Integral means of analytic functions

Shigeyoshi Owa, Tadayuki Sekine

Abstract

For analytic functions \( f(z) \) and \( g(z) \) which satisfy the subordination \( f(z) \prec g(z) \), J.E. Littlewood [Proc. London Math. Soc. 23 (1925) 481–519] has shown some interesting results for integral means of \( f(z) \) and \( g(z) \). The object of the present paper is to derive some applications of integral means by J.E. Littlewood. We also show interesting examples for our theorems.

Keywords: Integral means; Analytic function; Subordination; Starlike; Convex

1. Introduction

Let \( A_n \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\})
\]

that are analytic in the open unit disk \( U = \{z \in \mathbb{C} \mid |z| < 1\} \). Let \( S^*_n(\alpha) \) be the subclass of \( A_n \) consisting of all functions \( f(z) \) satisfying...
Re\left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}) \tag{1.2}

for some \( \alpha \) \((0 \leq \alpha < 1)\). A function \( f(z) \) in \( S^*_n(\alpha) \) is said to be starlike of order \( \alpha \) in \( \mathbb{U} \).

Let \( K_n(\alpha) \) denote the subclass of \( A_n \) consisting of functions \( f(z) \) which satisfy

\[
\operatorname{Re}\left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}) \tag{1.3}
\]

for some \( \alpha \) \((0 \leq \alpha < 1)\). A function \( f(z) \) belonging to \( K_n(\alpha) \) is called a convex function of order \( \alpha \) in \( \mathbb{U} \). Note that \( f(z) \in K_n(\alpha) \) if and only if \( zf''(z) \in S^*_n(\alpha) \).

For the classes \( S^*_n(\alpha) \) and \( K_n(\alpha) \), Chatterjea [1] (also see Srivastava, Owa, and Chatterjea [10]) has given the following results.

**Theorem A.** If a function \( f(z) \in A_n \) satisfies

\[
\sum_{k=n+1}^{\infty} (k-\alpha)|a_k| \leq 1-\alpha \tag{1.4}
\]

for some \( \alpha \) \((0 \leq \alpha < 1)\), then \( f(z) \in S^*_n(\alpha) \).

**Theorem B.** If a function \( f(z) \in A_n \) satisfies

\[
\sum_{k=n+1}^{\infty} k(k-\alpha)|a_k| \leq 1-\alpha \tag{1.5}
\]

for some \( \alpha \) \((0 \leq \alpha < 1)\), then \( f(z) \in K_n(\alpha) \).

**Remark.** As already observed earlier by Srivastava, Owa, and Chatterjea [10, p. 117], Theorems A and B would follow immediately from some results of Silverman [8, p. 110, Theorem 2; p. 111, Corollary 2] by merely setting the first \( n-1 \) coefficients in Silverman’s results equal to 0.

For analytic functions \( f(z) \) and \( g(z) \), the function \( f(z) \) is said to be subordinate to \( g(z) \) in \( \mathbb{U} \) if there exists a function \( w(z) \) analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = g(w(z)) \). We denote this subordination by

\[ f(z) \prec g(z) \]

(cf. Duren [2]). For subordinations, Littlewood [4] has given the following integral mean.

**Theorem C.** If \( f(z) \) and \( g(z) \) are analytic in \( \mathbb{U} \) with \( f(z) \prec g(z) \), then, for \( \mu > 0 \) and \( z = re^{i\theta} \)(0 < r < 1),

\[
\int_{0}^{2\pi} |f(z)|^\mu d\theta \leq \int_{0}^{2\pi} |g(z)|^\mu d\theta.
\]
Applying Theorem C by Littlewood [4], Silvermann [9], Kim and Choi [3], Sekine, Tsurumi, and Srivastava [6], and Owa et al. [5] have considered some interesting properties for integral means of analytic functions. More recently, Sekine et al. [7] have discussed the integral means inequalities for fractional derivatives of some general subclasses of analytic functions \( f(z) \) in the open unit disk \( U \). In the present paper, we discuss some conditions of coefficients for integral means.

2. Integral means for \( f(z) \) and \( g(z) \)

In this section, we discuss the integral means for \( f(z) \in \mathcal{A}_n \) and \( g(z) \) defined by

\[
g(z) = z + b_j z^j + b_{2j-1} z^{2j-1} \quad (j \geq n + 1).
\]

(2.1)

Our first result for integral means is contained in the following theorem.

**Theorem 2.1.** Let \( f(z) \in \mathcal{A}_n \) and \( g(z) \) be given by (2.1). If \( f(z) \) satisfies

\[
\sum_{k=n+1}^{\infty} |a_k| \leq |b_{2j-1}| - |b_j| \quad (|b_j| < |b_{2j-1}|),
\]

(2.2)

then, for \( \mu > 0 \) and \( z = re^{i\theta} \) (0 < \( r < 1 \)),

\[
\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta.
\]

(2.3)

**Proof.** By putting \( z = re^{i\theta} \) (0 < \( r < 1 \)), we see that

\[
\int_0^{2\pi} |f(z)|^\mu d\theta = r^{\mu \int_0^{2\pi} \left| 1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \right|^\mu d\theta}
\]

and

\[
\int_0^{2\pi} |g(z)|^\mu d\theta = r^{\mu \int_0^{2\pi} \left| 1 + b_j z^{j-1} + b_{2j-1} z^{2j-2} \right|^\mu d\theta}.
\]

Applying Theorem C, we have to show that

\[
1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \prec 1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}.
\]

Let us define the function \( w(z) \) by

\[
1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} = 1 + b_j \left( w(z) \right)^{j-1} + b_{2j-1} \left( w(z) \right)^{2j-2}.
\]
or, by
\[ b_{2j-1}(w(z))^{2j-2} + bj(w(z))^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}. \]  
(2.4)

Since, for \( z = 0 \),
\[ \left( w(0) \right)^{j-1} \left( b_{2j-1}(w(0))^{j-1} + bj \right) = 0, \]
there exists an analytic function \( w(z) \) in \( U \) such that \( w(0) = 0 \).

Next, we prove the analytic function \( w(z) \) satisfies \( |w(z)| < 1 \) for \( z \in U \) for
\[ \sum_{k=n+1}^{\infty} |a_k| \leq |b_{2j-1}| - |bj| \quad (|bj| < |b_{2j-1}|). \]

By the equality (2.4), we know that
\[ |b_{2j-1}(w(z))^{2j-2} + bj(w(z))^{j-1}| \leq \left| \sum_{k=n+1}^{\infty} a_k z^{k-1} \right| < \sum_{k=n+1}^{\infty} |a_k|, \]
for \( z \in U \), hence,
\[ |b_{2j-1}| |w(z)|^{2j-2} - |bj| |w(z)|^{j-1} - \sum_{k=n+1}^{\infty} |a_k| < 0. \]  
(2.5)

Letting \( t = |w(z)|^{j-1} \) (\( t > 0 \)) in (2.5), we define the function \( G(t) \) by
\[ G(t) = |b_{2j-1}| t^2 - |bj| t - \sum_{k=n+1}^{\infty} |a_k| \quad (t > 0). \]

If \( G(1) \geq 0 \), then we have \( t < 1 \) for \( G(t) < 0 \). Therefore, for \( |w(z)| < 1 \) \( (z \in U) \), we need
\[ G(1) = |b_{2j-1}| - |bj| - \sum_{k=n+1}^{\infty} |a_k| \geq 0, \]
that is,
\[ \sum_{k=n+1}^{\infty} |a_k| \leq |b_{2j-1}| - |bj|. \]

Consequently, if the inequality (2.2) holds true, there exists an analytic function \( w(z) \) with \( w(0) = 0 \), \( |w(z)| < 1 \) \( (z \in U) \) such that \( f(z) = g(w(z)) \). This completes the proof of Theorem 2.1. \( \Box \)

**Corollary 2.1.** Let \( f(z) \in A_n \) and \( g(z) \) be given by (2.1). If \( f(z) \) satisfies (2.2), then, for
\( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) \( (0 < r < 1) \),
\[ \int_{0}^{2\pi} |f(z)|^\mu d\theta \leq 2\pi r^\mu \left\{ 1 + |bj|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right\}^{\mu/2} \]
\[ < 2\pi \left\{ 1 + |bj|^2 + |b_{2j-1}|^2 \right\}^{\mu/2}. \]  
(2.6)
Further, we have that \( f(z) \in H^p(U) \) for \( 0 < p \leq 2 \), where \( H^p \) denotes the Hardy space (cf. Duren [2]).

**Proof.** Since,
\[
\int_0^{2\pi} |g(z)|^\mu d\theta = \int_0^{2\pi} |z|^\mu |1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}|^\mu d\theta,
\]
applying Hölder’s inequality for \( 0 < \lambda < 2 \), we obtain that
\[
\int_0^{2\pi} |g(z)|^\mu d\theta \leq \left( \int_0^{2\pi} (|z|^\mu)^{(2-\mu)/2} d\theta \right)^{2-\mu/2} \left( \int_0^{2\pi} \left| 1 + b_j z^{j-1} + b_{2j-1} z^{2j-2} \right|^2 d\theta \right)^{\mu/2}
\]
\[
= \left( 2\pi r^{2\mu/(2-\mu)} \right)^{(2-\mu)/2} \left( 2\pi \left( 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right) \right)^{\mu/2}
\]
\[
= 2\pi r^\mu \left( 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right)^{\mu/2}
\]
\[
< 2\pi \left( 1 + |b_j|^2 + |b_{2j-1}|^2 \right)^{\mu/2}.
\]

Further, it is easy to see that, for \( \mu = 2 \),
\[
\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left( 1 + |b_j|^2 r^{2j-1} + |b_{2j-1}|^2 r^{4(j-1)} \right)
\]
\[
< 2\pi \left( 1 + |b_j|^2 + |b_{2j-1}|^2 \right).
\]

From the above, we also have that, for \( 0 < \mu \leq 2 \),
\[
\sup_{z \in U} \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^\mu d\theta < \left( 1 + |b_j|^2 + |b_{2j-1}|^2 \right)^{\mu/2} < \infty,
\]
which observe that \( f(z) \in H^2(U) \). Noting that \( H^q \subset H^p \) (\( 0 < p < q < \infty \)), we complete the proof. \( \square \)

**Example 2.1.** Let \( f(z) \in A_n \) satisfy the coefficient inequality (1.4) in Theorem A and
\[
g(z) = z + \frac{n}{n + 1 - \alpha} \varepsilon z^j + \delta z^{2j-1} \quad (|\varepsilon| = |\delta| = 1)
\]
with \( 0 \leq \alpha < 1 \). Then \( b_j = (n\varepsilon)/(n + 1 - \alpha) \) and \( b_{2j-1} = \delta \).
By virtue of (1.4), we observe that
\[ \sum_{k=n+1}^{\infty} |a_k| \leq \frac{1 - \alpha}{n + 1 - \alpha} = 1 - \frac{n}{n + 1 - \alpha} = |b_{2j-1}| - |b_j|. \]

Therefore, \( f(z) \) and \( g(z) \) satisfy the conditions in Theorem 2.1. Thus, we have, for \( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),
\[
\int_0^{2\pi} |f(z)|^\mu d\theta = 2\pi r^\mu \left\{ 1 + \left( \frac{n}{n + 1 - \alpha} \right)^2 r^2(j-1) + r^{4(j-1)} \right\}^{\mu/2} < 2\pi \left\{ 2 + \left( \frac{n}{n + 1 - \alpha} \right)^2 \right\}^{\mu/2}.
\]

Using the same technique as in the proof of Theorem 2.1, we derive the following theorem.

**Theorem 2.2.** Let \( f(z) \in A_n \) and \( g(z) \) be given by (2.1). If \( f(z) \) satisfies
\[
\sum_{k=n+1}^{\infty} k|a_k| \leq (2j - 1)|b_{2j-1}| - j|b_j| \quad (j|b_j| < (2j - 1)|b_{2j-1}|), \tag{2.8}
\]
then, for \( \mu > 0 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),
\[
\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |g'(z)|^\mu d\theta. \tag{2.9}
\]

Further, with the help of Hölder’s inequality, we have

**Corollary 2.2.** Let \( f(z) \in A_n \) and \( g(z) \) be given by (2.1). If \( f(z) \) satisfies (2.8), then, for \( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),
\[
\int_0^{2\pi} |f'(z)|^\mu d\theta \leq 2\pi \left\{ 1 + j^2|b_j|^2 r^{2(j-1)} + (2j - 1)^2|b_{2j-1}|^2 r^{4(j-1)} \right\}^{\mu/2} < 2\pi \left\{ 1 + j^2|b_j|^2 + (2j - 1)^2|b_{2j-1}|^2 \right\}^{\mu/2}. \tag{2.10}
\]

**Example 2.2.** Let \( f(z) \in A_n \) satisfy the coefficient inequality (1.5) in Theorem B and
\[
g(z) = z + \frac{ne}{j(n + 1 - \alpha)} z^j + \frac{\delta}{2j - 1} z^{2j-1} \quad (|\varepsilon| = |\delta| = 1) \tag{2.11}
\]
with \( 0 \leq \alpha < 1 \). Then,
\[
b_j = \frac{ne}{j(n + 1 - \alpha)} \quad \text{and} \quad b_{2j-1} = \frac{\delta}{2j - 1}.
\]
Since
\[ \sum_{k=n+1}^{\infty} k|a_k| \leq \frac{1 - \alpha}{n + 1 - \alpha} = 1 - \frac{n}{n + 1 - \alpha} = (2j - 1)|b_{2j-1}| - j|b_j|, \]

if \( f(z) \) and \( g(z) \) satisfy the conditions in Theorem 2.2. Thus, by Corollary 2.2, we have, for \( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),

\[
\int_{0}^{2\pi} |f'(z)|^\mu d\theta = 2\pi \left\{ 1 + \left( \frac{n}{n + 1 - \alpha} \right)^2 r^{2(j-1)} + r^{4(j-1)} \right\}^{\mu/2} < 2\pi \left\{ 2 + \left( \frac{n}{n + 1 - \alpha} \right)^2 \right\}^{\mu/2}.
\]

3. Integral means for \( f(z) \) and \( h(z) \)

In this section, we introduce an analytic function \( h(z) \) given by
\[
h(z) = z + b_j z^j + b_{2j-1} z^{2j-1} + b_{3j-2} z^{3j-2} \quad (j \geq n + 1).
\]

\[ (3.1) \]

**Theorem 3.1.** Let \( f(z) \in A_n \) and \( h(z) \) be given by (3.1), if \( f(z) \) satisfies
\[
\sum_{k=n+1}^{\infty} |a_k| \leq |b_{3j-2}| - |b_{2j-1}| - |b_j| \quad (|b_j| + |b_{2j-1}| < |b_{3j-2}|),
\]

then, for \( \mu > 0 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),
\[
\int_{0}^{2\pi} |f(z)|^\mu d\theta \leq \int_{0}^{2\pi} |h(z)|^\mu d\theta \quad (\mu > 0).
\]

**Proof.** In the same way as in the proof of Theorem 2.1, we have to show that there exists an analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in \mathbb{U} \)) such that \( f(z) = h(w(z)) \).

Note that this function \( w(z) \) is defined by
\[
b_{3j-2} \left( w(z) \right)^{3j-3} + b_{2j-1} \left( w(z) \right)^{2j-2} + b_j \left( w(z) \right)^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}.
\]

Since, for \( z = 0 \),
\[
(w(0))^{j-1} \left( b_{3j-2} \left( w(0) \right)^{3j-3} + b_{2j-1} \left( w(0) \right)^{2j-2} + b_j \left( w(0) \right)^{j-1} + b_j \right) = 0,
\]

we consider \( w(z) \) such that \( w(0) = 0 \).

On the other hand, we have that
\[
|b_{3j-2}| |w(z)|^{3(j-1)} - |b_{2j-1}| |w(z)|^{2(j-1)} - |b_j| |w(z)|^{j-1} - \sum_{k=n+1}^{\infty} |a_k| < 0. \]
Putting \( t = |w(z)|^{j-1} \) \((t \geq 0)\), we define the function \( H(t) \) by

\[
H(t) = |b_{3j-2}|t^3 - |b_{2j-1}|t^2 - |b_j|t - \sum_{k=n+1}^{\infty} |a_k| \quad (t \geq 0).
\]

It follows that \( H(0) \leq 0 \), and

\[
H'(t) = 3|b_{3j-2}|t^2 - 2|b_{2j-1}|t - |b_j|.
\]

Since the discriminant of \( H'(t) = 0 \) is greater than 0, if \( H'(1) \geq 0 \), then \( t < 1 \) for \( H(t) < 0 \).

Therefore, we need the following inequality:

\[
H(1) = |b_{3j-2}| - |b_{2j-1}| - |b_j| - \sum_{k=n+1}^{\infty} |a_k| \geq 0,
\]

or

\[
\sum_{k=n+1}^{\infty} |a_k| \leq |b_{3j-2}| - |b_{2j-1}| - |b_j|.
\]

This completes the proof of Theorem 3.1.

**Corollary 3.1.** Let \( f(z) \in A_n \) and \( h(z) \) be given by (3.1). If \( f(z) \) satisfies (3.2), then, for \( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) \((0 < r < 1)\),

\[
2\pi \int_{0}^{2\pi} |f(z)|^{\mu} d\theta \leq 2\pi r^{\mu} \left( 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} + |b_{3j-2}|^2 r^{6(j-1)} \right)^{\mu/2}
\]

\[
< 2\pi \left( 1 + |b_j|^2 + |b_{2j-1}|^2 + |b_{3j-2}|^2 \right)^{\mu/2}.
\]

Further, we have that \( f(z) \in H^p(U) \) for \( 0 < p \leq 2 \).

**Example 3.1.** Let \( f(z) \in A_n \) satisfy the coefficient inequality (1.4) in Theorem A and \( h(z) \) be given by

\[
h(z) = z + \frac{nt}{n+1-\alpha} \varepsilon z^j + \frac{n(1-t)}{n+1-\alpha} \delta z^{2j-1} + \sigma z^{3j-2}
\]

\[(0 \leq t \leq 1, \ |\varepsilon| = |\delta| = |\sigma| = 1)\]

(3.7)

with \( 0 \leq \alpha < 1 \). Then

\[
b_j = \frac{nt}{n+1-\alpha} \varepsilon, \quad b_{2j-1} = \frac{n(1-t)}{n+1-\alpha} \delta, \quad \text{and} \quad b_{3j-2} = \sigma.
\]

In view of (1.4), we see that

\[
\sum_{k=n+1}^{\infty} |a_k| \leq \frac{1 - \alpha}{n+1-\alpha} = 1 - \frac{n(1-t)}{n+1-\alpha} - \frac{nt}{n+1-\alpha}
\]

\[
= |b_{3j-2}| - |b_{2j-1}| - |b_j|.
\]
This shows us that \( f(z) \) and \( h(z) \) satisfy the conditions in Theorem 3.1. Therefore, applying Corollary 3.1, we have, for \( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),

\[
\frac{2\pi}{\mu} \left| f(z) \right|^\mu d\theta = 2\pi r^\mu \left( 1 + \left( nt \frac{n(1-t)}{n+1-\alpha} r^{2(j-1)} + \frac{n(1-t)}{n+1-\alpha} r^{4(j-1)} + r^{6(j-1)} \right)^{\mu/2} \right.
\]

\[
< 2\pi r^\mu \left( 2 + (2j^2 - 2t - 1) n \frac{n}{n+1-\alpha} \right)^{\mu/2}.
\]

Finally, for the integral means of \( f'(z) \) and \( h'(z) \), we derive the following theorem.

**Theorem 3.2.** Let \( f(z) \in A_n \) and \( h(z) \) be given by (3.1). If \( f(z) \) satisfies

\[
\sum_{k=n+1}^{\infty} k|a_k| \leq (3j-2)|b_{3j-2}| - (2j-1)|b_{2j-1}| - j|b_j| \]

\[
(j|b_j| + (2j-1)|b_{2j-1}|(2j-1)| < (3j-2)|b_{3j-2}|),
\]

then, for \( \mu > 0 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),

\[
\frac{2\pi}{\mu} \left| f'(z) \right|^\mu d\theta \leq \frac{2\pi}{\mu} \left| h'(z) \right|^\mu d\theta.
\]

The proof of this theorem is similar to that of Theorem 2.2, hence we omit it.

**Corollary 3.2.** Let \( f(z) \in A_n \) and \( h(z) \) be given by (3.1). If \( f(z) \) satisfies (3.8), then, for \( 0 < \mu \leq 2 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)),

\[
\frac{2\pi}{\mu} \left| f'(z) \right|^\mu d\theta \leq \frac{2\pi}{\mu} \left| h'(z) \right|^\mu d\theta.
\]

**Example 3.2.** Let \( f(z) \in A_n \) satisfy the coefficient inequality (1.5) in Theorem B and \( h(z) \) be given by

\[
h(z) = z + nt \frac{n(1-t)}{j(n+1-\alpha)} \zeta^j + \frac{n(1-t)}{n+1-\alpha} \delta \zeta^{2j-1} + \frac{\sigma}{3j-2} \zeta^{3j-2}
\]

\[
(0 \leq t \leq 1, |\epsilon| = |\delta| = |\sigma| = 1)
\]

(3.11)
with $0 \leq \alpha < 1$. It follows that

$$
\begin{align*}
b_j &= \frac{nt\varepsilon}{j(n+1-\alpha)}, \quad b_{2j-1} = \frac{n(1-t)\delta}{(2j-1)(n+1-\alpha)}, \quad \text{and} \quad b_{3j-2} = \frac{\sigma}{3j-2}.
\end{align*}
$$

By the coefficient inequality (1.5), we obtain that

$$
\begin{align*}
\sum_{k=n+1}^{\infty} k|a_k| &\leq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} \\
&= (3j-2)|b_{3j-2}| - (2j-1)|b_{2j-1}| - j|b_j|.
\end{align*}
$$

This gives us that $f(z)$ and $h(z)$ satisfy the conditions in Theorem 3.2. Thus, applying Corollary 3.2, we see, for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$
\begin{align*}
\int_{0}^{2\pi} |f'(z)|^\mu d\theta &= 2\pi r^{\mu} \left\{ 1 + \left( \frac{nt}{n+1-\alpha} \right)^2 r^{2(j-1)} + \frac{n(1-t)}{(n+1-\alpha)} r^{4(j-1)} + r^{6(j-1)} \right\}^{\mu/2} \\
&< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left( \frac{n}{n+1-\alpha} \right)^2 \right\}^{\mu/2}.
\end{align*}
$$

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Appendix A

Applying Hölder’s inequality for analytic functions $F(z)$ and $G(z)$, we obtain, for $z = re^{i\theta}$ ($0 < r < 1$),

$$
\begin{align*}
\int_{a}^{b} |F(z)G(z)| \, d\theta &\leq \left( \int_{a}^{b} |F(z)|^p \, d\theta \right)^{1/p} \left( \int_{a}^{b} |G(z)|^q \, d\theta \right)^{1/q} \quad (A.1)
\end{align*}
$$

with $p > 1$ and $1/p + 1/q = 1$. Note that the inequality (4.1) gives

$$
\begin{align*}
\int_{0}^{2\pi} |F(z)|^p \, d\theta &\geq \frac{\left( \int_{0}^{2\pi} |F(z)G(z)| \, d\theta \right)^p}{\left( \int_{0}^{2\pi} |G(z)|^q \, d\theta \right)^{p/q}}. \quad (A.2)
\end{align*}
$$

Considering $p = \mu/2$, $q = \mu/(\mu - 2)$, and $\mu > 2$ in (4.2), we have, for $f(z)$ in the class $A_n$, 

\[ \int_0^{2\pi} |f(z)|^\mu \, d\theta = \int_0^{2\pi} \left( |f(x)|^2 \right)^{\mu/2} \, d\theta \geq \frac{\int_0^{2\pi} |f(z)|^2 \, d\theta}{\left( \int_0^{2\pi} d\theta \right)^{\mu/2}} \]
\[ = (2\pi)^{(2-\mu)/2} \left\{ 2\pi \left( r^2 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2k} \right) \right\}^{\mu/2} \]
\[ = 2\pi r\mu \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right)^{\mu/2}. \]

When \( \mu = 2 \), we also have that, for \( z = re^{i\theta} \) (\( 0 < r < 1 \)),
\[ \int_0^{2\pi} |f(z)|^2 \, d\theta = 2\pi r^2 \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right) < 2\pi \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 \right). \]

Thus, we conclude that

**Theorem 3.3.** Let \( f(z) \in A_n \) and \( \mu \geq 2 \). Then, for \( z = re^{i\theta} \) (\( 0 < r < 1 \)),
\[ \int_0^{2\pi} |f(z)|^\mu \, d\theta \geq 2\pi r^\mu \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right)^{\mu/2}. \]

**References**