf algebras from Definition 5.1.

THEOREM 5.5. *There is a Hopf algebra isomorphism from* $H = H(C, (n), c, (g^*, g^{*-1}), 0, 1)$ *to* $H' = H(C', (n'), c', (g^{*'}, (g^{*'})^{-1}), 0, 1)$ *if and only if* $C = C'$, $n = n'$ *and there is an automorphism* f *of* C *such that*

(i) $f(c_1) = c'_1, f(c_2) = c'_2$ *and* $g^* = g^{*'} \circ f$; or

(ii) $f(c_1) = c'_2, f(c_2) = c'_1$ *and* $g^* = (g^{*'})^{-1} \circ f$.

Proof. If $H \cong H'$, then exactly as in the proof of Theorem 2.1, there exists an automorphism f of C and a bijection π of $\{1, 2\}$ such that $f(c_i) = c'_{\pi(i)}$ and $c_i^* = c^{*'}_{\pi(i)} \circ f$. The conditions (i) and (ii) in the statement correspond to π the identity and π the nonidentity permutation.

Conversely, if (i) holds, define an isomorphism from H to H' by mapping g to $f(g)$ and x_i to x'_i. If (ii) holds, define an isomorphism from H to H' by mapping g to $f(g)$, x_1 to x'_2 and x_2 to $-\langle g^*, c_1 \rangle x'_1$. ∎

COROLLARY 5.6. *If* $C = \langle g \rangle$ *is cyclic, then the Hopf algebras* H *and* H' *above are isomorphic if and only if* $C = C'$, $n = n'$, *and there is an integer* h *such that the map taking* g *to* g^h *is an automorphism of* C *and either*

(i) $c_i^* = c_i^{*'h}$ *and* $c_i^h = g^{u_i h} = g^{u'_i} = c'_i$ *for* $i = 1, 2$; or

(ii) $c_i^* = (c_i^{*'})^{-h}$ *and* $g^{u_1 h} = g^{u'_2}, g^{u_2 h} = g^{u'_1}$.

For the Hopf algebras of Definition 5.2 there is a similar classification result.

THEOREM 5.7. *There is a Hopf algebra isomorphism from* $H = H(C, (2), c, (g^*), 0, 1)$ *to* $H' = H(C', (2), c', (g^{*'}), 0, 1)$ *if and only if* $C = C'$, $t = t'$ *and there is a permutation* $\pi \in S_t$ *and an automorphism* f *of* C *such that* $f(c_i) = c'_{\pi(i)}$ *and* $g^* = g^{*'} \circ f$.

COROLLARY 5.8. *Suppose* $C = \langle g \rangle$ *is cyclic. Then* H *and* H' *as above are isomorphic if and only if* $C = C'$, $t = t'$ *and there exists a permutation* $\pi \in S_t$ *and an automorphism of* C *taking* g *to* g^h, *such that* $c_i^h = g^{u_i h} = c'_{\pi(i)}$ *for all* i.

In Example 2.5 we saw that if $a \neq 0$, Ore extension Hopf algebras with non-zero derivations may be isomorphic to Ore extension Hopf algebras with zero derivations. The following theorem shows that if $a = 0$, this is impossible.

THEOREM 5.9. *Hopf algebras of the form* $H(C, n, c, c^*) = H(C, n, c, c^*, 0, 0)$ *cannot be isomorphic to either the Hopf algebras of Definition* 5.1 *or Definition* 5.2.

Proof. Suppose that $f: H(C', (n'), c', (g^{*'}, g^{*'-1}), 0, 1) \to H(C, n, c, c^*)$ is an isomorphism of Hopf algebras. Then, as in the proof of Theorem 2.1, we see that $C = C'$, $f(x'_1) = \sum_i \alpha_i x_i$ and $f(x'_2) = \sum_i \beta_i x_i$ for scalars α_i, β_i. But f applied to the relation

$$x'_2 x'_1 = \langle (g^{*'})^{-1}, c'_1 \rangle x'_1 x'_2 + c'_1 c'_2 - 1$$

yields $\sum_{i,j} \alpha_i \beta_j (x_j x_i - \langle (g^{*'})^{-1}, c'_1 \rangle x_i x_j) = l - 1$ in $H(C, n, c, c^*)$, where $l \neq 1$ is a group-like element. The relations of an Ore extension with zero derivations show that this is impossible. Similarly, $H(C, n, c, c^*)$ cannot be isomorphic to a Hopf algebra as in Definition 5.2. ∎

6. POINTED HOPF ALGEBRAS OF DIMENSION p^3

We noted in Sect. 1 that the Hopf algebras of dimension p^2, p a prime, constructed from kC_p are just the Taft Hopf algebras. The purpose of this final section is to list the pointed Hopf algebras of dimension p^3 that can be obtained using constructions from this paper, and to count how many types there are. If H is a pointed Hopf algebra of dimension p^3, then by the Nichols–Zoeller Theorem [15, Theorem 3.1.5], $\dim(\mathrm{Corad}(H)) \in \{1, p, p^2, p^3\}$. By the Taft–Wilson theorem, $\dim(\mathrm{Corad}(H)) \neq 1$. Thus $G(H)$ is one of the groups $C_p, C_p \times C_p, C_{p^2}, C_p \times C_p \times C_p, C_{p^2} \times C_p, C_{p^3}, G_1, G_2$, where $G_1 = C_{p^2} \rtimes C_p$ and $G_2 = C_p \rtimes C_{p^2}$ are the two types of nontrivial semidirect products. If $G(H)$ is one of the five groups of order p^3, then H is just $kG(H)$.

First, we consider the examples with $G(H) = C_p \times C_p$. By [4, Proposition 4], if a Hopf algebra H over an algebraically closed field k has dimension p^n and coradical isomorphic to $k(C_p^{n-1}) = kC_p \otimes \cdots \otimes kC_p$, then H is isomorphic to $k(C_p^{n-2}) \otimes T$ where T is a Taft Hopf algebra. We present the next result as an application of Theorem 2.1 for the non-cyclic group $C_p \times C_p$.

PROPOSITION 6.1. *$H = H(C_p \times C_p, p, c, c^*)$ is isomorphic to $H(C_p, p, c', c^{*'}) \otimes kC_p$ for some $c' \in C_p$, $c^{*'} \in C_p^*$, and thus there are $p-1$ isomorphism classes of such Hopf algebras, corresponding to the $p-1$ isomorphism classes of the Taft Hopf algebras of dimension p^2.*

Proof. Let $C_p \times C_p = \langle g_1 \rangle \times \langle g_2 \rangle$ and $\langle c^*, g_1 \rangle = \lambda_1$, $\langle c^*, g_2 \rangle = \lambda_2$. If $c = g_1^{u_1} g_2^{u_2}$, then $\langle c^*, c \rangle = \lambda_1^{u_1} \lambda_2^{u_2}$, a primitive pth root of unity.

We distinguish two cases. If $\lambda_1, \lambda_2 \neq 1$, choose h such that $\lambda_2 = \lambda_1^h$ and let f be the automorphism of $C_p \times C_p$ mapping g_1 to $g_1 g_2^{-hu_2}$ and g_2 to $g_1^h g_2^{hu_1}$. (Note that $\lambda_1^{u_1+hu_2} \neq 1$ implies that f is a bijection.) By Theorem 2.1, f induces an isomorphism from H to $H(C_p \times C_p, p, f(c), c^{*'})$ where $f(c) = g_1^{u_1} g_2^{-hu_1u_2} g_1^{hu_2} g_2^{hu_1u_2} = g_1^{u_1+hu_2}$, and $\langle c^{*'}, g_1 \rangle = \lambda_1$, $\langle c^{*'}, g_2 \rangle = 1$ so that $c^* = c^{*'} \circ f$. Clearly this last Hopf algebra is isomorphic to $H(C_p, p, g_1^{u_1+hu_2}, c^*) \otimes kCp$, i.e., the tensor product of a Taft Hopf algebra and a group algebra.

If λ_1 or λ_2 is 1, say $\lambda_2 = 1$, then the automorphism of $C_p \times C_p$ mapping g_1 to $g_1 g_2^{u_2}$ and g_2 to $g_2^{-u_1}$ induces in a similar way an isomorphism of Hopf algebras from H to $H(C_p \times C_p, p, g_1^{u_1}, c^{*'})$, $c^{*'}$ as above.

It is easy to see that the Hopf algebras $H(C_p \times C_p, p, g_1^u, c^*)$ and $H(C_p \times C_p, p, g_1^v, c^*)$ where $\langle c^*, g_1 \rangle = \lambda \neq 1$, $\langle c^*, g_2 \rangle = 1$, are isomorphic if and only if $u = v$. Therefore we obtain exactly $p-1$ types of Hopf algebras in this way. ∎

Examples with $G(H) = C_p$ are obtained by starting with kC_p and making a double Ore extension with zero or non-zero derivation.

PROPOSITION 6.2. *If p is an odd prime then there exist precisely $(p-1)^2/2$ non-isomorphic Hopf algebras of the form $H(C_p, (p, p), c, c^*)$. For $p = 2$ there is only one such Hopf algebra.*

Proof. Let p be an odd prime, let $C_p = \langle g \rangle$, and let $C_p^* = \langle g^* \rangle$ where $\langle g^*, g \rangle = \lambda$, $\lambda \neq 1$ a pth root of unity. Let $c_1 = g^{u_1}$, $c_2 = g^{u_2}$. By Proposition 4.5, we may assume $c_1^* = g^*$ and $c_2^* = g^{*d}$. Since $u_2 \equiv -du_1 \bmod p$, d and u_1 determine u_2. Thus we have $(p-1)^2$ Hopf algebras $H(C_p, (p, p), c, c^*)$, and we must determine which are isomorphic. Fix $H = H(C_p, (p, p), c, c^*)$ as above and suppose there is an isomorphism f from H to $H' = H(C_p, (p, p), c', c^{*'})$ where $c'_i = g^{u'_i}$, $c'^*_1 = g^*$ and $c'^*_2 = g^{*d'}$.

Suppose $f(g) = g^h$. If the permutation π of $\{1, 2\}$ associated with f is the identity, then $\langle g^*, g\rangle = \langle g^*, g^h\rangle$ so $h = 1$. If π is the nontrivial permutation of $\{1, 2\}$, then $\langle g^{*d}, g\rangle = \langle g^*, g^h\rangle$ so $d = h$. Then $(c_1', c_2') \neq (c_1, c_2)$. For $u_2' = du_1 = -u_2$ and for p odd, $u_2 \not\equiv -u_2 \mod p$.

If $p = 2$, it is clear that there is only one choice for d and c. ∎

PROPOSITION 6.3. *For any odd prime p there exist $p - 1$ types of Hopf algebras of the form $H(C_p, (p, p), (g^{u_1}, g^{u_2}), (g^{*d_1}, g^{*-d_1}), 0, 1)$ where $< g^*, g >= \lambda$, a nontrivial pth root of 1. If $p = 2$, then there are no double Ore extension with non-zero derivations.*

Proof. Since $\lambda^{u_1 d_1} = \lambda^{u_2 d_1}$, we may assume $u_1 = u_2$, and by Corollary 5.6 we may assume $c = (g, g)$. Again by Corollary 5.6, $H(C_p, (p, p), (g, g), (g^*, g^{*-1}), 0, 1)$ is isomorphic to $H(C_p, (p, p), (g, g), (g^{*h}, g^{*-h}), 0, 1)$ if and only if $h = 1$. ∎

Examples with $G(H) = C_{p^2}$ can be obtained by a single Ore extension starting with kC_{p^2}. The following is an immediate consequence of Theorem 2.1.

PROPOSITION 6.4. *There exist $2(p - 1)$ types of Hopf algebras of the form $H(C_{p^2}, p, c, c^*)$, and $p - 1$ types of the form $H(C_{p^2}, p, c, c^*, 1, 0)$.*

Adding all the types described, we have a total of $\frac{(p-1)(p+9)}{2} + 5$ types when p is odd, and 10 types when $p = 2$. In fact the results of [4] describing pointed Hopf algebras of dimension p^n with coradical kC, C abelian of order p^{n-1}, together with those of [7] or [2] which classify pointed Hopf algebras of dimension p^3 with coradical of dimension p, combine to show that these are all the types of pointed Hopf algebras of dimension p^3.

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