On the Oscillation of Nonlinear Hyperbolic Equations

WU-TEH HSIANG and MAN KAM KWONG

Department of Mathematical Sciences,
Northern Illinois University, DeKalb, Illinois 60115

Submitted by Ky Fan

1. INTRODUCTION

We are interested in the oscillatory behavior of the solutions of the nonlinear hyperbolic problem in the first quadrant $Q = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}$

$$u_{xy}(x, y) + g(x, y)f(u(x, y)) = 0 \quad (x, y) \in Q,$$

$$u(x, 0) = \phi(x), \ u(0, y) = \psi(y).$$

(1.1)

The following assumptions are made:

$g: Q \to [0, \infty)$ is a continuous function;

$f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(u) > 0$ when $u > 0$ and $f$ is continuously differentiable in $[0, \infty)$; and

$\phi, \ \psi: [0, \infty) \to (0, \infty)$ are continuously differentiable non-increasing functions with $\phi' \leq 0$ and $\psi' \leq 0$.

The smoothness conditions on $g, f, \phi$ and $\psi$ are imposed so as to guarantee existence, uniqueness and continuous dependence on the coefficient $g$ and the initial conditions of the solution in any open rectangle $[0, a] \times [0, b]$ in which the solution remains positive. These results can be proved in the same way as in the theory of ordinary differential equations making use of an equivalent integral equation formulation.

In the theory of oscillation of differential equations, it is customary to assume also that $f(u) < 0$ when $u < 0$. This condition is essential if the solution is to have higher-order zeros or nodal lines, or if solutions with negative initial values are considered. Since in this paper we are primarily interested in first nodal lines of solutions with positive initial values, we find this condition unnecessary.

DEFINITION. We say that a solution $u$ of (1.1) changes sign if $u(x, y) = 0$ for some $(x, y) \in Q$. We point out that this need not imply that $u < 0$ somewhere in $Q$. 
In [8], a solution $u$ is called oscillatory if $u$ has arbitrarily large zeros, in other words, if $u$ changes sign in the exterior of any bounded set in $Q$. This seems at first sight to be a better generalization of the usual oscillatory property of ordinary differential equations than our definition. That this is only apparent, under the assumption that $\phi', \psi' \leq 0$, is shown in Corollary 2. A truly better but still unsatisfactory definition of oscillation is to require $u$ to have zeros $(x, y)$ for which both $x$ and $y$ are arbitrarily large. We shall discuss this further at the end of the section.

It is shown by Pagan [4] that in the linear case, $f(u) = u$, $u$ changes sign if $g(x, y) \geq k^2 > 0$ under weaker conditions on $\phi$ and $\psi$. Estimates on the location of the zeros are also given. In a later paper [5], Pagan extends the result to $g(x, y)$ being independent of $x$ or $y$. A Sturm-type comparison theorem for the linear problem is proved in [1]. The result proves to be a useful tool from which can be derived improved versions of Pagan's results under the stricter assumption that $\phi', \psi' \leq 0$. It has been noted recently by Kreith [3] that the comparison theorem can be deduced from the theory of hyperbolic differential inequality as expounded in Walter [6]. In [8], Yosida considers what is essentially the "superlinear" case in which $f$ in (1.1) is an odd convex function. A comparison theorem is also established comparing (1.1) to an associated ordinary differential equation. As corollaries, "oscillation" criteria are derived for the case $f(u) = u^\gamma$, when $u > 0$ and $\gamma$ any real number $\geq 1$. In an attempt to include both positive and negative initial values, Yosida restricts $\gamma$ to be a quotient of two odd integers to avoid the difficulty of defining $u^\gamma$ when $u$ is negative. This is in fact not necessary because the case of negative initial values can be reduced to one of positive values by changing the dependent variable from $u$ to $-u$ ($f$ is assumed odd in [8]). That some of the results in [8] are not the best possible is shown in [2], in which rather general criteria involving non-integrable $g$ are established for the very general non-linear problem (1.1). There, besides requiring no convexity condition on $f$, less restrictive assumptions on $\phi$ and $\psi$ are also imposed. However a counterexample is given in the same paper showing that the condition $\int_{Q} g(x, y) \, dx \, dy = \infty$ alone is not sufficient to guarantee that $u$ changes sign in the linear case.

In this paper our main result is the rather surprising fact that $\int_{Q} g(x, y) \, dx \, dy = \infty$ is both necessary and sufficient for $u$ to change sign for a class of non-linear functions $f$, including $f(u) = u^\gamma$, $0 < \gamma < 1$. This certainly reminds us of the well-known result on the oscillation of non-linear second-order differential equations due to Atkinson, Belohorec and others. See Wong [7] for references on this subject.

In the rest of this section we shall establish some facts concerning the first nodal line and discuss "higher nodal lines." In Section 2 we prove the necessity part of our main result. In Section 3 we first extend the comparison theorem proved in [1] to cover "sublinear" problems. We then state as an
application of the comparison theorem a criterion of changing sign, which is
needed in the proof of the sufficiency part of our main result to be given in
Section 4. The rest of Section 3 is devoted to some examples, in particular
improvements of Yosida's result [8, Corollary 4], for the linear case.

Suppose now that the solution \( u \) of \( (1.1) \) changes sign in \( Q \). We define the
extended real-valued function

\[
    r(x) = \inf \{ \bar{y} > 0 : u(x, \bar{y}) = 0 \}, \quad x > 0.
\]

We adopt the usual convention that the infimum of the empty set is \( \infty \).

**Theorem 1.** The function \( r \) as defined above is a non-increasing
function. Suppose furthermore that for any \( y > 0 \), either \( u'(y) < 0 \) or
\( g(x, y) \neq 0 \) for \( x \) in any open interval \( (0, \varepsilon), \varepsilon > 0 \), then \( r \) is a differentiable
function of \( x \) in \( \{ x : r(x) \neq \infty \} \).

**Proof:** Let \( x_1 < x_2 \), such that \( r(x_1) < \infty \). We need to show that
\( r(x_2) \leq r(x_1) \). Suppose the contrary, i.e., \( r(x_2) > r(x_1) \). By the continuity of \( u \),
the set \( \Gamma = \{ (x, y) : u(x, y) = 0 \} \) is closed in \( Q \). Thus the intersection of \( \Gamma \)
with the rectangle \( R = [x_1, x_2] \times [0, r(x_2)] \) is compact. It follows that there exists
a point \( (\bar{x}, \bar{y}) \in \Gamma \cap R \) such that \( \bar{y} = \min \{ y : (x, y) \in \Gamma \cap R \} \leq r(x_1) < r(x_2) \).
We must have \( x \neq x_2 \), otherwise by definition, \( r(x_2) \leq \bar{y} \), contradicting the
previous inequality. By definition, \( \bar{y} = r(\bar{x}) \) and \( u(x, y) \geq 0 \) in the rectangle
\( R_1 = [\bar{x}, x_2] \times [0, \bar{y}] \). From \( (1.1) \) it follows that \( u_{yy}(x, y) \leq 0 \) in \( R_1 \).

Integrating over \( R_1 \) yields

\[
    u(x_1, y_1) + \phi(\bar{x}) - u(\bar{x}, \bar{y}) - \phi(x_2) \leq 0
\]

from which we obtain

\[
    u(x_2, \bar{y}) \leq \phi(x_2) - \phi(\bar{x}) \leq 0,
\]

a contradiction.

For the proof of the second part, observe that the additional hypothesis
implies that \( u_y(x, r(x)) < 0 \). Since \( y = r(x) \) satisfies \( u(x, r(x)) = 0 \), the
conclusion is an immediate consequence of the implicit function theorem.

**Corollary 2.** If \( u(x_0, y_0) = 0 \) for some \( (x_0, y_0) \in Q \), then given any
\( x_1 > x_0, y_2 > y_0 \), there exists \( y_1(x_2) \) such that \( u(x_1, y_1) = 0 \) \( \forall u(x_2, y_2) = 0 \).

**Proof:** Without loss of generality we may assume that \( r(x_0) = y_0 \). By the
monotonicity of \( r \), \( r(x_1) < y_0 < \infty \). Simply take \( y_1 = r(x_1) \). The part
concerning \( (x_2, y_2) \) is proved analogously by changing the roles of \( x \) and \( y \).

**Corollary 3.** For each fixed \( x \), \( u(x, y) \) is a non-negative non-increasing
function of \( y \) for \( 0 \leq y \leq r(x) \). Hence if \( u(x, y) \geq 0 \) in a rectangle \( R = [0, a] \times [0, b] \), \( u \) attains its minimum in \( R \) at the corner \((a, b)\).

**Proof.** Since \( u(x, y) \geq 0 \) in the rectangle \( R_1 = [0, x] \times [0, r(x)] \), \( u_{xy} \leq 0 \) in \( R_1 \). Integrating over \( R_1 \) yields the conclusion.

The curve \( y = r(x) \) may be called the first nodal line of \( u \). The solution of the simple problem \( u_{xy} + u = 0, \phi(x) \equiv 1 \equiv \psi(y) \) has an infinite number of disconnected curves of zeros, or nodal lines. This is analogous to the existence of an infinite number of zeros of an oscillating solution of an ordinary differential equation. The existence of higher nodal lines thus seems to be the most appropriate extension of the concept of oscillation to hyperbolic problems. Unfortunately no qualitative results on higher nodal lines are known to date. In general the zero set of a solution \( u \) of (1.1), even when \( f(u) = u \), need not consist solely of disconnected simple curves. The possibility of having isolated zeros, closed curves of zeros, branch points, or more complicated patterns has not been excluded. Indeed very little is known. The requirement of having zeros \((x, y)\) with both \( x \) and \( y \) large by no means guarantees the existence of higher nodal lines because the solution \( u \) may very well have only two nodal lines, the second one looking like the graph of \( y = x + 1/x \). The situation is even more complicated when the initial conditions are not necessarily non-decreasing. In the present paper we restrict our attention only to the first nodal line.

2. **S-LINEAR AND S-LINEAR PROBLEMS: A NECESSARY CONDITION**

**Definition.** Problem (1.1) is said to be \( s \)-linear (\( S \)-linear) iff \( (f(u)/u \) is a non-increasing (non-decreasing) function of \( u \) in \((0, \infty)\).

For example, with \( f(u) = u^\gamma \), (1.1) is \( s \)-linear if \( 0 < \gamma \leq 1 \) and is \( S \)-linear if \( \gamma \geq 1 \).

**Theorem 4.** Suppose that (1.1) is either \( s \)-linear or \( S \)-linear and that
\[
\inf_{u>0} (f(u)/u) = 0.
\]
Then a necessary condition that the solutions of (1.1) always change sign with all choices of \( \phi \) and \( \psi \) is that
\[
\int_0^1 g(x, y) \, dx \, dy = \infty.
\]

**Proof.** Suppose the contrary, that \( A = \int_0^1 g(x, y) \, dx \, dy < \infty \). Let us first treat the \( S \)-linear case. Let \( \phi(x) \equiv \alpha \equiv \psi(y) \) be constant functions such that \( Af(\alpha)/\alpha < \frac{1}{2} \). We claim that the solution of (1.1) with this choice of \( \phi \) and \( \psi \) does not change sign. Indeed we claim that \( u(x, y) > \alpha/2 \) for all \((x, y) \in Q\). Suppose that this is not true. Then (see Lemma 1 of [1]) there exists \((\bar{x}, \bar{y}) \in Q\) such that \( u(\bar{x}, \bar{y}) = \alpha/2 \) but \( u(x, y) > \alpha/2 \) for all \((x, y) \in [0, \bar{x}] \times [0, \bar{y}] - \{(\bar{x}, \bar{y})\}\). Integrating over \([0, \bar{x}] \times [0, \bar{y}]\) gives
\[
\alpha - u(\tilde{x}, \tilde{y}) = \int_0^{\tilde{y}} \int_0^{\tilde{x}} g(x, y) f(u(x, y)) \, dx \, dy \\
= \int_0^{\tilde{y}} \int_0^{\tilde{x}} g(x, y) \frac{f(u(x, y))}{u(x, y)} u(x, y) \, dx \, dy \\
\leq A \frac{f(\alpha)}{\alpha} \alpha < \frac{1}{2} \alpha,
\]
contradicting the assumption that \( u(\tilde{x}, \tilde{y}) = \alpha/2 \). In the above estimation we used the S-linearity of (1.1) and Corollary 3.

In the s-linear case we have \( \lim_{u \to \infty} f(u)/u = 0 \). We can therefore choose \( \alpha \) so large that \( 2Af(\alpha/2)/\alpha < 1/2 \). As in the S-linear case, we choose \( \phi(x) \equiv \phi(x) \equiv \psi(y) \) and claim that \( u(x, y) > \alpha/2 \) for all \( (x, y) \in Q \). If this is false, then there exists \( (\tilde{x}, \tilde{y}) \in Q \) such that \( u(\tilde{x}, \tilde{y}) = \alpha/2 \) but \( u(x, y) > \alpha/2 \) for all \( (x, y) \in [0, \tilde{x}] \times [0, \tilde{y}] - \{ (\tilde{x}, \tilde{y}) \} \). Integrating over \( [0, \tilde{x}] \times [0, \tilde{y}] \) yields

\[
\alpha - u(\tilde{x}, \tilde{y}) = \int_0^{\tilde{y}} \int_0^{\tilde{x}} g(x, y) \frac{f(u(x, y))}{u(x, y)} u(x, y) \, dx \, dy \\
\leq A \frac{f(\alpha/2)}{\alpha/2} \alpha < \frac{1}{2} \alpha,
\]
a contradiction as before. This completes the proof of the theorem.

Examples of \( f \) that satisfy the hypotheses of the theorem are \( f(u) = u^g, g > 0, \gamma \neq 1 \), and \( u(\log(u + 1))^\gamma, \gamma \in (-\infty, \infty), \gamma \neq 0 \).

3. A COMPARISON THEOREM FOR S-LINEAR CASE:  
A SUFFICIENT CONDITION AND EXAMPLES

The following comparison theorem for linear problems is a special case of Theorem 1 in \[1\].

**THEOREM.** Let \( u \) be the solution of \((1.1)\) with \( f(u) = u^\gamma \), \( \gamma > 0, \gamma \neq 1 \), and \( u(\log(u + 1))^\gamma, \gamma \in (-\infty, \infty), \gamma \neq 0 \).

\[
v_{xy}(x, y) + \tilde{g}(x, y) v(x, y) = 0 \quad (x, y) \in Q, \\
v(x, 0) = \tilde{\phi}(x), v(0, y) = \tilde{\psi}(y) \quad x \geq 0, y \geq 0.
\]

The following inequalities are assumed:

\[
g(x, y) \geq \tilde{g}(x, y) > 0 \quad (x, y) \in Q, \\
\phi'(t)/\phi(t) \leq \tilde{\phi}'(t)/\tilde{\phi}(t) \leq 0 \quad t \geq 0,
\]
and

\[
\psi'(t)/\psi(t) \leq \tilde{\psi}'(t)/\tilde{\psi}(t) \leq 0 \quad t \geq 0.
\]
Then if \( u(x, y) > 0 \) for all \((x, y)\) in a rectangle \([0, a] \times [0, b]\), it follows that \( v(x, y) > 0 \) for \((x, y)\) in the same rectangle.

**Remark 1.** Condition (3.2) can be replaced by the weaker condition: for any \( x \) and \( y \geq 0 \)

\[
\int_0^y g(x, \bar{y}) \, d\bar{y} \geq \int_0^y \bar{g}(x, \bar{y}) \, d\bar{y}
\]

and

\[
\int_0^x g(\bar{x}, y) \, d\bar{x} \geq \int_0^x \bar{g}(\bar{x}, y) \, d\bar{x}.
\]  

(3.4)

Condition (3.2) has been used twice in the proof in [1] of the comparison theorem, once in establishing Lemma 2 and a second time in the very last step of the proof. Closer examination of the arguments will reveal that their validity are not affected if (3.2) is replaced by (3.4). However the same remark does not apply in the s-linear case treated in Theorem 5.

**Remark 2.** Although not stated explicitly in the theorem, the inequalities \( u(x, y)/u(x, y) \leq v(x, y)/v(x, y) \) and \( u(x, y)/u(x, y) \leq v(x, y)/v(x, y) \) for \((x, y)\) in \([0, a] \times [0, b]\) have been established in the proof in [1]. If we now integrate the first inequality along the straight line from \((0, 0)\) to \((x, 0)\) and then integrate the second one along the straight line from \((x, 0)\) to \((x, y)\), we obtain \( u(x, y)/\phi(0) \leq v(x, y)/\tilde{\phi}(0) \). Thus under the additional hypothesis \( \phi(0) \leq (\leq \phi(0) \), we can conclude that \( u(x, y) \leq (\leq v(x, y) \) for \((x, y)\) in \([0, a] \times [0, b]\).

**THEOREM 5** (Comparison Theorem for the s-linear case). Let \( u \) be a solution of (1.1) which is assumed to be s-linear and \( v \) be a solution of a similar problem

\[
v_{xy}(x, y) + \tilde{g}(x, y)f(v(x, y)) = 0 \quad (x, y) \in \mathcal{Q},
\]

(3.5)

\[
v(x, 0) = \tilde{\phi}(x), \quad v(0, y) = \tilde{\psi}(y) \quad x \geq 0, \, y \geq 0.
\]

The following inequalities are assumed:

\[
g(x, y) \geq \tilde{g}(x, y) > 0 \quad (x, y) \in \mathcal{Q},
\]

(3.2)

\[
\phi'(t)/\phi(t) \leq \tilde{\phi}'(t)/\tilde{\phi}(t) \leq 0 \quad t \geq 0,
\]

(3.3)

\[
\psi'(t)/\psi(t) \leq \tilde{\psi}'(t)/\tilde{\psi}(t) \leq 0 \quad t \geq 0,
\]

and

\[
\phi(0) = \psi(0) \leq \tilde{\psi}(0) = \tilde{\phi}(0).
\]

(3.6)
Then if $u(x, y) > 0$ in a rectangle $R = [0, a] \times [0, b]$, it follows that for $(x, y) \in R$

$$u(x, y) \leq v(x, y), \quad (3.7)$$

$$u_x(x, y)/u(x, y) \leq v_x(x, y)/v(x, y), \quad (3.8)$$

and

$$u_y(x, y)/u(x, y) \leq v_y(x, y)/v(x, y). \quad (3.9)$$

**Proof.** A continuity argument (similar ones have been employed in [1]) enables us to assume without loss of generality that $\phi(0) < \tilde{\phi}(0)$ and $\psi(0) < \tilde{\psi}(0)$. We claim that it then follows that $u(x, y) < v(x, y)$ in $R$. Suppose the contrary. Either using Lemma 1 in [1] or directly we can show without difficulty that there exists a point $(\bar{x}, \bar{y}) \in R$ such that $u(\bar{x}, \bar{y}) = v(\bar{x}, \bar{y})$ but $u(x, y) < v(x, y)$ for all $(x, y) \in R - \{(\bar{x}, \bar{y})\}$. That $\bar{x} \neq 0$ follows from the fact that

$$u(0, y) = \psi(y) = \psi(0) \exp \left\{ \int_0^y \left( \psi'(t)/\psi(t) \right) dt \right\}$$

$$< \tilde{\psi}(0) \exp \left\{ \int_0^y \left( \tilde{\psi}'(t)/\tilde{\psi}(t) \right) dt \right\} = \tilde{\psi}(y) = v(0, y).$$

Similarly $\bar{y} \neq 0$. We can now regard $u$ as the solution of the "linear" hyperbolic equation $u_{xy}(x, y) + \left[ g(x, y)f(u(x, y))/u(x, y) \right] u(x, y) = 0$ with potential $gf(u)/u$. Similarly $v$ is the solution of a similar "linear" equation with potential $g\tilde{f}(v)/v$. Since the original equations are s-linear, $f(u)/u \geq f(v)/v$ in $[0, \bar{x}] \times [0, \bar{y}]$, implying that $gf(u)/u \geq g\tilde{f}(v)/v$ in $[0, \bar{x}] \times [0, \bar{y}]$. Thus the Comparison Theorem for the linear case applies and (3.8), (3.9) hold in $[0, \bar{x}] \times [0, \bar{y}]$. By integrating these inequalities just as in Remark 2, we obtain $u(\bar{x}, \bar{y}) < v(\bar{x}, \bar{y})$, a contradiction. A repetition of the same arguments, but now over the rectangle $R$, completes the proof of the theorem.

As an application of the Comparison Theorem we give the following criterion for changing signs.

**Theorem 6.** Suppose $u$ is the solution of (1.1) which is assumed to be s-linear. Denote $a = \phi(0)$. If there exists a point $(\bar{x}, \bar{y}) \in Q$ such that

$$\left( \int_0^\infty \int_{\bar{y}}^{\bar{x}} g(x, y) \, dx \, dy \right) \left( \int_0^\infty \int_{\bar{y}}^{\bar{x}} g(x, y) \, dx \, dy \right) > \frac{\alpha^2}{f^2(a)}, \quad (3.10)$$

then $u$ changes sign.
Denote \( R_1 = [0, \bar{x}] \times [\bar{y}, \infty] \) and \( R_2 = [\bar{x}, \infty) \times [0, \bar{y}] \). Observe that (3.10) is equivalent to
\[
1 + \frac{f(\alpha)}{\alpha} \left[ \int_{R_1} g(x, y) \, dx \, dy \right]^{-1} + \left[ \int_{R_2} g(x, y) \, dx \, dy \right]^{-1} < 1.
\]
Define
\[
\tilde{g}(x, y) = g(x, y) \quad (x, y) \in R_1 \cup R_2
= 0 \quad \text{otherwise}.
\]
By the Comparison Theorem, we see that it suffices to prove that the solution of the following problem
\[
\begin{align*}
uxy + \tilde{g}(u) &= 0 \quad (x, y) \in Q, \\
v(x, 0) &= \alpha = v(0, y) \quad x > 0, y \geq 0
\end{align*}
\]
changes sign. Since \( \tilde{g} = 0 \) in \([0, \bar{x}] \times [0, \bar{y}] \cup [\bar{x}, \infty) \times [\bar{y}, \infty)\), it is easy to see that
\[
v(x, y) = \alpha \quad (x, y) \in [0, \bar{x}] \times [0, \bar{y}]
= v(\bar{x}, y) + v(x, \bar{y}) - \alpha \quad (x, y) \in [\bar{x}, \infty) \times [\bar{y}, \infty).
\]
If \( v \) changes sign already in \( R_1 \cup R_2 \), there is nothing more to prove. Hence we assume that \( v(x, y) > 0 \) in \( R_1 \cup R_2 \). From (3.12) we see that \( v \) will change sign if
\[
\lim_{x \to \infty} v(\bar{x}, y) + \lim_{x \to \infty} v(x, \bar{y}) < \alpha.
\]
Let us estimate \( \lim_{x \to \infty} v(x, y) \). Take \( x_1 > \bar{x} \). By Corollary 3, \( v(x, y) \) attains its minimum in \( R_3 = [\bar{x}, x_1] \times [0, \bar{y}] \) at the corner \((x_1, \bar{y})\). Let \( \beta = u(x_1, \bar{y}) \). Integrating the differential equation (3.11) over \([\bar{x}, x_1] \times [0, \bar{y}]\) gives
\[
\alpha - \beta = \int_{R_3} g(x, y) f(v(x, y)) \, dx \, dy.
\]
The \( s \)-linearity of (3.11) implies that
\[
f(v(x, y)) = \frac{f(v(x, y))}{v(x, y)} v(x, y) \geq \frac{f(\alpha)}{\alpha} \beta.
\]
Estimating the right-hand side of (3.14) using this inequality and then solving for $\beta$, we have

$$\beta \leq \alpha \left[ 1 + \frac{f(\alpha)}{\alpha} \iint_{R_1} g(x, y) \, dx \, dy \right]^{-1}. \quad (3.15)$$

Thus

$$\lim_{x \to \infty} v(x, \bar{y}) \leq \alpha \left[ 1 + \frac{f(\alpha)}{\alpha} \iint_{R_1} g(x, y) \, dx \, dy \right]^{-1}.$$

A similar estimate holds for $\lim_{y \to \infty} v(\bar{x}, y)$. It is now easy to see that the hypothesis of the theorem implies (3.12) and thus $v$ changes sign.

**Example 1.** In [8] potentials of the form

$$g(x, y) \geq p(x + y) \quad (3.16)$$

are considered, where $p(t)$ is a continuous function of $t \geq 0$. The following "oscillation" criterion was established for the linear case, $f(u) = u$, along with similar criteria for the superlinear cases, $f(u) = u^y$, $y \geq 1$: $u$ changes sign if

$$\int_0^\infty \psi(t) \, p(t) \, dt = \infty, \quad (3.17)$$

where

$$\psi(t) = t^2 \left\{ \left[ l_0(t) \right]^u \prod_{i=0}^m l_i(t)^{l_i(t)} \right\}^{-1}. \quad (3.18)$$

for some positive number $\mu$, some non-negative integer $m$, $l_0(t) = t$ and $l_i(t) = \log(l_{i-1}(t) + 1)$.

In fact a more general result follows from Theorem 6. If (1.1) is $s$-linear, in particular linear, and (3.16), (3.17) hold for some strictly increasing function $\psi$ such that

$$\int_0^\infty \frac{dw(t)}{t} = \int_1^\infty \frac{dt}{\psi^{-1}(t)} < \infty, \quad (3.19)$$

then $u$ changes sign. Here $\psi^{-1}$ is the inverse function of $\psi$. In particular, functions of the form (3.18) satisfy (3.19).

It is not obvious but true that (3.17), (3.19) imply that

$$t \int_t^\infty p(s) \, ds \text{ is unbounded.} \quad (3.20)$$

This much simpler condition is sufficient to guarantee changing sign.
For the proof, we claim that

$$\limsup_{t \to \infty} \left( \int_{t}^{\infty} g(x, y) \, dx \, dy \right) \left( \int_{0}^{t} g(x, y) \, dx \, dy \right) = \infty. \quad (3.21)$$

Then the conclusion follows from Theorem 6. Inequality (3.16) implies that

$$\int_{t}^{\infty} \int_{0}^{t} g(x, y) \, dx \, dy \geq t \int_{t}^{\infty} p(s) \, ds$$

and

$$\int_{0}^{t} \int_{t}^{\infty} g(x, y) \, dx \, dy \geq t \int_{t}^{\infty} p(s) \, ds$$

and so (3.21) follows from (3.20).

It remains to show that (3.20) follows from (3.16) and (3.19). Suppose the contrary, i.e., \( \int_{t}^{\infty} p(s) \, ds < K/t \) for some \( K > 0 \) and for all \( t > \psi^{-1}(1) \). We show that (3.17) will then be violated. Expressing the integral in (3.17) as a double integral and then changing the order of integration, we obtain

$$\int_{\phi^{-1}(t)}^{\infty} \psi(t) \, p(t) \, dt = \int_{\phi^{-1}(1)}^{\infty} \left( \int_{0}^{\phi(t)} p(s) \, ds \right) p(t) \, dt$$

$$= \int_{1}^{\infty} \left( \int_{\phi^{-1}(s)}^{\infty} p(t) \, dt \right) ds + \int_{0}^{1} \left( \int_{\phi^{-1}(1)}^{\infty} p(t) \, dt \right) ds$$

$$\leq \int_{1}^{\infty} \frac{K}{\psi^{-1}(s)} \, ds + \frac{K}{\psi^{-1}(1)} < \infty,$$

contradicting (3.17). That the two integrals in (3.19) are equal follows from a simple change of variable.

**Example 2.** Even more general results than that of Example 1 can be established. It suffices to require that (3.15) holds only in two small wedges of positive angle resting on the two axes, namely, \( \{(x, y): y \leq \varepsilon x \text{ or } x \leq \varepsilon y\} \) for some \( 0 < \varepsilon < 1 \). The proof is similar, using Theorem 6 with \( (\bar{x}, \bar{y}) = (t, et) \) and letting \( t \to \infty \). The second integral in (3.10) is greater than \( et \int_{t}^{\infty} p(s) \, ds \) which can be estimated as before. The first integral in (3.10) is \( \int_{et}^{\infty} \int_{0}^{t} g(x, y) \, dx \, dy \geq \int_{et}^{\infty} \int_{0}^{et} g(x, y) \, dx \, dy \geq et \int_{t}^{\infty} p(s) \, ds \) because \( [0, et] \times [t, \infty) \subseteq [0, t] \times [et, \infty) \). It then can be estimated as before.

It is also obvious from the proof that it suffices to require (3.16) with some \( p \), say \( p_1 \) in one of the two wedges and possibly with a different \( p \), say \( p_2 \), in the other wedges as long as we require that

$$t^2 \int_{t}^{\infty} p_1(s) \, ds \int_{t}^{\infty} p_2(s) \, ds$$

be unbounded.
EXAMPLE 3. The function $p(t)$ in (3.16) serves as a lower bound of $g(x, y)$ along the (family of) straight lines $x + y = t$. In fact other one-parameter families of curves can also be used. For instance let $r = r(\theta)$ be a continuously differentiable function of $\theta$ on $[0, \pi/2]$. It is the polar equation of a curve in the first quadrant $Q$. Assume that the curve is neither tangent to the $x$-axis nor to the $y$-axis. If we require that $g(x, y) \geq t$ for all $(x, y)$ with polar coordinate $(\theta, t r(\theta))$ and that (3.20) is satisfied, then $u$ changes sign. The remarks in Example 2 apply here also.

EXAMPLE 4. Let $h(x, y)$ be any continuous function such that $\lim_{x, y \to \infty} h(x, y) = \infty$. Then all solutions of the $s$-linear equation

$$u_{xy} + \frac{h(x, y)}{x^2 + y^2 + 1} f(u) = 0,$$

with non-increasing initial values, change sign. This result is not covered by any previously known criterion, but follows easily from Theorem 6 since the left-hand side of (3.10) tends to $\infty$ if $(\bar{x}, \bar{y})$ is chosen to be $(t, t)$, $t \to \infty$.

If we assume the stronger condition $\lim_{u \to \infty} f(u)/u = \infty$, then $u$ still changes sign even though the requirement on $h$ is relaxed to $\lim_{x, y \to \infty} h(x, y) > 0$. This weaker condition on $h$ implies that

$$l = \lim_{t \to \infty} \left( \int_0^t \int_0^t g(x, y) \, dx \, dy \right) \left( \int_0^t \int_0^t g(x, y) \, dx \, dy \right) > 0. \quad (3.22)$$

But $l$ may not be large enough for (3.10) to hold. However Theorem 6 can still be applied in the following way. It is easy to check that

$$\lim_{t \to \infty} \int_0^t \int_0^t g(x, y) \, dx \, dy = \infty.$$

Using the same argument that leads to (3.15), we can prove that $u(t, t) \to 0$ as $t \to \infty$, or else $u$ changes sign. Thus choose a $\tau$ large enough that $u^2(\tau, \tau)/f^2(u(\tau, \tau)) < l$. Now consider $u$ as the solution of the same equation in the smaller "quadrant" $Q_1 = [\tau, \infty) \times [\tau, \infty)$ with initial conditions given on the boundary of $Q_1$. If $u$ already changes sign on either the complement or the boundary of $Q_1$, there is nothing to prove. In the contrary case, by Corollary 3, the initial conditions on the boundary of $Q_1$ is non-increasing. Theorem 6 is applicable after a translation, provided the condition corresponding to (2.10), namely,

$$\left( \int_0^\infty \int_\tau^\infty g(x, y) \, dx \, dy \right) \left( \int_\tau^\infty \int_\tau^\infty g(x, y) \, dx \, dy \right) > \frac{u^2(\tau, \tau)}{f^2(u(\tau, \tau))}$$
holds. That the inequality indeed holds with \((\bar{x}, \bar{y}) = (t, t), \ t \) large enough follows from the fact that

\[
\lim_{t \to \infty} \left( \int_0^t \int_0^t g(x, y) \, dx \, dy - \int_0^t \int_\tau^t g(x, y) \, dx \, dy \right) = 0,
\]

and (3.22).

4. A CLASS OF S-LINEAR PROBLEMS: SUFFICIENT CONDITION

We consider those s-linear problems in which the non-linear function \(f\) satisfies the condition:

\[
\int_0^1 \frac{ds}{f(s)} < \infty.
\] (*)

Let \(F(s) = s/f(s)\) which is a non-decreasing function of \(s > 0\). Using the integral test for infinite series we see easily that (*) is equivalent to either one of the following conditions:

\[
\sum_{n=0}^{\infty} F(2^{-n}) < \infty
\] (**)

For any \(\delta > 1, \sum_{n=0}^{\infty} F(\delta^{-n}) < \infty.\) (***)

**THEOREM 7.** Suppose (1.1) is s-linear, \(f\) satisfies (*) and \(\int_0^x g(x, y) \, dx \, dy = \infty.\) Then \(u\) changes sign.

**Proof.** Without loss of generality we may assume that \(u(0,0) = 1\), otherwise we can apply a change of variable \(\bar{u}(x, y) = u(x, y)/u(0, 0)\). Let us use the method of contradiction. Suppose \(u\) does not change sign.

First note that there does not exist a finite point \(x_0 > 0\) such that

\[
\int_0^{\infty} \int_{x_0}^{x_1} g(x, y) \, dx \, dy = \infty
\]

for all \(x_1 > x_0\), because if such an \(x_0\) does exist, then Theorem 1 in [2] implies that \(u\) changes sign. From the observation follows that

\[
\int_0^{\infty} \int_0^{\bar{x}} g(x, y) \, dx \, dy
\]
is a continuous function of $\bar{x}$ in $[0, a)$, where

$$a = \sup \left\{ \bar{x} > 0: \int_0^{\bar{x}} \int_0^{\bar{x}} g(x, y) \, dx \, dy < \infty \right\}.$$  

The Intermediate Value Theorem gives a point $x_1 \in [0, a)$ such that

$$\int_0^{x_1} \int_0^{x_1} g(x, y) \, dx \, dy = \frac{2}{f(1)}.$$  

Let $y_1 > 0$ be such that

$$\int_0^{y_1} \int_0^{x_1} g(x, y) \, dx \, dy = \frac{1}{f(1)} = \int_0^{x_1} \int_0^{y_1} g(x, y) \, dx \, dy.$$  

By the same technique used to establish (3.15) we easily see that $u(x_1, y_1) \leq 2^{-1}$. That $u$ does not vanish implies, via Theorem 6,

$$\left( \int_0^{\infty} \int_0^{x_1} g(x, y) \, dx \, dy \right) \left( \int_0^{y_1} \int_0^{\infty} g(x, y) \, dx \, dy \right) \leq \frac{1}{f^2(1)}$$

from which we have $\int_0^{y_1} \int_0^{\infty} g(x, y) \, dx \, dy \leq 1/f(1)$. Thus the integral of $g(x, y)$ over the L-shaped region $\{(x, y): 0 \leq x \leq x_1 \text{ or } 0 \leq y \leq y_1\}$ is not more than $3/f(1)$.

We intend to construct, by induction, a sequence of points $\{(x_n, y_n): n = 1, 2, \ldots\}$ such that

$$x_{n-1} < x_n, y_{n-1} < y_n, \quad u(x_n, y_n) \leq 2^{-n}, \quad (4.1)$$

and the integral of $g(x, y)$ over the L-shaped region $\{(x, y): 0 \leq x \leq x_n \text{ or } 0 \leq y \leq y_n\}$ is not more than

$$3 \sum_{i=0}^{n-1} F(2^{-i}). \quad (4.3)$$

The first step has been described above. The $n$th step is in fact done in exactly the same way. So suppose we already have $(x_{n-1}, y_{n-1})$. Let $\alpha = u(x_{n-1}, y_{n-1})$. The Intermediate Value Theorem gives the point $(x_n, y_n)$ such that

$$\int_{y_{n-1}}^{y_n} \int_{x_{n-1}}^{x_n} g(x, y) \, dx \, dy = \frac{\alpha}{f(\alpha)} = \int_{y_n}^{\infty} \int_{x_{n-1}}^{x_n} g(x, y) \, dx \, dy.$$
An estimate for $u(x_n, y_n)$ can be obtained using the same technique that leads to (3.15), this time integrating over the rectangle $[x_{n-1}, x_n] \times [y_{n-1}, y_n]$. We have

$$u(x_n, y_n) \leq \frac{\alpha}{2} \leq 2^{-n}.$$ 

Theorem 6 (after a translation to the "quadrant" $[x_{n-1}, \infty) \times [y_{n-1}, \infty)$) gives

$$\left( \int_{y_{n-1}}^{y_n} \int_{x_{n-1}}^{x_n} g(x, y) \, dx \, dy \right) \left( \int_{y_{n-1}}^{y_n} \int_{x_{n-1}}^{x_n} g(x, y) \, dx \, dy \right) \leq \frac{\alpha^2}{f^2(\alpha)}$$

otherwise $u$ will change sign. From this

$$\int_{y_{n-1}}^{y_n} \int_{x_{n-1}}^{x_n} g(x, y) \, dx \, dy \leq \frac{\alpha}{f(\alpha)}.$$ 

Thus the integral over the L-shaped area $\{(x, y) : x_{n-1} \leq x \leq x_n \text{ or } y_{n-1} \leq y \leq y_n\}$ is not more than $3a/f(\alpha) \leq 3F(2^{-n+1})$. Summing up we obtain (4.3).

By (4.1) $\lim_{n \to \infty} x_n = \bar{x}$ and $\lim_{n \to \infty} y_n = \bar{y}$ exist. From condition (***) and (4.3) we see that $\int_0^\infty \int_0^\infty g(x, y) \, dx \, dy \leq \sum_{n=0}^\infty 3F(2^{-n}) < \infty$. Thus $\bar{x} < \infty$. Similarly $\bar{y} < \infty$. On the other hand the continuity of $u$ and (4.2) give $u(\bar{x}, \bar{y}) = \lim_{n \to \infty} u(x_n, y_n) = 0$, a contradiction. This completes the proof of the theorem.

**Corollary 8.** If (1.1) is $s$-linear, $\lim_{s \to \infty} f(s)/s = 0$ and $\int_0^\infty (ds/f(s)) < \infty$, then a necessary and sufficient condition for the solution $u$ of (1.1) to change sign for all choices of $\phi$ and $\psi$ is that $\int_0^\infty g(x, y) \, dx \, dy = \infty$.

Obvious examples of $f$ satisfying the hypotheses of the corollary are $f(u) = u^r$, $0 < r < 1$.

**References**


