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On the spanning tree packing number of a graph: a survey

E.M. Palmer

Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

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Dedicated to Paul A. Catlin

Abstract

The *spanning tree packing number* or *STP number* of a graph G is the maximum number of edge-disjoint spanning trees contained in G. We use an observation of Paul Catlin to investigate the STP numbers of several families of graphs including quasi-random graphs, regular graphs, complete bipartite graphs, cartesian products and the hypercubes. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

For any graph G of order n, the *spanning tree packing number* or *STP number*, denoted by $\sigma = \sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G. I first became interested in the STP number when my friend and colleague, Paul Catlin, explained to me why a graph with 2 edge-disjoint spanning trees, was collapsible and hence supereulerian (see [3]). Our discussions and the concept of collapsibility lead to a thesis for one of my students, Spencer [20], a couple of short papers, [16] and [10], many interesting and pleasurable conversations at conferences, and exchange visits between MSU and WSU. The last time we visited Paul (in November 1994) and traded graph theory results, he told me about some of his latest observations on edge-disjoint spanning trees. After the memorial service for him in June of 1995, I recalled that conversation, located my notes and found that his insights provided an alternative and simple method for determining the STP number of a graph. His main result takes the form of a corollary of a theorem of Nash-Williams [14] that yields a useful lower bound for the STP number. It can be applied successfully to several families of graphs

E-mail address: palmer@math.msu.edu (E.M. Palmer)

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including quasi-random graphs, regular graphs, complete bipartite graphs, and cartesian products. I found 29 papers using the MathSciNet search on 'edge-disjoint spanning trees' and learned of a few others from the editors. So, we will be brief with STP numbers that have already been determined.

We use $\kappa = \kappa(G)$ and $\lambda = \lambda(G)$ for the vertex and edge connectivity of the graph G. And $\delta = \delta(G)$ denotes the minimum degree. Most other definitions can be found in the book [8].

2. Catlin's corollary

Here is the sufficient condition Paul found that can be so useful in determining the STP number:

If G is a graph with $\lambda(G) \geqslant 2k$ and we delete *any* set of k edges from G, then the resulting graph H has k edge-disjoint spanning trees.

The proof follows nicely from Nash-Williams [14] and so we give a brief sketch. First we need a bit of notation. For any graph G with vertex set $V(G)$ and any subset $S \subseteq V(G)$, we denote by $e_G(S, \bar{S})$ the set of edges of G with one vertex in S and the other in $\bar{S} = V(G) \setminus S$. Thus $e_G(S, \bar{S})$ is sometimes called the *edge boundary* of S. Now suppose H is a graph obtained by deleting any k edges from G. Let

$$V(H) = V_1 \dot{\cup} \cdots \dot{\cup} V_t$$

be any partition P of $V(H) = V(G)$ with t parts. Then by hypothesis we have for each $i = 1$–t

$$|e_G(V_i, \bar{V}_i)| \geqslant \lambda(G) \geqslant 2k. \tag{2.1}$$

Therefore, the number of external edges in H determined by the partition P satisfies

$$\begin{aligned} |E_P(H)| &= \frac{1}{2} \sum_{i=1}^{t} |e_H(V_i, \bar{V}_i)| \\ &\geqslant \left\{ \frac{1}{2} \sum_{i=1}^{t} |e_G(V_i, \bar{V}_i)| \right\} - k \\ &\geqslant k(t-1). \end{aligned} \tag{2.2}$$

On applying the theorem of Nash-Williams [14] we obtain the sufficient condition in the theorem below for H to have k edge-disjoint spanning trees. Later I found this result stated in Paul's survey paper [4] (see Theorem 5.1). It was also attributed to him in [12]. The proof of the necessity follows right from the definition of λ.

Theorem (Catlin [4]). *The edge connectivity of G satisfies $\lambda(G) \geqslant 2k$ if and only if for any set E_k of k edges of G, the subgraph $H = G - E_k$ has k edge disjoint spanning trees.*

Corollary (Catlin [4]). *If $\lambda(G) \geqslant 2k$, then G has k edge-disjoint spanning trees.*

It follows immediately that for any graph G

$$\lfloor \lambda(G)/2 \rfloor \leqslant \sigma(G). \tag{2.3}$$

As for an upper bound, from the definition of σ we have $\sigma(G) \leqslant \lambda(G)$. But a more useful and equally obvious upper bound is

$$\sigma(G) \leqslant \lfloor |E(G)|/(n-1) \rfloor, \tag{2.4}$$

where $E(G)$ denotes the edge set of G and $n = |V(G)|$ is the number of vertices or order of G. When equality holds in (2.4), the value of σ is optimal in the sense that there are simply not enough edges left to form another spanning tree. All of these results hold for multigraphs too.

There are some important related theorems that can be used to determine σ that must be mentioned.

On examining the papers [21] and [14] the reader will find that the main theorems, though equivalent, take slightly different forms and use different proofs. As has often been observed (see [12] for further references), an immediate consequence of Tutte's formulation is an expression for the STP number of G, namely

$$\sigma(G) = \left\lfloor \min_{E \subseteq E(G)} |E|/(c(G-E)-1) \right\rfloor, \tag{2.5}$$

where $c(G-E)$ is the number of components of $G-E$. Formula (2.4) follows from (2.5) by taking $E = E(G)$.

In a sequel to [14] we find in [15] an expression for another related invariant. The *edge arboricity* of a graph G, denoted by $\alpha_1 = \alpha_1(G)$, is the minimum number of acyclic, mutually edge-disjoint subgraphs whose union is G. Thus $\sigma(G) \leqslant \alpha_1(G)$. Although it is easy to concoct examples for which σ and α_1 differ wildly, they are sometimes very close and the formula for α_1 found by Nash-Williams can be useful in determining α_1 or σ:

$$\alpha_1(G) = \left\lceil \max_{H<G} |E(H)|/(|V(H)|-1) \right\rceil, \tag{2.6}$$

where the maximum is taken over all non-trivial induced subgraphs H of G. Occasionally the maximum is achieved for $H = G$. In this case if (perhaps this should be IF)

$$\alpha_1(G) = |E(G)|/(|V(G)|-1), \tag{2.7}$$

then, of course, $\alpha_1 = \sigma$. We shall see several instances of this phenomenon.

These fundamental theorems of Tutte and Nash-Williams were extended to matroids by Catlin et al. [7] where the reader will find a number of useful related results and references.

3. Applications of Catlin's corollary

There are efficient algorithms for determining sets of edge-disjoint trees in a given graph [18]. But the aim of this section is to indicate how Catlin's Theorem and Corollary can be used to determine σ quickly and easily for several important families of graphs.

3.1. Complete graphs

First, we consider the complete graph of order n, denoted by K_n. It is well-known (see [8, Theorem 8.12, p. 237]) that when n is odd K_n is an edge-disjoint union of spanning cycles, whereas for n even, it can be factored in paths. These constructions imply that $\sigma(K_n)$ is given by the formula (3.1).

$$\sigma(K_n) = \lfloor n/2 \rfloor. \tag{3.1}$$

See also Corollary 8 of [7]. But (3.1) can also be verified using Catlin's theorem. We observe that $\lambda(K_n) = n - 1$ for all n and so from (2.3) and (2.4) we have

$$\left\lfloor \frac{n-1}{2} \right\rfloor \leqslant \sigma(K_n) \leqslant \left\lfloor \frac{n}{2} \right\rfloor. \tag{3.2}$$

These bounds are equal for all odd n and so (3.1) holds in that case. For even n we use Catlin's theorem and an easy construction. Consider K_{n-1} and with $k = (n-2)/2$, choose a set E_k of k independent edges in K_{n-1}. Now add a new vertex v to K_{n-1} and form a spanning tree of K_n by adding one edge from v to each of the k edges in E_k as well as the one vertex not covered by E_k. Since $\lambda(K_{n-1}) = 2k$, there are k edge disjoint spanning trees, say $T_1, \ldots, T_k$ in $K_{n-1} - E_k$. Now add one edge from v to each of $T_1, \ldots, T_k$ to form k more spanning trees in K_n. Thus K_n has $k + 1 = n/2$ spanning trees.

3.2. Paley graphs

The Paley graphs (see [2]) make up another large family whose STP numbers can be determined by Catlin's corollary. Let q be a prime power congruent to $1 \pmod 4$. The Paley graph P_q has the q elements of the field F_q as its vertices and x is adjacent to y if $x - y$ is a square in F_q. These graphs are regular of degree $(q-1)/2$ and hence have diameter 2. Plesnick's theorem (see [8, p. 155]) implies that

$$\lambda(P_q) = (q-1)/2. \tag{3.3}$$

And so from Catlin's corollary and (2.4) we have

$$\left\lfloor \frac{q-1}{4} \right\rfloor \leqslant \sigma \leqslant \left\lfloor \frac{q}{q-1} \, \frac{q-1}{4} \right\rfloor. \tag{3.4}$$

But these two bounds are seen to be identical and so

$$\sigma(P_q) = (q-1)/4. \tag{3.5}$$

Table 1

Graph	STP no.
K_n $(n \geqslant 1)$	$\lfloor n/2 \rfloor$
$K_{m,n}$ $(1 \leqslant m \leqslant n)$	$\left\lfloor \dfrac{mn}{n+m-1} \right\rfloor$
Q_n $(n \geqslant 1)$	$\lfloor n/2 \rfloor$
$K_m \times K_n$ $(2 \leqslant m \leqslant n)$	$\left\lfloor \dfrac{m+n-2}{2} \right\rfloor$
$K_m \times C_n$	$\left\lfloor \dfrac{m+1}{2} \right\rfloor$
Paley P_q	$(q-1)/4$
Octahedron Icosahedron Maximal planar $C_m \times C_n$ $(3 \leqslant m \leqslant m)$	2
Connected cubic $(n \geqslant 6)$	1
Connected 4-regular $(n \geqslant 7)$	1 or 2
Random r-regular	$\lfloor r/2 \rfloor$
Harary $H_{n,M}$	$\lfloor \lfloor 2M/n \rfloor /2 \rfloor$
Quasi-random	$(1+\mathrm{o}(1))n/4$

At this point we observe another useful consequence of Catlin's corollary for graphs like the Paley family which are regular of degree r and have edge connectivity λ equal to r. For these we have from (2.3) and (2.4)

$$\left\lfloor \frac{r}{2} \right\rfloor \leqslant \sigma \leqslant \left\lfloor \frac{rn}{2(n-1)} \right\rfloor. \tag{3.6}$$

It is easy to see that these bounds are identical unless $r = n-1$ and n is even. Hence, the graph under consideration is complete, a case with which we have already dealt. Summarizing this principle we can state that if G is regular of degree $r = \lambda$ and $r \neq n-1$ when n is even, then

$$\sigma(G) = \lfloor r/2 \rfloor. \tag{3.7}$$

Then (3.7) applies to the Paley graphs and we will use it repeatedly in the next application.

3.3. Cartesian products

The STP numbers of cartesian products of various combinations of complete graphs and cycles can also be determined by Catlin's corollary. We simply use Sabidussi's lemma for connectedness of cartesian products [19] and formula (3.7). Some of these results are included in Table 1. The same approach works on the complements $\overline{K_m \times K_n}$, $\overline{K_m \times C_n}$, $\overline{C_m \times C_n}$ and $\overline{Q_n}$. Some of these results can also be deduced from the fact that the cartesian products are uniformly dense (see [13]).

3.4. Complete bipartite graphs

Now consider the complete bipartite graphs $K_{m,n}$ with $2 \leqslant m \leqslant n$. The edge connectivity here is $\lambda = m$ and so using Catlin's corollary and (2.4) we have

$$\lfloor m/2 \rfloor \leqslant \sigma(K_{m,n}) \leqslant \left\lfloor \frac{mn}{m+n-1} \right\rfloor . \tag{3.8}$$

When $m = n$, (3.7) applies and so for all $n \geqslant 1$

$$\sigma(K_{n,n}) = \lfloor n/2 \rfloor . \tag{3.9}$$

The case for $m < n$ is more complicated. Beineke [1] showed how to factor $K_{m,n}$ into acyclic subgraphs and determined that the upper bound in (3.8) gives the right answer. See [17] for a simple proof that (2.4) gives the right answer for *any* complete t-partite graph.

3.5. Planar graphs

For the graphs of all five regular polyhedra, we have seen that the tetrahedron (K_4) has STP number 2, whereas the cube (Q_3) and the dodecahedron, being 3-regular, have $\sigma = 1$. A bit of construction work or Catlin's corollary shows that the octahedron and the icosahedron both have $\sigma = 2$, as do all maximal planar graphs (see [7, Corollary 8]).

3.6. Harary graphs

Which graphs with n vertices and $M \geqslant n-1$ edges have the largest STP number? The Harary graphs $H_{n,M}$ form such a family [11]. They are all regular or nearly so. They are constructed by arranging the n vertices in circular order and spreading the M edges around the boundary in a nice way, keeping the chords as short as possible. They have the maximum connectivity for their size and

$$\kappa(H_{n,M}) = \lambda(H_{n,M}) = \delta(H_{n,M}) = \left\lfloor \frac{2M}{n} \right\rfloor . \tag{3.10}$$

When they are regular, of course (3.7) applies, unless $H_{n,M} = K_n$ with n even. For all other cases we have

$$\sigma(H_{n,M}) = \left\lfloor \left\lfloor \frac{2M}{n} \right\rfloor \Big/ 2 \right\rfloor . \tag{3.11}$$

3.7. Regular graphs

Next, we turn to arbitrary regular graphs of fixed degree r and order n. For cubic graphs ($r = 3$) we have $\sigma(K_4) = 2$ but for any connected cubic G of order $n \geqslant 6$, the upper bound (2.4) implies $\sigma(G) = 1$.

For 4-regular graphs G it follows from (2.4) that $\sigma(G)\leqslant 2$. Catlin's corollary shows that if $\lambda(G)=4$, then $\sigma(G)=2$. An easy degree argument shows that $\lambda\neq 1$ or 3 for these graphs. And we do find examples with $\sigma=1$ and $\lambda=2$. Take a cycle of order 6 and replace alternate edges uv by any connected graph H which is 4-regular except for u and v which have degree 3 in H. So H has at least 5 vertices. Two of the 3 single edges of the original cycle must be used in each spanning tree. Hence $\sigma=1$. We shall see that these are rare. The smallest example obtained from our construction has 15 vertices. Is there a smaller 4-regular graph with $\lambda=2$ and $\sigma=1$? The same strategy serves to show that connected 5-regular graphs of order $n\geqslant 7$ have $\sigma=1$ or 2. Of course, we have already seen that $\sigma(K_6)=3$.

For random r-regular graphs, there is Wormald's theorem [22] which states that $\kappa(G)=\lambda(G)=r$ almost surely. Then (3.7) applies almost surely.

3.8. Random graphs

We have seen that for any graph $\sigma\leqslant\lambda$ but σ can be greater or less than κ. For example, when $n\geqslant 3$

$$\kappa(K_n)=n-1>\lfloor n/2\rfloor=\sigma(K_n). \tag{3.12}$$

On the other hand, for odd $n\geqslant 7$ let G_n be the graph obtained from two copies of $K_{(n-1)/2}$ by adding a new vertex v adjacent to every other vertex. Then since v is a cut vertex,

$$\kappa(G_n)=1<\left\lfloor\frac{n+1}{4}\right\rfloor=\sigma(G_n). \tag{3.13}$$

In [16] we found that for fixed $k\geqslant 1$, the hitting times in a random graph process for $\sigma=k$ and for the minimum degree $\delta=k$ were almost surely identical. Hence when the edge probability of a random graph G is given by

$$pn=\log n+(k-1)\log\log n+\omega_n, \tag{3.14}$$

where $\omega_n\to\infty$ but $\omega_n=\mathrm{o}(\log\log n)$ then with probability approaching 1 as $n\to\infty$ the invariants σ,κ,λ and the minimum degree δ *all* have the same value, namely k (see [2, p. 152]). The main ingredient of our method was the powerful characterization theorem of Tutte [21] and Nash-Williams [14].

As the number of edges in a random graph increases, however, σ loses ground to κ. In the Catlin papers [5] and [6] it was found that if the edge probability was rather larger than (3.14), then almost surely (2.4) becomes an equality. This means, for example, that for $p=1/2$, a random graph G has almost surely

$$\sigma=\lfloor |E(G)|/(n-1)\rfloor \tag{3.15}$$

and since we know good approximations for the number of edges, we have

$$\sigma=(1+\mathrm{o}(1))n/4 \tag{3.16}$$

almost surely. Catlin's corollary provides an alternate derivation of this estimate. Since it is so easy we will sketch the steps in the slightly more general context of quasi-random graphs.

Following [9], let $\mathscr{F}_n$ be a family of graphs of order n with vertex set $[n] = \{1, 2, \dots, n\}$ and some specified probability distribution. The family is *quasi-random* if with probability approaching 1 as $n \to \infty$, we have

$$|e_G(S, \bar{S})| = \frac{|S||\bar{S}|}{2} + o(n^2) \tag{3.17}$$

for every subset $S \subseteq [n]$.

One of the remarkable discoveries of [9] is that there are quite a few other equivalent definitions, some of which are rather surprising. At any rate, by this definition a quasi-random graph is very much like an ordinary random graph with edge probability $p = \frac{1}{2}$ especially if we stipulate that the minimum degree is asymptotic to $n/2$, and so we make this assumption from here on.

The proof technique of [5,6] can also be applied to quasi-random graphs to prove (3.15) and hence (3.16). To use Catlin's corollary, on the other hand, we just need to estimate λ for quasi-random graphs. If S is any subset of the vertex set with $|S| = o(n)$, then

$$|e_G(S, \bar{S})| \geqslant \delta - (|S| - 1) \geqslant (1 + o(1))n/2, \tag{3.18}$$

but if

$$n/\log n \leqslant |S| \leqslant n/2 \tag{3.19}$$

then it follows from the quasi-random axioms (3.17) that

$$|e_G(S, \bar{S})| \geqslant \frac{n}{\log n} \binom{n}{2} \Big/ 2 + o(n^2) \geqslant (1 + o(1))n/2. \tag{3.20}$$

Therefore for quasi-random graphs

$$\lambda \geqslant (1 + o(1))n/2, \tag{3.21}$$

and so

$$\sigma \geqslant (1 + o(1))n/4. \tag{3.22}$$

Since quasi-random graphs have approximately $\binom{n}{2}/2$ edges, the upper bound (2.4) implies that equality holds in (3.22).

Note that this result is consistent with what we learned earlier about the Paley graphs, a well-known quasi-random family.

4. Conclusion

No doubt there are other interesting families for which the STP number has yet to be determined. For example, one could ask for the STP numbers of the line graphs or the nth powers of the entries in Table 1. There is also the problem of actually packing these graphs in an elegant way with edge-disjoint spanning trees.

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