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Almost Periodic Plane Wave Solutions for Reaction Diffusion Equations

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1. INTRODUCTION

The recent development in reaction-diffusion equation gives considerable attention to plane wave solutions as well as to periodic solutions. However, most of the discussions in the current literature are devoted either to plane wave solutions in the form $v(\lambda t - \beta \cdot x)$ or to periodic solutions (cf. [2, 5, 6, 9]). This form of plane wave or periodic solutions consists of a single moving coordinate and can often be analyzed by using the theory of ordinary differential equations. In this paper, we study plane wave solutions for a coupled system of reaction-diffusion equations in the form $v(t, \lambda t - \beta \cdot x)$ which depends on two independent variables t and $\xi \equiv \lambda t - \beta \cdot x$, where $\lambda > 0$ is a constant scalar and β is a vector in the n-dimensional space $\mathbb{R}^n$. This consideration leads to a suitable set of reaction diffusion equations rather than the usual two-point periodic boundary-value problem. The periodicity of the plane wave solution discussed in this paper is with respect to not only the variable t but also the moving coordinate ξ. A novelty of this consideration is that the periodic plane wave solution possesses the combined frequencies $\omega_0 \equiv 2\pi\lambda/L$ and $\omega_1 \equiv 2\pi/T$, where T and L are the respective time and spatial period of the solution.

Consider the coupled nonlinear reaction-diffusion equations

$$(u_j)_t - D_j^* \nabla^2 u_j + \sum_{l=1}^{n} c_{jl}(u_j)_{x_l} + \sigma_j u_j = f_j(u), \qquad (t, x) \in \mathbb{R}^{n+1}, j = 1,\dots, n, \tag{1.0}$$

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where $u=(u_1,\ldots,u_n)$, $D_j^*>0$, $\sigma_j\geqslant 0$, c_{jl} are constants and f_j are functions defined in $\mathbb{R}^n$. By introducing the moving coordinate $\xi=\lambda t-\beta\cdot x$ and writing $u(t,x)=v(t,\lambda t-\beta\cdot x)=v(t,\xi)$, Eq. (1.0) is reduced to the form

$$(v_j)_t-D_j(v_j)_{\xi\xi}+c_j(v_j)_\xi+\sigma_j v_j=f_j(v), \qquad (t,\xi)\in\mathbb{R}^2, \tag{1.1}$$

where $D_j=|\beta|^2D_j^*$, $c_j=\lambda-\sum_{l=1}^n c_{jl}\beta_l$. We require that for some given positive constants L, T the solution v satisfies the time-space periodic condition

$$\begin{aligned} v_j(t,\xi)&=v_j(t,\xi+L),\\ v_j(t,\xi)&=v_j(t+T,\xi), \end{aligned} \qquad (t,\xi)\in\mathbb{R}^2, j=1,\ldots,n. \tag{1.2}$$

Our objective is to show the existence of a periodic solution to the system (1.1)–(1.2) from which the existence of an almost periodic plane wave solution to the system (1.0) can be deduced. The basic approach in proving the existence problem is to establish an equivalent relation between the system (1.1)–(1.2) and a corresponding integral equation. This integral equation is obtained through the construction of a suitable Green's function. Upon deriving some properties of the Green's function we prove the existence of a periodic solution to the integral equation, or equivalently, an almost periodic plane wave solution to (1.0). The existence proof is based on the property of the Green's function which does not make use of the Hopf theorem for periodic solutions nor the usual "a priori" C^2-bound of the solution for almost periodic solutions (cf. [1, 5]).

2. Periodic Green's Function

Throughout this paper we assume that $f_j\in C^1(\mathbb{R}^n)$, where $C^1(\mathbb{R}^n)$ denotes the set of all continuously differentiable functions in $\mathbb{R}^n$. By adding a linear term on both sides of Eq. (1.0), if necessary, we may assume that $\sigma_j>0$ for $j=1,\ldots,n$. In order to investigate the existence of a periodic plane wave solution to the coupled nonlinear reaction diffusion equations we first construct a suitable periodic Green's function for the system (1.1)–(1.2). This Green's function will be used to establish an equivalent relation between the time-space periodic system (1.1)–(1.2) and a corresponding integral equation which is the basis for establishing our existence and uniqueness theorem. Due to the periodic nature of the problem, our integral equation is rather different from the usual integral representation of initial boundary value problems.

Motivated by the Green's functions obtained in [6, 7] we seek a Green's function for the present system in the form

$$G_j(t-\tau, \xi-\eta) = (LT)^{-1} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [i\omega_1 m + (D_j\omega_2^2 k^2 + \sigma_j + ic_j\omega_2 k)]^{-1}$$
$$\times \exp[i(\omega_1 m(t-\tau) + \omega_2 k(\xi-\eta))]$$
$$\text{for } t-\tau \neq NT,\ N \in Z, j = 1,..., n, \quad (2.1)$$

where $\omega_1 = 2\pi T^{-1}$, $\omega_2 = 2\pi L^{-1}$, $\omega_0 = \lambda\omega_2$. It is clear that the above series converges uniformly in every compact subset of $\mathbb{R}^2$ excluding the line $t = \tau + 2N\pi\omega_1^{-1}$ ($N \in Z$). By defining

$$g_{jk}(z) = \exp(-(D_j\omega_2^2 k^2 + \sigma_j) z), \quad (2.2)$$
$$\psi_{jk}(z) = [1 - g_{jk}(T)\exp(-ic_j\omega_2 kT)]^{-1}[g_{jk}(z)\exp(-ic_j\omega_2 kz)] \quad (2.3)$$

and using the relation (cf. [8])

$$\sum_{p=-\infty}^{\infty} \frac{e^{i\omega p(t-\tau)}}{i\omega p - \alpha} = \begin{cases} T(1-\exp(\alpha T))^{-1}\exp[\alpha(t-\tau)], & 0 \leqslant \tau < t \leqslant T, \\ T(1-\exp(\alpha T))^{-1}\exp[\alpha(T+t-\tau)], & 0 \leqslant t < \tau \leqslant T, \end{cases} \quad (2.4)$$

for any complex number $\alpha \neq i\omega p$, $\omega = 2\pi T^{-1}$, the Green's function may be written in the form

$$G_j(t-\tau, \xi-\eta) = L^{-1} \sum_{k=-\infty}^{\infty} \psi_{jk}(t-\tau)\exp(i\omega_2 k(\xi-\eta)) \quad \text{for } 0 \leqslant \tau < t \leqslant T,$$
$$G_j(t-\tau, \xi-\eta) = L^{-1} \sum_{k=-\infty}^{\infty} \psi_{jk}(T+t-\tau)\exp(i\omega_2 k(\xi-\eta))$$
$$\text{for } 0 \leqslant t < \tau \leqslant T, \quad (2.5)$$

where $\xi, \eta \in R$. Define

$$\left.\begin{aligned} a_{jk} &= [1-g_{jk}(T)]^2 + 2(1-\cos c_j\omega_2 kT)\, g_{jk}(T), \\ b_{jk}(z) &= (a_{jk})^{-1} g_{jk}(z), \end{aligned}\right\} \quad (2.6)$$

$$H_j(t-\tau) = \begin{cases} L^{-1}\psi_{j0}(t-\tau) & \text{for } 0 \leqslant \tau < t \leqslant T, \\ L^{-1}\psi_{j0}(T+t-\tau) & \text{for } 0 \leqslant t < \tau \leqslant T. \end{cases} \quad (2.7)$$

Then an elementary calculation gives

$$\psi_{jk}(z) = b_{jk}(z)\exp(-ic_j\omega_2 kz) - b_{jk}(T+z)\exp\{ic_j\omega_2 k(T-z)\}. \quad (2.8)$$

Substitution of (2.7), (2.8) into (2.5) the Green's function is expressed in the form

$$G_j(t-\tau, \xi-\eta) = H_j(t-\tau) + S_j(t-\tau, \xi-\eta) \qquad (t, \tau \in [0, T], \xi, \eta \in \mathbb{R}) \tag{2.9}$$

where

$$S_j(t-\tau, \xi-\mu) = \begin{cases} G_{j1}(t-\tau, \xi-\eta) - G_{j2}(t-\tau, \xi-\eta) & \text{for } 0 \leqslant \tau < t \leqslant T \\ G_{j3}(t-\tau, \xi-\eta) - G_{j4}(t-\tau, \xi-\eta) & \text{for } 0 \leqslant t < \tau \leqslant T, \end{cases} \tag{2.10}$$

and

$$\begin{aligned} G_{j1}(t-\tau, \xi-\eta) &= 2L^{-1} \sum_{k=1}^{\infty} b_{jk}(t-\tau) \cos\{\omega_2 k(\xi-\eta-c_j(t-\tau))\}, \\ & \qquad\qquad 0 \leqslant \tau < t \leqslant T, \\ G_{j2}(\cdot) &= 2L^{-1} \sum_{k=1}^{\infty} b_{jk}(T+t-\tau) \cos\{\omega_2 k(\xi-\eta-c_j(t-\tau-T))\}, \\ & \qquad\qquad 0 \leqslant \tau < t \leqslant T, \\ G_{j3}(\cdot) &= 2L^{-1} \sum_{k=1}^{\infty} b_{jk}(T+t-\tau) \cos\{\omega_2 k(\xi-\eta-c_j(t-\tau+T))\}, \\ & \qquad\qquad 0 \leqslant t < \tau \leqslant T, \\ G_{j4}(\cdot) &= 2L^{-1} \sum_{k=1}^{\infty} b_{jk}(2T+t-\tau) \cos\{\omega_2 k(\xi-\eta-c_j(t-\tau))\}, \\ & \qquad\qquad 0 \leqslant t < \tau \leqslant T. \end{aligned} \tag{2.11}$$

In the remaining of this section we establish some properties for the Green's function.

By considering the functions

$$\phi_j(\xi, z) \equiv b_{j\xi}(z) = (a_{j\xi})^{-1} g_{j\xi}(z), \quad a_j(\xi) \equiv a_{j\xi}, \quad g_j(\xi, z) \equiv g_{j\xi}(z) \tag{2.12}$$

as a continuous function of ξ and z and setting

$$\alpha_j(\xi, z) \equiv K_j \int_{\xi}^{+\infty} \phi_j(\eta, z)\, d\eta, \qquad \text{for } z > 0, \xi > 0, \tag{2.13}$$

where

$$K_j = (8e^{-1} + \sqrt{2}\, |c_j| (eD_j)^{-1/2} T^{1/2}) \exp(-\sigma_j T)(1 - \exp(-\sigma_j T))^{-2} + 1$$

we have the following estimate which is a generalization of the well-known classical trigonometric series theorem (cf. [3]).

LEMMA 1. *For every* $v \in R, j = 1, \ldots, n$,

$$\int_0^L \left| \sum_{k=1}^{\infty} \omega_2 k b_{jk}(z) \sin \omega_2 k(v-\eta) \right| d\eta \leqslant 4\pi \left(\sum_{k=1}^{\infty} b_{jk}(z) + 2 \sum_{k=1}^{\infty} k^{-1} \alpha_j(k, z) \right) \quad (z > 0). \tag{2.14}$$

Proof. By the definition of ϕ_j direct differentiation gives

$$\partial(\xi \phi_j(\xi, z))/\partial \xi = -\Phi_j(\xi, z) + \phi_j(\xi, z)(1 + r_j(\xi)), \tag{2.15}$$

where

$$\begin{aligned} \Phi_j(\xi, z) &= 2D_j \omega_2^2 z \xi^2 \phi_j(\xi, z) \\ &\quad + 4D_j \omega_2^2 T \xi^2 (1 - g_{j\xi}(T)) g_{j\xi}(T) \phi_j(\xi, z)(a_{j\xi})^{-1}, \\ r_j(\xi) &= (4D_j \omega_2^2 T \xi^2 g_{j\xi}(T)(1 - \cos c_j \omega_2 T \xi) \\ &\quad - 2c_j \omega_2 T \xi g_{j\xi}(T) \sin c_j \omega_2 T \xi)(a_{j\xi})^{-1}. \end{aligned}$$

Since $|\xi \exp(-a\xi^2)| \leqslant (2ae)^{-1/2}$ and $(\xi^2 \exp(-a\xi^2)) \leqslant (ae)^{-1}$ we have, from (2.2), (2.6) and (2.12),

$$\begin{aligned} &(1 - \exp(-\sigma_j T))^2 \leqslant a_{j\xi} < 5, \qquad \Phi_j(\xi, z) > 0, \\ &|r_j(\xi, z)| + 1 \leqslant K_j \qquad (\xi > 0). \end{aligned} \tag{2.16}$$

Let $I_j(\xi, z) = \xi \phi_j(\xi, z) + \alpha_j(\xi, z)$. Then by (2.13), (2.15) and (2.16)

$$\begin{aligned} &I_j(\xi, z) > 0, \qquad \alpha_j(\xi, z) > 0, \quad \partial I_j/\partial \xi < 0, \quad \partial \alpha_j/\partial \xi < 0 \quad \text{for } \xi > 0; \\ &\lim_{\xi \to \infty} I_j(\xi, z) = 0, \qquad \lim_{\xi \to \infty} \alpha_j(\xi, z) = 0. \end{aligned}$$

The above relations imply that I_j, α_j are monotone decreasing in ξ. It follows from a well-known theorem (cf. [3, p. 33]) that

$$(2\pi)^{-1} \int_0^{\pi} \left| \sum_{k=1}^{\infty} I_j(k, z) \sin k\eta \right| d\eta \leqslant \sum_{k=1}^{\infty} j^{-1} I_j(k, z) = \sum_{k=1}^{\infty} b_{jk}(z) + \sum_{k=1}^{\infty} k^{-1} \alpha_j(k, z), \tag{2.17}$$

$$(2\pi)^{-1} \int_0^{\pi} \left| \sum_{k=1}^{\infty} \alpha_j(k, z) \sin k\eta \right| d\eta \leqslant \sum_{k=1}^{\infty} k^{-1} \alpha_j(k, z), \quad \text{for any } z > 0.$$

Since by letting $\tilde{\mu} = \omega_2(\nu - \eta)$ and using the periodic condition,

$$\int_0^L \left| \sum_{k=1}^{\infty} \omega_2 k b_{jk}(z) \sin \omega_2 k(\nu - \eta) \right| d\eta = \int_{\omega_2 \nu - 2\pi}^{\omega_2 \nu} \left| \sum_{k=-\infty}^{\infty} k b_{jk}(z) \sin k\tilde{\eta} \right| d\tilde{\eta}$$

$$= 2 \int_0^{\pi} \left| \sum_{k=1}^{\infty} (I_j(k, z) - \alpha_j(k, z)) \sin k\eta \right| d\eta$$

the relation (2.14) follows immediately from (2.17).

Using the result of Lemma 1 we derive some estimates for $\partial G_j/\partial \xi$.

LEMMA 2. *Let* $\tilde{K}_j = 2K_j(1 - \exp(-\sigma_j T))^{-2}$. *Then*

$$\int_0^T d\tau \int_0^L |\partial G_j(t - \tau, \xi - \eta)/\partial \xi| \, d\eta$$

$$\leqslant 8\pi L^{-1} \left[\sum_{k=1}^{\infty} (D_j \omega_2^2 k^2 + \sigma_j)^{-1} (b_{jk}(0) - b_{jk}(2T)) + \tilde{K}_j \sum_{k=1}^{\infty} (D_j \omega_2^2 k^2)^{-1} \right]. \quad (2.18)$$

Proof. By the definition of the Green's function in (2.9), (2.11),

$$\int_0^T d\tau \int_0^L |\partial G_j(t - \tau, \xi - \eta)/\partial \xi| \, d\eta$$

$$\leqslant \int_0^t d\tau \int_0^L |(\partial/\partial \xi)(G_{j1} - G_{j2})| \, d\xi + \int_t^T d\tau \int_0^L |(\partial/\partial \xi)(G_{j3} - G_{j4})| \, d\xi. \quad (2.19)$$

Since

$$\partial G_{j1}/\partial \xi = 2L^{-1} \sum_{k=1}^{\infty} (-\omega_2) k b_{jk}(t - \tau) \sin[\omega_2 k(\xi - \eta - c_j(t - \tau))],$$

$$\int_0^t b_{jk}(t - \tau) \, d\tau = (D_j \omega_2^2 k^2 + \sigma_j)^{-1} (b_{jk}(0) - b_{jk}(t)),$$

$$2 \int_0^t \alpha_j(k, t - \tau) \, d\tau = 2K_j \int_0^t d\tau \int_k^{\infty} (a_j(\eta))^{-1} g_j(\eta, t - \tau) \, d\eta$$

$$\leqslant \tilde{K}_j \int_k^{\infty} (D_j \omega_2^2 \eta^2 + \sigma_j)^{-1} (1 - g_j(\eta, t)) \, d\eta,$$

we have from Lemma 1

$$\int_0^t d\tau \int_0^L |\partial G_{j1}/\partial \xi| \, d\eta \leqslant 8\pi L^{-1} \sum_{k=1}^{\infty} (D_j \omega_2^2 k^2 + \sigma_j)^{-1} (b_{jk}(0) - b_{jk}(t))$$

$$+ 8\pi L^{-1} \tilde{K}_j \sum_{k=1}^{\infty} k^{-1} \int_k^{\infty} (D_j \omega_2^2 \eta^2 + \sigma_j)^{-1} (1 - g_j(\eta, t)) \, d\eta. \quad (2.20)$$

A similar calculation leads to

$$\begin{aligned}
\int_0^t d\tau \int_0^L |\partial G_{j2}/\partial\xi|\, d\eta &\leqslant 8\pi L^{-1} \sum_{k=1}^{\infty} (D_j\omega_2^2 k^2+\sigma_j)^{-1}(b_{jk}(T)-b_{jk}(T+t)) \\
&\quad + 8\pi L^{-1}\tilde{K}_j \sum_{k=1}^{\infty} k^{-1}\int_k^{\infty} (D_j\omega_2^2\eta^2+\sigma_j)^{-1}(g_j(\eta,T)-g_j(\eta,T+t))\,d\eta, \\
\int_t^T d\tau \int_0^L |\partial G_{j3}/\partial\xi|\, d\eta &\leqslant 8\pi L^{-1} \sum_{k=1}^{\infty} (D_j\omega_2^2 k^2+\sigma_j)^{-1}(b_{jk}(t)-b_{jk}(T)) \\
&\quad + 8\pi L^{-1}\tilde{K}_j \sum_{k=1}^{\infty} k^{-1}\int_k^{\infty} (D_j\omega_2^2\eta^2+\sigma_j)^{-1}(g_j(\eta,t)-g_j(\eta,T))\,d\eta, \\
\int_t^T d\tau \int_0^L |\partial G_{j4}/\partial\xi|\, d\eta &\leqslant 8\pi L^{-1} \sum_{k=1}^{\infty} (D_j\omega_2^2 k^2+\sigma_j)^{-1}(b_{jk}(T+t)-b_{jk}(2T)) \\
&\quad + 8\pi L^{-1}\tilde{K}_j \sum_{k=1}^{\infty} k^{-1}\int_k^{\infty} (D_j\omega_2^2\eta^2+\sigma_j)^{-1}(g_j(\eta,T+t)-g_j(\eta,2T))\,d\eta.
\end{aligned} \tag{2.21}$$

The conclusion (2.18) follows from (2.19)–(2.21) and the relation

$$\int_k^{\infty} (D_j\omega_2^2\eta^2+\sigma_j)^{-1}(1-g_j(\eta,2T))\,d\eta \leqslant (D_j\omega_2^2)^{-1}\int_k^{\infty}\eta^{-2}d\eta = (D_j\omega_2^2 k)^{-1}.$$

We next give an estimate for G_j. For this purpose it is convenient to set

$$\begin{aligned}
K_j^* &= 2\omega_2[(8D_jTe^{-1})^{1/2}\exp(-\sigma_jT)+|C_j|\,T](1-\exp(-\sigma_jT))^{-4}, \\
R_j &= 4\left[\sum_{k=1}^{\infty} (D_j\omega_2^2k^3+\sigma_jk)^{-1}(b_{jk}(0)-b_{jk}(2T)) + 2K_j^*(D_j\omega_2^2k^2)^{-1}\right].
\end{aligned} \tag{2.22}$$

LEMMA 3. *Let G_j be given by* (2.9). *Then*

$$\begin{aligned}
\int_0^T d\tau\int_0^L |S_j(t-\tau,\xi-\eta)|\,d\eta &\leqslant R_j, \\
\int_0^T d\tau\int_0^L |G_j(t-\tau,\xi-\eta)|\,d\eta &\leqslant \sigma_j^{-1}+R_j.
\end{aligned} \tag{2.23}$$

Proof. It is readily seen by a simple calculation that

$$\partial\phi_j(\xi,z)/\partial\xi = -2D_j\omega_2^2\xi g_{j\xi}(z)(a_{j\xi})^{-2}\,r_{j1}(\xi,z) + g_{j\xi}(z)(a_{j\xi})^{-2}r_{j2}(\xi,z),$$

where

$$r_{j1}(\xi, z) = za_{j\xi} + 2T(1 - g_{j\xi}(T))\, g_{j\xi}(T) \geqslant 0 \qquad \text{for } \xi, z \geqslant 0,$$

$$r_{j2}(\xi, z) = (16D_j\omega_2^2 T)^{1/2}(1 - \cos c_j\omega_2 T\xi)((D_j\omega_2^2 T\xi^2)^{1/2} \times \exp(-D_j\omega_2^2 T\xi^2))\exp(-\sigma_j T) - 2c_j\omega_2 T g_{j\xi}(T) \sin c_j\omega_2 T\xi.$$

The above expression and the inequality $|\eta \exp(-a\eta^2)| \leqslant (2ae)^{-1/2}$ ensure that

$$|r_{j2}(\xi, z)| \leqslant 2\omega_2[(8D_j Te^{-1})^{1/2} \exp(-\sigma_j T) + |C_j|\, T] \equiv \bar{K}_j.$$

Let

$$I_j^*(\xi, z) = \phi_j(\xi, z) + \bar{K}_j \int_\xi^\infty [a_j(\eta)]^{-2} g_j(\eta, z)\, d\eta \equiv \phi_j + a_j^*.$$

Then from the relation

$$[a_j(\xi)]^{-2} \leqslant (1 - \exp(-\sigma_j T))^{-4}$$

the same argument as in the proof of Lemma 1 leads to the estimate

$$2L^{-1} \int_0^L \left| \sum_{k=1}^\infty b_{jk}(z) \cos \omega_2 k(v - \eta) \right| d\eta \leqslant 4 \sum_{k=1}^\infty k^{-1}\{b_{jk}(z) + 2a_j^*(k, z)\}.$$

Using the above inequality and (2.7), the conclusion (2.23) follows from the same argument as in the proof of Lemma 2.

LEMMA 4. *Let $TL^{-1}c_j = N_j$ be integers. Then*

$$\int_0^T d\tau \int_0^L |S_j(t - \tau, \xi - \eta)|\, d\eta \leqslant 4 \sum_{k=1}^\infty (D_j\omega_2^2 k^3 + \sigma_j k)^{-1}. \qquad (2.24)$$

Proof. In view of the hypothesis on c_j the function S_j in (2.10) becomes

$$S_j(t - \tau, \xi - \eta) = 2L^{-1} \sum_{k=1}^\infty b_{jk}(t - \tau) \cos \omega_2 k\{\xi - \eta - c_j(t - \tau)\} \quad \text{for } 0 \leqslant \tau < t \leqslant T, \qquad (2.25)$$

$$S_j(t - \tau, \xi - \eta) = 2L^{-1} \sum_{k=1}^\infty b_{jk}(T + t - \tau) \cos \omega_2 k\{\xi - \eta - c_j(t - \tau)\} \quad \text{for } 0 < t < \tau \leqslant T,$$

where

$$b_{jk}(z) = \{1 - \exp(-(D_j\omega_2^2 k^2 + \sigma_j) T)\}^{-1} \exp\{-(D_j\omega_2^2 k^2 + \sigma_j) z\} \qquad (z > 0). \qquad (2.26)$$

It is easily seen that

$$b_{jk+1}(z) < b_{jk}(z) \text{ for } z > 0, k = 1, 2,..., \qquad \text{and} \qquad \lim b_{jk}(z) = 0 \text{ as } k \to \infty.$$

Using the transformation $\eta \to \tilde{\eta} = \omega_2[\xi - \eta - c_j(t - \tau)]$ a similar argument as in the proof of Lemmas 1 and 2 gives

$$\int_0^T d\tau \int_0^L |S_j(t-\tau, \xi-\eta)| \, d\eta$$
$$\leqslant 4 \sum_{k=1}^{\infty} k^{-1} \left\{ \int_0^t b_{jk}(t-\tau) \, d\tau + \int_t^T b_{jk}(T+t-\tau) \, d\tau \right\}.$$

The conclusion in (2.24) follows by an integration on the right-side expression, using the relation (2.26).

3. Integral Representation

Using the estimates for the Green's function in Section 2 we now establish an equivalence relation between the system (1.1), (1.2) and the integral equation

$$v_j(t, \xi) = \int_0^T d\tau \int_0^L G_j(t-\tau, \xi-\eta) \, f_j(v(\tau, \eta)) \, d\eta,$$
$$(t, \xi) \in \bar{\Omega} = [0, T] \times [0, L], j = 1,..., n. \tag{3.1}$$

Our main effort in establishing this equivalence relation is the differentiability of a solution to the integral equation (3.1).

Lemma 5. *Let $f_j \in C^1(\mathbb{R}^n)$ and let v be a continuous solution of the integral equation* (3.1). *Then the derivatives $v_t, v_\xi, v_{\xi\xi}$ all exist in Ω.*

Proof. Let $q_j(t, \xi) \equiv f_j(v(t, \xi))$ so that q_j is bounded in $\bar{\Omega}$. In view of Lemma 2 and the well-known differentiability theorem (cf. [10, p. 794]), $\partial v_j/\partial \xi$ exists and is continuous. By the hypothesis on f_j, $P_j \equiv \partial q_j/\partial \xi$ also exists and is continuous. We show that v_j is continuously differentiable in t. Using (2.9), (2.10) and setting

$$h_j(t) = L^{-1} \int_0^t \psi_{j0}(t-\tau) \left(\int_0^L q_j(\tau, \eta) \, d\eta \right) d\tau$$
$$+ L^{-1} \int_t^T \psi_{j0}(T+t-\tau) \left(\int_0^L q_j(\tau, \eta) \, d\eta \right) d\tau, \tag{3.2}$$

where according to (2.3), $\psi_{j0}(z) = (1 - \exp(-\sigma_j T))^{-1} \exp(-\sigma_j z)$, Eq. (3.1) is equivalent to

$$v_j(t,\xi) = h_j(t) + \int_0^t d\tau \int_0^L (G_{j1}(t-\tau, \xi-\eta) - G_{j2}(t-\tau, \xi-\eta))\, f_j(v(\tau,\eta))\, d\eta$$
$$+ \int_t^T d\tau \int_0^L (G_{j3}(t-\tau, \xi-\eta) - G_{j4}(t-\tau, \xi-\eta))\, f_j(v(\tau,\eta))\, d\eta. \quad (3.3)$$

Let $Q_{jl}(t-\tau, \xi-\eta)$, $l = 1, 2, 3, 4$, be any L-periodic functions such that $\partial Q_{jl}/\partial\eta = -G_{jl}$. Then by integration by parts and using the L-periodicity of Q_{jl} and q_j, Eq. (3.3) may be expressed in the form

$$\begin{aligned} v_j(t,\xi) = h_j(t) &+ \int_0^t d\tau \int_0^L (Q_{j1}(t-\tau, \xi-\eta) - Q_{j2}(\cdot))\, P_j(\tau,\eta)\, d\eta \\ &+ \int_t^T d\tau \int_0^L (Q_{j3}(\cdot) - Q_{j4}(\cdot))\, P_j(\tau,\eta)\, d\eta, \quad j = 1,..., n. \end{aligned} \quad (3.4)$$

For definiteness, we choose

$$Q_{j1}(t-\tau, \xi-\eta) = 2L^{-1} \sum_{k=1}^{\infty} (\omega_2 k)^{-1} b_{jk}(t-\tau) \times \sin\{\omega_2 k(\xi - \eta - c_j(t-\tau))\}, \qquad 0 \leqslant \tau < t \leqslant T,$$

$$Q_{j2}(t-\tau, \xi-\eta) = 2L^{-1} \sum_{k=1}^{\infty} (\omega_2 k)^{-1} b_{jk}(T+t-\tau) \times \sin\{\omega_2 k(\xi - \eta - c_j(t-\tau-T))\}, \qquad 0 \leqslant \tau < t \leqslant T,$$

$$Q_{j3}(t-\tau, \xi-\eta) = 2L^{-1} \sum_{k=1}^{\infty} (\omega_2 k)^{-1} b_{jk}(T+t-\tau) \times \sin\{\omega_2 k(\xi - \eta - c_j(t-\tau+T))\}, \qquad 0 \leqslant t < \tau \leqslant T,$$

$$Q_{j4}(t-\tau, \xi-\eta) = 2L^{-1} \sum_{k=1}^{\infty} (\omega_2 k)^{-1} b_{jk}(2T+t-\tau) \times \sin\{\omega_2 k(\xi - \eta - c_j(t-\tau))\}, \qquad 0 \leqslant t < \tau \leqslant T,$$

and set

$$\begin{aligned} Q_j(t-\tau, \xi-\eta) &= -Q_{j1}(t-\tau, \xi-\eta) + Q_{j2}(t-\tau, \xi-\eta) \qquad \text{for } 0 \leqslant \tau < t \leqslant T, \\ Q_j(t-\tau, \xi-\eta) &= -Q_{j3}(t-\tau, \xi-\eta) + Q_{j4}(t-\tau, \xi-\eta) \qquad \text{for } 0 \leqslant t < \tau \leqslant T. \end{aligned} \quad (3.5)$$

It is easily seen by direction calculation that

$$\left.\begin{aligned}\partial Q_j(t-\tau,\xi-\eta)/\partial\xi &= -S_j(t-\tau,\xi-\eta),\\ \partial^2 Q_j(t-\tau,\xi-\eta)/\partial\xi^2 &= -\partial G_j(t-\tau,\xi-\eta)/\partial\xi,\end{aligned}\right\} t\neq\tau\in[0,T], \tag{3.6}$$

$$\partial Q_j(t-\tau,\xi-\eta)/\partial t = -D_j\partial G_j(t-\tau,\xi-\eta)/\partial\xi + c_jS_j(\cdot)-\sigma_jQ_j(\cdot),$$
$$0\leqslant\tau<t\leqslant T, \tag{3.7}$$
$$\partial Q_j(t-\tau,\xi-\eta)/\partial t = -D_j\partial G_j(t-\tau,\xi-\eta)/\partial\xi + c_jS_j(\cdot)-\sigma_jQ_j(\cdot),$$
$$0\leqslant t<\tau\leqslant T.$$

Since by (3.4), (3.5),

$$v_j(t,\xi)-h_j(t) = -\int_0^T d\tau\int_0^L Q_j(t-\tau,\xi-\eta)\,P_j(\tau,\eta)\,d\eta,$$
$$(t,\xi)\in\Omega, j=1,\ldots,n, \tag{3.8}$$

and since by Lemma 2, the integrals

$$\int_0^t d\tau\int_0^L |\partial G_j/\partial\xi|\,|P_j|\,d\eta \qquad\text{and}\qquad \int_t^T d\tau\int_0^L |\partial G_j/\partial\xi|\,|P_j|\,d\eta$$

exist, we see from the differentiability theorem that $\partial v_j/\partial t$ exists. In fact

$$\begin{aligned}(v_j)_t = (h_j)_t &+ D_j\int_0^t d\tau\int_0^L \partial G_j(t-\tau,\xi-\eta)/\partial\xi P_j(\tau,\eta)\,d\eta - c_j\int_0^t d\tau\int_0^L S_j(\cdot)\,P_j(\cdot)\,d\eta\\ &+ D_j\int_t^T d\tau\int_0^L \partial G_j(\cdot)/\partial\xi P_j(\cdot)\,d\eta - c_j\int_t^T d\tau\int_0^L S_j(\cdot)\,P_j(\cdot)\,d\eta\\ &+ \sigma_j\int_0^T d\tau\int_0^L Q_u(\cdot)\,P_j(\cdot)\,d\eta + \int_0^L \tilde{I}_j(\xi-\eta)\,P_j(t,\eta)\,d\eta,\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\tilde{I}_j(\xi-\eta) &= Q_{j1}(0,\xi-\eta)+Q_{j4}(0,\xi-\eta)\\ &\quad - Q_{j2}(0,\xi-\eta)-Q_{j3}(0,\xi-\eta).\end{aligned} \tag{3.10}$$

A similar argument shows that $\partial^2 v_j/\partial\xi^2$ exists and

$$\partial^2 v_j/\partial\xi^2 = \int_0^T d\tau\int_0^L \partial G_j(\cdot)/\partial\xi P_j(\tau,\eta)\,d\eta. \tag{3.11}$$

This proves the differentiability of v_j as stated in the lemma.

Based on the result of Lemma 3 we now establish the integral representation of the system (1.1), (1.2).

THEOREM 1. *Let $f_j \in C^1(\mathbb{R}^n)$. Then a continuous function $v(t, \xi)$ in $\mathbb{R}^2$ is a solution of the periodic system* (1.1)–(1.2) *if and only if it is a solution of the integral equation* (3.1).

Proof. Suppose that v is a solution of the integral equation (3.1). Then by the representation of G_j in (2.1), v satisfies the periodic condition (1.2). Since

$$(v_j)_\xi = -\int_0^T d\tau \int_0^L \partial Q_j(\cdot)/\partial\xi P_j(\tau, \eta)\, d\eta = \int_0^T d\tau \int_0^L S_j(\cdot)\, P_j(\tau, \eta)\, d\eta, \quad (3.12)$$

$$v_j = h_j(t) - \int_0^T d\tau \int_0^L Q_j(\cdot)\, P_j(\tau, \eta)\, d\eta, \quad (3.13)$$

the relations (3.9), (3.11), (3.12) and (3.13) ensure that

$$(v_j)_t - D_j(v_j)_{\xi\xi} + c_j(v_j)_\xi + \sigma_j v_j = (h_j)_t + \sigma_j h_j + \int_0^L \tilde{I}_j(\xi - \eta)\, P_j(t, \eta)\, d\eta. \quad (3.14)$$

In view of the definition of $Q_{jl}(0, \xi - \eta)$, the functions $\tilde{I}_j$ are given by

$$\tilde{I}_j(\xi - \eta) = 2L^{-1} \sum_{k=1}^{\infty} (\omega_2 k)^{-1} \sin(\omega_2 k(\xi - \eta)), \qquad j = 1, \ldots, n.$$

It follows that

$$\begin{aligned}
&\int_0^L \tilde{I}_j(\xi - \eta)\, P_j(t, \eta)\, d\eta \\
&\quad = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} L^{-1}(i\omega_2 k)^{-1} \int_0^L P_j(t, \eta) \exp\{i\omega_2 k(\xi - \eta)\}\, d\eta \\
&\quad = \sum_{k=-\infty}^{\infty} L^{-1} \int_0^L q_j(t, \eta) \exp\{i\omega_2 k(\xi - \eta)\}\, d\eta - L^{-1} \int_0^L q_j(t, \eta)\, d\eta \\
&\quad = q_j(t, \xi) - L^{-1} \int_0^L q_j(t, \eta)\, d\eta \\
&\quad = f_j(v(t, \xi)) - L^{-1} \int_0^L q_j(t, \eta)\, d\eta. \qquad (3.15)
\end{aligned}$$

It is easily seen by a direct calculation that

$$(h_j)_t = -\sigma_j h_j + L^{-1} \int_0^L q_j(t, \eta)\, d\eta. \quad (3.16)$$

Addition of (3.15) and (3.16) shows that the right side of (3.14) is $f_j(v(t, \xi))$. This proves that $v = (v_1, \ldots, v_n)$ is a solution of (1.1)–(1.2).

Conversely, if $v(t, \xi)$ is a solution of (1.1)–(1.2) then $q_j(t, \xi) \equiv f_j(v(t, \xi))$ is continuous differentiable in (t, ξ). Define

$$v_j^*(t, \xi) = \int_0^T d\tau \int_0^L G_j(t - \tau, \xi - \eta)\, q_j(\tau, \eta)\, d\eta. \tag{3.17}$$

Then by the above argument, v_j^* satisfies the periodic condition (1.2) and the linear equation (1.1) with the given function $f_j(v(t, \xi))$. Hence the function $w_j \equiv v_j^* - v_j$ is a solution of the homogeneous system

$$\begin{aligned} (w_j)_t - D_j(w_j)_{\xi\xi} + c_j(w_j)_\xi + \sigma_j w_j &= 0, \\ w_j(t + T, \xi) &= w_j(t, \xi), \\ w_j(t, \xi + L) &= w_j(t, \xi). \end{aligned} \tag{3.18}$$

Let $\bar{w}_{jk}(t)$ be the Fourier coefficients of $w_j(t, \xi)$, that is,

$$\bar{w}_{jk}(t) = L^{-1} \int_0^L w_j(t, \xi) \exp(-i\omega_2 k\xi)\, d\xi, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Then by an elementary calculation, $\bar{w}_{jk}(t)$ satisfies the equation

$$\frac{\partial}{\partial t} (\bar{w}_{jk}(t)) = -(D_j \omega_2^2 k^2 + \sigma_j + i c_j \omega_2 k)\, t.$$

An integration of the above equation gives

$$\bar{w}_{jk}(t) = \alpha_{jk} \exp(-(D_j w_2^2 k^2 + \sigma_j + i c_j \omega_2 k)\, t), \tag{3.19}$$

where $\alpha_{jk} = \bar{w}_{jk}(0)$. But by the periodic condition $\bar{w}_{jk}(t + T) = \bar{w}_{jk}(t)$ the relation (3.19) can hold only when $\alpha_{jk} = 0$ for all k. This shows that $\bar{w}_{jk}(t) = 0$, for $t \in \mathbb{R}$, $j = 1, \ldots, n$; $k = 0$, ± 1, $\pm 2, \ldots$. Since the set $\{\exp(-i\omega_2 k\xi)\}$ is complete and w is continuous we obtain $w_j(t, x) \equiv 0$. This proves that $v = v^*$ and thus v is also a solution of the integral equations (3.1). The proof of the theorem is completed.

We next show that every solution of (3.1) is an almost time periodic solution of (1.0) and satisfies the spatial periodic condition

$$u(t, x + e_j) = u(t, x), \qquad (t, x) \in \mathbb{R}^{n+1}, \tag{3.20}$$

where e_j is the vector given by

$$e_j = (0, \ldots, 0, L/\beta_j, 0, \ldots, 0).$$

This almost periodic solution possesses two time frequencies ω_0, ω_1 which is a unique feature of our integral representation.

THEOREM 2. *If* $v(t, \xi)$ *is a continuous solution of the integral equation* (3.1) *then* $u(t, x) \equiv v(t, \lambda t - \beta \cdot x)$ *is an almost periodic plane wave solution of* (1.0). *Furthermore this almost periodic solution possesses two frequencies* $\omega_0 = 2\pi\lambda/L$, $\omega_1 = 2\pi/T$ *and satisfies the spatial periodic condition* (3.20).

Proof. In view of Theorem 1, $u(t, x) = v(t, \lambda t - \beta \cdot x)$ is a solution of (1.0). Define

$$\bar{v}_{jmk} = (TL)^{-1} \int_0^T d\tau \int_0^L [i\omega_1 m + (D_j \omega_2^2 k^2 + \sigma_j + ic_j \omega_2 k)]^{-1}$$
$$\times \exp(-i(\omega_1 m\tau + w_2 k\eta)) f_j(v(\tau, \eta))\, d\eta.$$

Then by (2.9), (2.23) and the Lebesque theorem, a termwise integration of the right side of (3.1) gives

$$v_j(t, \xi) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \bar{v}_{jmk} \exp(i(\omega_1 mt + \omega_2 k\xi)).$$

Replacing ξ by $(\lambda t - \beta \cdot x)$ yields

$$u_j(t, x) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \bar{v}_{jmk} \exp[i(\omega_1 mt + \omega_0 kt - \omega_2 k\beta \cdot x)].$$

The above representation of the solution shows that u is almost periodic in t with combined frequencies ω_1, ω_0 and satisfies the spatial periodic condition (3.20).

4. EXISTENCE OF PERIODIC PLANE WAVE SOLUTIONS

In this section we establish the existence of an almost periodic plane wave solution to the system (1.0). In view of Theorems 1 and 2 it suffices to show the existence of a continuous solution to the integral equation (3.1). To achieve this we use the representation (2.9) for the Green's functions G_j and seek a solution in the Banach space

$$\mathscr{X} \equiv \{v = (v_1, ..., v_n);\ v_j \in C(\mathbb{R}^2),\ v_j(t+T, \xi) = v_j(t, \xi) = v_j(t, \xi + L), j = 1, ..., n\},$$

where the norm in $\mathscr{X}$ is given by

$$\|v\| = \sum_{j=1}^{n} \max\{|v_j(t, \xi)|;\ (t, \xi) \in \bar{\Omega}\}.$$

Let

$$(A_j v)(t) = \int_0^T d\tau \int_0^L H_j(t-\tau) f_j(v(\tau, v))\, d\eta,$$

$$j = 1,..., n, t \in [0, T], \quad (4.1)$$

$$(B_j v)(t, \xi) = \int_0^T d\tau \int_0^L S_j(t-\tau, \xi-\eta) f_j(v(\tau, \eta))\, d\eta,$$

and define operators A, B from χ into inself by

$$Av = (A_1 v,..., A_n v), \qquad Bv = (B_1 v,..., B_n v),$$

where H_j, S_j are given by (2.7) and (2.10). Then Eq. (3.1) is equivalent to

$$v = (A + B)v \qquad (v \in \chi). \tag{4.2}$$

Thus the existence problem of (3.1) is resolved if the operator $(A + B)$ has a fixed point in χ. To ensure this, we need the following assumption:

(H_1): For each $j = 1,..., n, f_j \in C^1(\mathbb{R}^n)$ and $TL^{-1}c_j \equiv N_j$ are integers, where $c_j \equiv \lambda - \sum_{l=1}^n c_{jl}\beta_l$.

The requirement of $TL^{-1}c_j$ being integers can always be fulfilled by a suitable choice of β. For notational convenience, we set

$$\begin{gathered} D^* = \sum_{j=1}^n (D_j^*)^{-1}, \qquad K_0 = \sum_{k=1}^\infty k^{-3}, \qquad \rho = \sum_{j=1}^n \sigma_j^{-1}, \\ \bar{M} = \max\{|\text{grad}. f_j(v)|; v \in S, j = 1,..., n\}, \end{gathered} \tag{4.3}$$

where S is a bounded subset in χ. Notice that $K_0 \cong 1.20$.

THEOREM 3. *Let hypothesis* (H_1) *hold and let S be a bounded closed convex subset of χ such that*

$$(Av_1 + B_{v_2}) \in S \qquad \text{for} \quad v_1, v_2 \in S. \tag{4.4}$$

Then for any λ, β with $|\beta| > (L/\pi)(K_0 D^* \bar{M})^{1/2}$, Eq. (1.0) *has an almost periodic plane wave solution in the form $u(t, x) \equiv v(t, \lambda t - \beta \cdot x)$ and $v \in S$.*

Proof. We prove the theorem by showing that A is a compact operator and B is a contraction mapping on S. It is easily seen by direct calculation, using (2.2), (2.3), (2.7), that

$$\int_0^T d\tau \int_0^L |H_j(t-\tau)|\, d\eta = \int_0^t \psi_{jo}(t-\tau)\, d\tau + \int_t^T \psi_{jo}(T+t-\tau)\, d\tau = \sigma_j^{-1}. \tag{4.5}$$

The above relation and (H_1) yield

$$\|Av\| \leqslant M \sum_{j=1}^{n} \sigma_j^{-1} = M\rho \qquad (v \in S),$$

where M is an upper bound of $|f_j(v)|$ on S for all j. In view of (2.7) we also have

$$\begin{aligned}
&L\,|(Av)(t+\Delta t) - (Av)(t)| \\
&\quad\leqslant \sum_{j=1}^{n} \left| \int_0^t d\tau \int_0^L (\psi_{j0}(t+\Delta t-\tau) - \psi_{j0}(t-\tau))\, f_j(v(\tau,\eta))\, d\eta \right| \\
&\quad\quad + \sum_{j=1}^{n} \left| \int_t^{t+\Delta t} d\tau \int_0^L (\psi_{j0}(t+\Delta t-\tau) - \psi_{j0}(T+t-\tau)) f_j(v(\tau,\eta))\, d\eta \right| \\
&\quad\quad + \sum_{j=1}^{n} \left| \int_{t+\Delta t}^{T} d\tau \int_0^L (\psi_{j0}(T+t+\Delta t-\tau) - \psi_{j0}(T+t-\tau)) f_j(v(\tau,\eta))\, d\eta \right|,
\end{aligned} \tag{4.6}$$

where $\Delta t > 0$. Since by (2.8) and (H_1)

$$\begin{aligned}
&L^{-1}|d\psi_{j0}(z)/dz| \leqslant (\sigma_j/L)(1-\exp(-\sigma_j T))^{-1} \equiv M_1, \\
&L^{-1}\,|\psi_{j0}(z)| \leqslant M_1/\sigma_j, && \text{for } z \geqslant 0, \\
&|f_j(v)| \leqslant M, && \text{for } v \in S_M,
\end{aligned} \tag{4.7}$$

we see from (4.6) that

$$\begin{aligned}
|(Av)(t+\Delta t) - (Av)(t)| &\leqslant nLM_1M(T + 2\sigma_j^{-1} + T)\,|\Delta t| \\
&= 2nLM_1M(T+\sigma_j^{-1})\,|\Delta t|.
\end{aligned} \tag{4.8}$$

A similar expression as in (4.6) shows that (4.8) holds when $\Delta t < 0$. The above relations and (4.4) imply that $(Av)(t)$ is equicontinuous and is uniformly bounded on $[0, T]$ for every $v \in S$. It follows from Arzelà's theorem that A is a compact operator on S. Clearly, A is a continuous operator on S.

To show the contraction property of B we observe from (4.1) and Lemma 4 that

$$\begin{aligned}
\|Bv - Bv^*\| &\leqslant 4\bar{M} \sum_{j=1}^{n} \sum_{k=1}^{\infty} (|\beta|^2 \omega_2^2 k^3 D_j^* + \sigma_j k)^{-1}\, \|v - v^*\| \\
&\leqslant 4\bar{M}K_0(|\beta|\,\omega_2)^{-2} \sum_{j=1}^{n} (D_j^*)^{-1}\, \|v - v^*\| \qquad (v, v^* \in \chi).
\end{aligned} \tag{4.9}$$

Using the notations in (4.3) and the relation $\omega_2 = 2\pi L^{-1}$, the above inequality is equivalent to

$$\|Bv - Bv^*\| \leqslant \bar{M} K_0 |\beta|^{-2} (L/\pi)^2 D^* \|v - v^*\| \qquad (v, v^* \in \chi). \quad (4.10)$$

It follows from the hypothesis on β that B is a contraction on S. By a well-known theorem in [12, p. 31], the operator $(A + B)$ has a fixed point in S. This proves the theorem.

An immediate consequence of Theorem 3 is the following.

COROLLARY. *Let* (H_1) *hold and let* $f_j(v)$ *be uniformly bounded by a constant* M *for each* $j = 1, \ldots, n$. *Denote by* B_r *the ball in* χ *with center origin radius* $r \equiv M[\rho + (L/\pi)^2 K_0 D^*]$. *Then for any* λ, β *with* $|\beta| > \max\{1, (L/\pi)(K_0 D^* \bar{M})^{1/2}\}$, *where* $\bar{M}$ *is given by* (4.3) *with* $S = B_r$, *Eq.* (1.0) *has an almost periodic plane wave solution in the form* $u(t, x) \equiv v(t, \lambda t - \beta \cdot x)$ *and* $v \in B_r$.

Proof. It is easily seen from (4.3), (4.5), Lemma 4 and the uniform boundedness of f_j that for any $v_1, v_2 \in B_r$,

$$\begin{aligned} \|Av_1 + Bv_2\| &\leqslant M \sum_{j=1}^{n} \sigma_j^{-1} + 4M \sum_{j=1}^{n} \sum_{k=1}^{\infty} (|\beta|^2 \omega_2^2 k^3 D_j^* + \sigma_j k)^{-1} \\ &\leqslant M[\rho + (L/(\pi |\beta|))^2 K_0 D^*] \leqslant r. \end{aligned}$$

The above relation shows that $(Av_1 + Bv_2) \in B_r$ for all $v_1, v_2 \in B_r$. The conclusion of the corollary follows from Theorem 3 with $S = B_r$.

Remark. It is easily seen from the proofs of Theorems 1 and 3 that all the conclusions in these theorems hold when the requirement $f_j \in C^1(\mathbb{R}^n)$ is replaced by that f_j satisfies a Lipschitz condition in every bounded subset of $\mathbb{R}^n$.

EXAMPLE. To give an application of our results in the previous section we consider an immobilized enzyme reaction problem which involves a coupled system of two equations in the form (1.0) with

$$\begin{aligned} f_1(u_1, u_2) = f_2(u_1, u_2) &= \sigma_1 (u_2 + v_2) f(u_1), \\ f(u_1) &= (u_1 + v_1)/[1 + (u_1 + v_1) + \gamma (u_1 + v_1)^2], \end{aligned} \quad (4.11)$$

where γ, σ_i, v_i, $i = 1, 2$, are positive constants (cf. [4]). For physical reasons we assume that $f_1(u_1, u_2) = f_2(u_1, u_2) = 0$ when either $u_1 + v_1 < 0$ or $u_2 + v_2 < 0$. Clearly, f_1 and f_2 are continuous in $\mathbb{R}^2$ and satisfy a Lipschitz condition in every bounded subset of $\mathbb{R}^2$. In view of Theorem 3, Eq. (1.0) with the function (4.11) has a periodic traveling wave solution if there is a bounded closed convex set S such that $(Av_1 + Bv_2) \in S$ when $v_1, v_2 \in S$. We seek such a set in the form

$$S_0 \equiv \{(v_1, v_2);\ \|v_1 + \nu_1\| \leqslant R_1, \|v_2 + \nu_2\| \leqslant R_2\}$$

by a suitable choice of R_1, R_2. Since by direct calculation,

$$\sup\{f(v_1); 0 \leqslant v_1 + \nu_1 < \infty\} = (2\gamma^{1/2} + 1)^{-1},$$

Eq. (4.5) and Lemma 4 imply that for any $v = (v_1, v_2)$, $v^* = (v_1^*, v_2^*)$ in S_0.

$$\begin{aligned}\|A_1 v + B_1 v^*\| &\leqslant \sigma_1^{-1}(\sigma_1 \|v_2 + \nu_2\|\, \|f(v_1)\|) \\ &\quad + 4K_0(|\beta|\, \omega_2)^{-2}(D_1^*)^{-1}\sigma_1 \|v_2^* + \nu_2\|\, \|f(v_1^*)\| \\ &\leqslant (2\gamma^{1/2} + 1)^{-1}[1 + 4K_0(|\beta|\, \omega_2)^{-2}(D_1^*)^{-1}\sigma_1]\, R_2 \equiv \rho_1 R_2, \\ \|A_2 v + B_2 v^*\| &\leqslant (2\gamma^{1/2} + 1)^{-1}[(\sigma_1/\sigma_2) + 4K_0(|\beta|\, \omega_2)^{-2}(D_2^*)^{-1}\sigma_1]\, R_2.\end{aligned}$$

Hence if

$$\sigma_1/\sigma_2 < (2\gamma^{1/2} + 1) \tag{4.12}$$

then there exists $\beta_0 \in \mathbb{R}^n$ such that for all β with $|\beta| \geqslant |\beta_0|$

$$\begin{aligned}&\|A_2 v + B_2 v^*\| \leqslant \|v_2 + \nu_2\| \leqslant R_2, \\ &\|A_1 v + B_1 v^*\| \leqslant R_1 \qquad (v, v^* \in S_0),\end{aligned}$$

where we have chosen $R_1 = \rho_1 R_2$. This shows that $(Av + Bv^*) \in S_0$ when $v, v^* \in S_0$. In view of Theorem 3, we have the following conclusion: if $TL^{-1}c_j$ are integers and if (4.12) holds then there exists $\beta_0 \in \mathbb{R}^n$ such that for all λ, β with $|\beta| \geqslant |\beta_0|$ the enzyme reaction system has a periodic traveling wave solution $u = (u_1, u_2)$ with $u_i(t, x) = v_i(t, \lambda t - \beta \cdot x)$ and $v = (v_1, v_2) \in S_0$.

References

1. L. Amerio and G. Prouse, "Almost-Periodic Functions and Functional Equations," Van Nostrand–Reinhold, New York, 1971.
2. D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. in Math.* **30** (1978), 33–76.
3. G. H. Hardy and W. W. Rogosinski, "Fourier Series," Cambridge Univ. Press, London, 1962.
4. J. P. Kernevez, G. Joly, M. C. Duban, and D. Thomas, Hysteresis, oscillations, and pattern formation in realistic immobilized enzyme systems, *J. Math. Biol.* **7** (1979), 41–56.

5. N. Kopell and L. N. Howard, Plane wave solutions to reaction-diffusion equations, *Stud. Appl. Math.* **52** (1973), 291–328.
6. Liu Bao-ping and C. V. Pao, Periodic solutions of coupled semilinear parabolic boundary value problem, *Nonlinear Anal. TMA* **6** (1982), 237–252.
7. Liu Bao-ping, Mixed problems for one-dimensional nonlinear parabolic systems (in Chinese), *Acta Math. Sinica* **23** (1980), 870–882.
8. Liu Bao-ping, The integral equation method in the nonlinear vibrations of systems with several degree of freedom (in Chinese), *Acta Math. Sinica* **21** (1978), 80–85.
9. K. Maginu, Stability of periodic travelling wave solutions with large spatial periods in reaction-diffusion systems, *J. Differential Equations* **39** (1981), 73–99.
10. L. Schwartz, "Analysis," Vol. I, Press "Peace," Moscow, 1972 (in Russian).
11. W. Pogorzelski, "Integral Equations and Their Applications," Vol. 1, Pergamon, New York, 1966.
12. D. R. Smart, "Fixed Point Theorems," Cambridge Univ. Press, London, 1974.