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Topology and its Applications 69 (1996) 205–209

TOPOLOGY AND ITS APPLICATIONS

A note on the homotopy invariance of Pontrjagin classes

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Received 16 September 1994; revised 10 August 1995, 21 September 1995

Abstract

We prove that the classical result of Wu that, for every $n \geqslant 1$, the integral Pontrjagin class p_n modulo 3 is homotopy invariant is the only and best possible result, 'only' in the sense that no other Pontrjagin classes of stable vector bundles—rational, integral, multiple of integral or integral mod p, $p \neq 3$, are homotopy invariant and 'best' in the sense that p_n mod 3^r is not homotopy invariant if $r > 1$.

Keywords: Integral Pontrjagin classes; Rational Pontrjagin classes; Pontrjagin classes modulo p^r; Homotopy invariance; Adams operations

AMS classification: 55R40; 55S25; 57R20

1. Introduction

A classical result of Wu [14] states that integral Pontrjagin classes p_n mod 3, $n = 1, 2, 3, \ldots$ are homotopy invariant. Singh [12] proved that this is the best possible result in the sense that for every $n \geqslant 1$, p_n mod 3^r is homotopy invariant only if $r = 1$. One of the aims of this paper is to show that Wu's result is the best possible in another sense, namely, p_n mod p, where p is an odd prime, is homotopy invariant only if $p = 3$. In other words, integral Pontrjagin classes modulo an odd prime p, $p \neq 3$, are not homotopy invariant. In fact, we prove the following:

Theorem. *Let p be an odd prime. For every $n \geqslant 1$, p_n mod p^r is homotopy invariant iff $p = 3$ and $r = 1$.*

The above result is the only positive result on the homotopy invariance of Pontrjagin classes. In contrast to results on topological invariance [8,6,10,11], we shall see that integral or multiples of integral Pontrjagin classes are not homotopy invariant. Rational Pontrjagin classes of stable bundles are not homotopy invariant [12], in fact, Kahn [3]

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SSDI 0166-8641(95)00115-8

proved that up to rational multiple the only polynomial in rational Pontrjagin classes of a manifold whose value on the fundamental class of the manifold is homotopy invariant is the Hirzebruch polynomial L_k. We show that L_k itself is not homotopy invariant.

2. Proof of the theorem

Let BO and BG denote the stable classifying spaces for real vector bundles and spherical fibrations respectively. Let $J: BO \to BG$ denote the canonical map, G/O the homotopy fibre of J and $\zeta: G/O \to BO$ the inclusion of the fibre. Since $J \circ \zeta$ is homotopic to a constant map, ζ may be thought of as a vector bundle which is homotopically trivial with a given trivialization.

We recall from [12] that a universal characteristic class $x \in H^*(BO; \Lambda)$, where Λ is any coefficient ring, is homotopy invariant if for every pair of homotopy equivalent bundles ξ_0 and ξ_1 over X, $x(\xi_0) = x(\xi_1)$, i.e., $\xi_0^*(x) = \xi_1^*(x)$. We say that two vector bundles ξ_0 and ξ_1 over the base X with total spaces E_0, E_1 and projections π_0, π_1 respectively are homotopy equivalent if there is a fibre homotopy equivalence $h: E_0 \to E_1$ such that $\pi_1 \circ h = \pi_0$.

Let v be the composite map: $BO \times G/O \xrightarrow{1\times\zeta} BO \times BO \xrightarrow{m} BO$ where 1 represents the universal stable vector bundle γ and m is the Hopf space multiplication. We assume that in the ring Λ, 2 is invertible, so that $H^*(BO; \Lambda)$ has no torsion.

Proposition 2.1. *A characteristic class $x \in H^*(BO; \Lambda)$ is homotopy invariant iff $v^*x = x \otimes 1$.*

The proof is same as in the case of topological invariance [10].

Consider the fibration

$$G/O[p] \xrightarrow{\zeta[p]} BO[p] \xrightarrow{J[p]} BG[p]$$

where the spaces and maps are localized at an odd prime p. Let k be a positive integer which reduces to a generator of the group of units in $\mathbb{Z}/p^2$. From May [5, p. 124] we deduce from the splitting of $G/O[p]$ that calculation of $\zeta[p]$ is the same as calculation of $\Psi^k - 1$, where Ψ^k is the Adams operation and the calculation of $(\Psi^k - 1)^*$ is standard; see, e.g., [9] or [2]. Thus the necessary and sufficient condition for homotopy invariance can now be stated in the following form:

Proposition 2.2. *A characteristic class x is homotopy invariant iff $x \otimes 1 = v^*x = (1 \otimes (\Psi^k - 1)^*) \circ m^*x$.*

We require the following lemmas for the proof of the theorem.

Lemma 2.3.

$$\begin{aligned}(\Psi^k - 1)^* L_n = \sum \big\{ &\alpha_{i_1 i_2 \cdots i_r}(k^{2i_1} - 1) L_{i_1}^{t_1} L_{i_2}^{t_2} \cdots L_{i_r}^{t_r} + \alpha_{i_2 i_1 \cdots i_r}(k^{2i_2} - 1) \\ &\times L_{i_1}^{t_1} L_{i_2}^{t_2} \cdots L_{i_r}^{t_r} + \cdots + \alpha_{i_r i_1 \cdots i_{r-1}}(k^{2i_r} - 1) L_{i_1}^{t_1} L_{i_2}^{t_2} \cdots L_{i_r}^{t_r} \big\},\end{aligned}$$

where the summation runs over $i_1, i_2, \ldots, i_r$ such that $i_1 \neq i_2 \neq \cdots \neq i_r$ and $i_1 t_1 + \cdots + i_r t_r = n$. All i's and t's are positive integers and α's are nonzero integers.

Proof. We recall from [4, p. 111] that $(\Psi^k - 1)^* L = (\Psi^k)^* L/L$ and from [4, p. 103] that $(\Psi^k)^* : H^{4n}(BO;\mathbb{Q}) \to H^{4n}(BO;\mathbb{Q})$ is multiplication by k^{2n}. The lemma follows by comparing terms of degree $4n$ on both the sides. □

Lemma 2.4. *Let p be an odd prime. If $(p-1)/2$ divides n then p divides $(k^{2n}-1)$, and if $(p-1)/2$ does not divide n then p will not divide $(k^{2n}-1)$.*

Proof. Let $n = (p-1)\cdot q/2$ where q is an integer. Since k is a generator of $(\mathbb{Z}/p^2)^*$, $(k, p^2) = 1$ which implies $(k, p) = 1$. By Fermat's theorem, $k^{p-1} \equiv 1 (\operatorname{mod} p)$. It follows that $k^{(p-1)\cdot q} \equiv 1 (\operatorname{mod} p)$ or $k^{2n} \equiv 1 (\operatorname{mod} p)$, thus p divides $(k^{2n}-1)$. Since k generates $(\mathbb{Z}/p^2)^*$ which is a group of order $p(p-1)$, $k^{p(p-1)} \equiv 1 (\operatorname{mod} p^2)$. To show that k also generates $(\mathbb{Z}/p)^*$, let $k^m \equiv 1 (\operatorname{mod} p)$ for $m < (p-1)$. It follows that $k^{mp} \equiv 1 (\operatorname{mod} p^2)$ which is a contradiction. Hence $k^{p-1} \equiv 1 (\operatorname{mod} p)$ implies that k is a generator of $(\mathbb{Z}/p)^*$. Now if $(p-1)/2$ does not divide n then $2n$ is not a multiple of $(p-1)$. Hence $k^{2n} \not\equiv 1 (\operatorname{mod} p)$ or p does not divide $(k^{2n}-1)$. □

Proof of the theorem. (Sufficient part) By [14], if $p = 3$ and $r = 1$, $p_n \bmod p^r$ is homotopy invariant. An alternative proof is given in [12, Theorem B].

(Necessary part) It is sufficient to prove that $p_n \bmod p$ is not homotopy invariant if p is an odd prime different from 3. In fact, we shall see that it suffices to prove that $p_1 \bmod p$, $p \neq 3$ is not homotopy invariant.

Applying Lemma 2.3 for $n = 1$, we get $(\Psi^k - 1)^* L_1 = (k^2 - 1) L_1$. Now since $L_1 = \frac{1}{3} p_1$, if we take coefficients in $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ we have $(\Psi^k - 1)^* p_1 = (k^2 - 1) p_1$. Lemma 2.4 for $n = 1$ implies that 3 will divide $(k^2 - 1)$ and no odd prime different from 3 will divide $(k^2 - 1)$. Since $\mathbb{Z}_{(p)} \otimes \mathbb{Z}/p \cong \mathbb{Z}/p$, for $p \neq 3$, $(\Psi^k - 1)^*(p_1 \bmod p) \neq 0$. Now

$$\begin{aligned} v^*(p_1 \bmod p) &= \big(1 \otimes (\Psi^k - 1)^*\big) m^*(p_1 \bmod p) \\ &= \big(1 \otimes (\Psi^k - 1)^*\big)(p_1 \bmod p \otimes 1 + 1 \otimes p_1 \bmod p) \\ &= (p_1 \bmod p) \otimes 1 + 1 \otimes (\Psi^k - 1)^*(p_1 \bmod p) \\ &\neq (p_1 \bmod p) \otimes 1. \end{aligned}$$

By the necessary and sufficient condition, this implies that $p_1 \bmod p$ is not homotopy invariant. In general

$$\begin{aligned} v^*(p_n \bmod p) &= (p_n \bmod p) \otimes 1 + (p_{n-1} \bmod p) \otimes (\Psi^k - 1)^*(p_1 \bmod p) \\ &\quad + \sum_{i+j=n,\ j>1} (p_i \bmod p) \otimes (\Psi^k - 1)^*(p_j \bmod p) \\ &\neq (p_n \bmod p) \otimes 1 \quad \big(\text{since } (\Psi^k - 1)^*(p_1 \bmod p) \neq 0\big). \end{aligned}$$

Again, using the necessary and sufficient condition for $p_n \bmod p$ we get the result. □

Remark 2.5. From the above argument it follows that the universal integral Pontrjagin classes and their multiples are not homotopy invariant.

Theorem 2.6. *Hirzebruch classes L_n are not homotopy invariant.*

Proof. By Lemma 2.3, we have

$$(\Psi^k - 1)^* L_n = \sum \{\alpha_{i_1 i_2 \cdots i_r}(k^{2i_1} - 1) L_{i_1}^{t_1} L_{i_2}^{t_2} \cdots L_{i_r}^{t_r} + \cdots + \alpha_{i_r i_1 \cdots i_{r-1}}(k^{2i_r} - 1) L_{i_1}^{t_1} L_{i_2}^{t_2} \cdots L_{i_r}^{t_r}\}$$

where

$$\alpha_{i_1 \cdots i_r} = \begin{cases} 1 & \text{if } i_1, i_2, \ldots, i_r = n, 0, \ldots, 0, \\ (-1)^{(n+1)} & \text{if } i_1, i_2, \ldots, i_r = 1, 1, \ldots, 1. \end{cases}$$

Since by definition of k, $k \neq 1$, $(k^{2n} - 1) \neq 0$, implying $(\Psi^k - 1)^* L_n \neq 0$. Hence by Proposition 2.2 $v^* L_n \neq L_n \otimes 1$. Now by applying the necessary and sufficient condition for homotopy invariance we get the result. □

Acknowledgments

The author is supported by the University Grants Commission as Research Scientist A. The author wishes to thank Prof. B.L. Sharma for his help in the preparation of this paper and the referees for their useful comments and suggestions.

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