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Representation theorems for quadratic $\mathcal{F}$-consistent nonlinear expectations☆

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Abstract

In this paper we extend the notion of "filtration-consistent nonlinear expectation" (or "$\mathcal{F}$-consistent nonlinear expectation") to the case when it is allowed to be dominated by a g-expectation that may have a quadratic growth. We show that for such a nonlinear expectation many fundamental properties of a martingale can still make sense, including the Doob–Meyer type decomposition theorem and the optional sampling theorem. More importantly, we show that any quadratic $\mathcal{F}$-consistent nonlinear expectation with a certain domination property must be a quadratic g-expectation as was studied in [J. Ma, S. Yao, Quadratic g-evaluations and g-martingales, 2007, preprint]. The main contribution of this paper is the finding of a domination condition to replace the one used in all the previous works (e.g., [F. Coquet, Y. Hu, J. Mémin, S. Peng, Filtration-consistent nonlinear expectations and related g-expectations, Probab. Theory Related Fields 123 (1) (2002) 1–27; S. Peng, Nonlinear expectations, nonlinear evaluations and risk measures, in: Stochastic Methods in Finance, in: Lecture Notes in Math., vol. 1856, Springer, Berlin, 2004, pp. 165–253]), which is no longer valid in the quadratic case. We also show that the representation generator must be deterministic, continuous, and actually must be of the simple form $g(z) = \mu(1 + |z|)|z|$, for some constant $\mu > 0$.

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1. Introduction

In this paper we study a class of *filtration-consistent nonlinear expectations* (or *$\mathcal{F}$-consistent nonlinear expectations*), first introduced by Coquet, Hu, Mémin and Peng [6]. Such nonlinear expectations are natural extensions of the so-called g-expectation, initiated in [13], and therefore have direct relations with a fairly large class of risk measures in finance. The main point of interest of this paper is that the nonlinear expectations are allowed to have possible quadratic growth, and our ultimate goal is to prove a representation theorem that characterizes the nonlinear expectations in terms of a class of quadratic BSDEs. We should note that the class of "quadratic nonlinear expectations" under consideration contains many convex risk measures that are not necessarily "coherent". The most notable example is the entropic risk measure (see, e.g., Barrieu and El Karoui [2]), which is known to have a representation as the solution to a quadratic BSDE, but falls outside the existing theory of $\mathcal{F}$-consistent nonlinear expectations. We refer the reader to [1] and [7] for the basic concepts of coherent and convex risk measures, respectively, to [14] for detailed accounts of the relationship between the risk measures and nonlinear expectations. A brief review of the basic properties of $\mathcal{F}$-consistent nonlinear expectations will be given in Section 2 for ready reference.

An interesting result so far in the development of the notion of $\mathcal{F}$-consistent nonlinear expectations is its relationship with the backward stochastic differential equations (BSDEs). Although as an extension of the so-called *g-expectation*, which is defined directly via the BSDE, it is conceivable that an $\mathcal{F}$-consistent nonlinear expectation should have some connection to BSDEs, the proof is by no means easy. For the case where g has only linear growth, it was shown in [6] that if an $\mathcal{F}$-consistent nonlinear expectation is "*dominated*" by a g^μ-expectation in the sense that

$$\mathcal{E}[X] - \mathcal{E}[Y] \le \mathcal{E}^{g^\mu}[X - Y], \quad \forall X, Y \in L^2(\mathcal{F}_T), \tag{1.1}$$

where $g^\mu = \mu|z|$ for some constant $\mu > 0$, then it has to be a g-expectation. The significance of such a result might be more clearly seen from the following consequence in finance: *any time-consistent risk measure satisfying the required domination condition can be represented by the solution of a simple BSDE*(!). In an accompanying paper by Ma and Yao [11], the notion of g-expectation was generalized to the quadratic case, along with some elementary properties of the g-expectations including the Doob–Meyer decomposition and upcrossing inequalities. However, the representation property for general (even convex) risk measure seems to be much more subtle. One of the immediate obstacles is that the "domination" condition (1.1) breaks down in the quadratic case. For example, one can check that a quadratic g-expectation with $g = \mu(|z| + |z|^2)$ *cannot* be dominated by itself(!). Therefore some new ideas for replacing the domination condition (1.1) are in order.

The main purpose of this paper is to generalize the notion of $\mathcal{F}$-consistent nonlinear expectation to the quadratic case and prove at least a version of the representation result for such nonlinear expectations. An important contribution of this paper is the finding of a new domination condition for the quadratic nonlinear expectation, stemming from the reverse Hölder inequality in BMO theory [9]. More precisely, we observe that there exists an L^p estimation for the difference of quadratic g-expectations by using the reverse Hölder inequality. Extending such an estimate to the general nonlinear expectations, we then obtain an L^p-type domination which turns out to be sufficient for our purpose. Following the idea in [14], with the help of the new domination condition, we then prove the optional sampling, and a Doob–Meyer type decomposition theorem for quadratic $\mathcal{F}$-martingales. Like for the linear case, we can then prove that the representation property for the quadratic $\mathcal{F}$-consistent nonlinear expectation remains valid under such a domination condition. That is, one can always find a quadratic g-expectation with g being of the form: $g = \mu(1 + |z|)|z|$, to represent the given nonlinear expectation.

Our discussion on quadratic nonlinear expectation benefited greatly from the recent development on the theory of BSDEs with quadratic growth, initiated by Kobylanski [10], and the subsequent results on such BSDEs with unbounded terminal conditions by Briand and Hu [4,5]. In particular, we need to identify an appropriate subset of exponentially integrable random variables with certain algebraic properties on which a quadratic $\mathcal{F}$-consistent nonlinear expectation can be defined. It is worth noting that such a set will have to contain all the random variables of the form $\xi + zB_\tau$, where B is the driving Brownian motion, $\xi \in L^\infty(\mathcal{F}_T)$, and τ is any stopping time, which turns out to be crucial in proving the representation theorem and the continuity of the representation function g. We should remark that although most of the steps towards our final result look quite similar to the linear growth case, some special treatments are necessary along the way to overcome various technical subtleties caused by the quadratic BSDEs, especially those with unbounded terminal conditions. We believe that many of the results are interesting in their own right. We therefore present full details for future reference.

This paper is organized as follows. In Section 2 we give the preliminaries and review some basics of quadratic g-expectations and the BMO martingales. In Section 3 we introduce the notion of quadratic $\mathcal{F}$-consistent nonlinear expectations and several new notions of the dominations. In Section 4, we show some properties of quadratic $\mathcal{F}$-expectations including the optional sampling theorem, which pave the ways for the later discussions. In Section 5, we prove a Doob–Meyer type decomposition theorem for quadratic $\mathcal{F}$-submartingales. The last section is devoted to the proof of the representation theorem of the quadratic nonlinear expectations.

2. Preliminaries

Throughout this paper we consider a filtered, complete probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$ on which is defined a d-dimensional Brownian motion B. We assume that the filtration $\mathbf{F} \stackrel{\triangle}{=} \{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion B, augmented by all the P-null sets in $\mathcal{F}$, so that it satisfies the *usual hypotheses* (cf. [15]). We denote as $\mathscr{P}$ the progressive measurable σ-field on $\Omega\times[0, T]$; and as $\mathcal{M}_{0,T}$ the set of all $\mathbf{F}$-stopping times τ such that $0 \leq \tau \leq T$, P-a.s., where $T > 0$ is some fixed time horizon.

In what follows we fix a finite time horizon $T > 0$, and denote as $\mathbb{E}$ a generic Euclidean space, whose inner products and norms will be denoted as the same $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively; and denote as $\mathbb{B}$ a generic Banach space with norm $\|\cdot\|$. Moreover, we shall denote as $\mathcal{G} \subseteq \mathcal{F}$ any sub-σ-field, and for any $x \in \mathbb{R}^d$ and any $r > 0$ we denote as $\overline{B}_r(x)$ the closed ball with center x and radius r. Furthermore, the following spaces of functions will be frequently used in the sequel. We define:

- for $0 \leq p \leq \infty$, $L^p(\mathcal{G}; \mathbb{E})$ to be the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables ξ, with $E(|\xi|^p) < \infty$; in particular, if $p = 0$, then $L^0(\mathcal{G}, \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables; and if $p = \infty$, then $L^\infty(\mathcal{G}; \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables ξ such that $\|\xi\|_\infty \stackrel{\triangle}{=} \mathrm{esssup}_{\omega\in\Omega}|\xi(\omega)| < \infty$;

- for $0 \le p \le \infty$, $L^p_{\mathbf{F}}([0,T];\mathbb{B})$ to be the space of all $\mathbb{B}$-valued, $\mathbf{F}$-adapted processes ψ, such that $E\int_0^T \|\psi_t\|^p \mathrm{d}t < \infty$; in particular, $p = 0$ stands for all $\mathbb{B}$-valued, $\mathbf{F}$-adapted processes; and $p = \infty$ denotes all processes $X \in L^0_{\mathbf{F}}([0,T];\mathbb{B})$ such that $\|X\|_\infty \triangleq \operatorname{esssup}_{t,\omega}|X(t,\omega)| < \infty$;
- $\mathbb{D}^\infty_{\mathbf{F}}([0,T];\mathbb{B}) = \{X \in L^\infty_{\mathbf{F}}([0,T];\mathbb{B}) : X \text{ has càdlàg paths}\}$;
- $\mathbb{C}^\infty_{\mathbf{F}}([0,T];\mathbb{B}) = \{X \in \mathbb{D}^\infty_{\mathbf{F}}([0,T];\mathbb{B}) : X \text{ has continuous paths}\}$;
- $\mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{B}) = \{X \in L^2_{\mathbf{F}}([0,T];\mathbb{B}) : X \text{ is predictably measurable}\}$.

The following two spaces are variations of the L^p spaces defined above; they will be important for our discussions regarding quadratic BSDEs with unbounded terminal conditions. For any $p > 0$, we define $\mathcal{M}^p(\mathbb{R}^d)$ to be the space of all $\mathbb{R}^d$-valued predictable processes X such that

$$\|X\|_{\mathcal{M}^p} \triangleq \left(E\left(\int_0^T |X_s|^2 \mathrm{d}s\right)^{p/2}\right)^{1\wedge 1/p} < \infty. \tag{2.1}$$

We note that for $p \ge 1$, $\mathcal{M}^p(\mathbb{R}^d)$ is a Banach space with the norm $\|\cdot\|_{\mathcal{M}^p}$, and for $p \in (0,1)$, $\mathcal{M}^p(\mathbb{R}^d)$ is a complete metric space with the distance defined through (2.1) Finally, if $d = 1$, we shall drop $\mathbb{E} = \mathbb{R}$ from the notation (e.g., $L^p_{\mathbf{F}}([0,T]) = L^p_{\mathbf{F}}([0,T];\mathbb{R})$, $L^\infty(\mathcal{F}_T) = L^\infty(\mathcal{F}_T;\mathbb{R})$, and so on).

2.1. *Quadratic g-expectations on $L^\infty(\mathcal{F}_T)$*

We now give a brief review of the notion of *quadratic g-expectations* studied in Ma and Yao [11]. First recall that for any $\xi \in L^2(\mathcal{F}_T)$, and a given "generator" $g = g(t,\omega,y,z) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfying the standard conditions (e.g., it is Lipschitz in all spatial variables, and is of linear growth, etc.), the g-expectation of ξ is defined as $\mathcal{E}^g(\xi) \triangleq Y_0$, where $Y = \{Y_t : 0 \le t \le T\}$ is the solution to the following BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad \forall t \in [0,T]. \tag{2.2}$$

We shall denote (2.2) as BSDE(ξ, g) in the sequel for notational convenience.

In [11] the g-expectation was extended to the quadratic case, based on the well-posedness result of the quadratic BSDEs given by Kobylanski [10], and under rather general conditions on the generator g. In this paper, however, we shall content ourselves with a slightly simplified form of the generator g that is sufficient for our purpose. More precisely, we assume that the generator g is independent of the variable y, and satisfies the following *Standing Assumptions*:

(H1) The function $g : [0,T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable and $g(t,\omega,\cdot)$ is continuous for all $(t,\omega) \in [0,T] \times \Omega$.

(H2) There exists a constant $\ell > 0$ such that for $\mathrm{d}t \times \mathrm{d}P$-a.s. $(t,\omega) \in [0,T] \times \Omega$ and any $z \in \mathbb{R}^d$

$$|g(t,\omega,z)| \le \ell(|z| + |z|^2) \quad \text{and} \quad \left|\frac{\partial g}{\partial z}(t,\omega,z)\right| \le \ell(1 + |z|). \tag{2.3}$$

In the light of the results of [10] we know that under the assumptions (H1) and (H2), for any $\xi \in L^\infty(\mathcal{F}_T)$ the BSDE (2.2) has a unique solution $(Y,Z) \in \mathbb{C}^\infty_{\mathbf{F}}([0,T]) \times \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. We can then define the quadratic g-expectation of ξ as $\mathcal{E}^g(\xi) = Y_0$ and the *conditional g-expectation* as

$$\mathcal{E}^g[\xi|\mathcal{F}_t] \stackrel{\triangle}{=} Y_t^\xi, \quad \forall t \in [0,T], \forall \xi \in L^\infty(\mathcal{F}_T). \tag{2.4}$$

It is easy to see that $g|_{z=0} = 0$ from (H2). So by the uniqueness of the solution to the quadratic BSDE, one can show that all the fundamental properties of nonlinear expectations are still valid for quadratic g-expectations:

(i) (time-consistency) $\mathcal{E}^g[\mathcal{E}^g[\xi|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}^g[\xi|\mathcal{F}_s]$, P-a.s. $\forall 0 \le s \le t$;
(ii) (constant-preserving) $\mathcal{E}^g[\xi|\mathcal{F}_t] = \xi$, P-a.s. $\forall \xi \in L^\infty(\mathcal{F}_t)$;
(iii) ("zero–one law") $\mathcal{E}^g[\mathbf{1}_A\xi|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}^g[\xi|\mathcal{F}_t]$, P-a.s. $\forall A \in \mathcal{F}_t$.

Furthermore, since g is independent of y, then we know that the quadratic g-expectation is also "*translation invariant*" in the sense that

$$\mathcal{E}^g[\xi+\eta|\mathcal{F}_t] = \mathcal{E}^g[\xi|\mathcal{F}_t] + \eta, \quad P\text{-a.s. } \forall t \in [0,T], \forall \eta \in L^\infty(\mathcal{F}_t). \tag{2.5}$$

Along the same lines as [14] we can define the "quadratic g-martingales" as usual. For example, a process $X \in L^\infty_{\mathbf{F}}([0,T])$ is called a g-submartingale (resp. g-supermartingale) if for any $0 \le s < t \le T$, it holds that

$$\mathcal{E}^g[X_t|\mathcal{F}_s] \ge (\text{resp. } \le) X_s, \quad P\text{-a.s.}$$

The process X is called a quadratic g-martingale if it is both a g-submartingale and a g-supermartingale.

Similar to the cases studied in [14] where g is Lipschitz continuous and of linear growth, it was shown in [11] that the quadratic g-sub(super)martingales also admit the Doob–Meyer type decomposition, and an upcrossing inequality holds (cf. [11, Theorem 4.6]). The next theorem summarizes some results of [11], adapted to the current setting, which will be used in our future discussion. The proof of these results can be found in [11, Theorem 4.2 and Corollary 4.7].

Theorem 2.1. *Assume* (H1) *and* (H2). *Then, for any right-continuous g-submartingale (resp. g-supermartingale) $Y \in L^\infty_{\mathbf{F}}([0,T])$, there exist an increasing (resp. decreasing) càdlàg process A null at* 0 *and a process $Z \in \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$, such that*

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - A_T + A_t - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0,T].$$

Furthermore, if g vanishes as z vanishes, then any g-submartingale (resp. g-supermartingale) X must satisfy the following continuity property: For any dense subset $\mathcal{D}$ of $[0,T]$*, P-almost surely, the limit* $\lim_{r\nearrow t, r\in\mathcal{D}} X_r$ *(resp.* $\lim_{r\searrow t, r\in\mathcal{D}} X_r$*) exists for any $t \in (0,T]$ (resp. $t \in [0,T)$).* ∎

2.2. BMO and exponential martingales

To end this section, we recall some important facts regarding the so-called "BMO martingales" and the properties of the related stochastic exponentials. We refer the reader to the monograph of Kazamaki [9] for a complete exposition of the theory of continuous BMO and exponential martingales. Here we shall be content with just some facts that are useful in our future discussions.

To begin with, we recall that a uniformly integrable martingale M null at zero is called a "BMO martingale" on $[0,T]$, if for some $1 \le p < \infty$, it holds that

$$\|M\|_{\mathrm{BMO}_p} \stackrel{\triangle}{=} \sup_{\tau\in\mathcal{M}_{0,T}} \|E\{|M_T - M_{\tau-}|^p|\mathcal{F}_\tau\}^{1/p}\|_\infty < \infty. \tag{2.6}$$

In such a case we define $M \in \text{BMO}(p)$. It is important to note that $M \in \text{BMO}(p)$ if and only if $M \in \text{BMO}(1)$, and all the $\text{BMO}(p)$ norms are equivalent. Therefore in what follows we shall say that a martingale M is BMO without specifying the index p; and we shall use only the BMO(2) norm and denote it simply by $\|\cdot\|_{\text{BMO}}$. Note also that for a *continuous* martingale M one has

$$\|M\|_{\text{BMO}} = \|M\|_{\text{BMO}_2} = \sup_{\tau \in \mathcal{M}_{0,T}} \|E\{\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\}\|_\infty.$$

Now, for a given Brownian motion B, we say that a process $Z \in L^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ is a BMO process, denoted by $Z \in \text{BMO}$ with a slight abuse of notation, if the stochastic integral $M \stackrel{\triangle}{=} Z \bullet B$ is a BMO martingale. We remark that the space of BMO martingales is smaller than any $\mathcal{M}^p(\mathbb{R}^d)$ space (see (2.1) for definition). To wit, it holds that $\text{BMO} \subset \bigcap_{p>0} \mathcal{M}^p(\mathbb{R}^d)$. Furthermore, by the so-called "energy inequality" [9, p. 29], one checks that

$$(\|Z\|_{\mathcal{M}_{2n}})^{2n} = E\left(\int_0^T |Z_s|^2 \mathrm{d}s\right)^n \leq n!\|Z\|^{2n}_{\text{BMO}}, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

We now turn our attention to the stochastic exponentials of the BMO martingales. Recall that for a continuous martingale M, the Doléans–Dade stochastic exponential of M, denoted customarily by $\mathscr{E}(M)$, is defined as $\mathscr{E}(M)_t \stackrel{\triangle}{=} \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}$, $t \geq 0$. Note that if $\mathscr{E}(M)$ is a uniformly integrable martingale, then the Hölder inequality implies that

$$\mathscr{E}(M)^p_\tau \leq E[\mathscr{E}(M)^p_T | \mathcal{F}_\tau], \quad P\text{-a.s.} \tag{2.8}$$

for any stopping time $\tau \in \mathcal{M}_{0,T}$ and any $p \geq 1$. However, if M is further a BMO martingale, then the stochastic exponential $\mathscr{E}(M)$ is itself a uniform integrable martingale (see [9, Theorem 2.3]). Moreover, the so-called "reverse Hölder inequality" (cf. [9, Theorem 3.1]) holds for $\mathscr{E}(M)$. We note that this inequality plays a fundamental role in the new domination condition for the nonlinear expectations, which leads to the representation theorem and its continuity; we give the complete statement here for ready reference. For any $\alpha > 2$, define

$$\phi_\alpha(x) \stackrel{\triangle}{=} \left\{1 + x^{-2}\log\left[(1 - 2\alpha^{-x})\frac{2x-1}{2x-2}\right]\right\}^{\frac{1}{2}} - 1, \quad x \in (1,\infty). \tag{2.9}$$

Theorem 2.2 (*Reverse Hölder Inequality*). *Suppose that $M \in$ BMO. If it satisfies that $\|M\|_{\text{BMO}} \leq \phi_\alpha(p)$, then one has*

$$E[\mathscr{E}(M)^p_T | \mathcal{F}_\tau] \leq \alpha^p \mathscr{E}(M)^p_\tau, \quad \forall \tau \in \mathcal{M}_{0,T}. \tag{2.10}$$

Finally, we give a result that relates the solution to a quadratic BSDE to the BMO processes. Let us consider the BSDE (2.2) in which the generator g has a quadratic growth. For simplicity, we assume there is some $k > 0$ (we may assume without loss of generality that $k \geq \frac{1}{2}$) such that for $\mathrm{d}t \times \mathrm{d}P$-a.s. $(t,\omega) \in [0,T] \times \Omega$,

$$|g(t,\omega,y,z)| \leq k(1 + |z|^2), \quad \forall (y,z) \in \mathbb{R} \times \mathbb{R}^d. \tag{2.11}$$

Let $(Y,Z) \in \mathbb{C}^\infty_{\mathbf{F}}([0,T]) \times \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ be a solution to (2.2). Applying Itô's formula to e^{4kY_t} from t to T one has

$$\begin{aligned}\mathrm{e}^{4kY_t}+8k^2\int_t^T \mathrm{e}^{4kY_s}|Z_s|^2\mathrm{d}s &= \mathrm{e}^{4kY_T}+4k\int_t^T \mathrm{e}^{4kY_s}g(s,Y_s,Z_s)\mathrm{d}s-4k\int_t^T \mathrm{e}^{4kY_s}Z_s\mathrm{d}B_s\\ &\le \mathrm{e}^{4kY_T}+4k^2\int_t^T \mathrm{e}^{4kY_s}(1+|Z_s|^2)\mathrm{d}s-4k\int_t^T \mathrm{e}^{4kY_s}Z_s\mathrm{d}B_s.\end{aligned}$$

Taking the conditional expectation $E\{\cdot|\mathcal{F}_t\}$ on both sides above, and then using some standard manipulations one derives fairly easily that

$$E\left[\int_t^T |Z_s|^2\mathrm{d}s|\mathcal{F}_t\right]\le \mathrm{e}^{4k\|Y\|_\infty}E[\mathrm{e}^{4k\xi}-\mathrm{e}^{4kY_t}|\mathcal{F}_t]+\mathrm{e}^{8k\|Y\|_\infty}(T-t).$$

In other words, we have proved the following result.

Proposition 2.3. *Suppose that $(Y,Z)\in\mathbb{C}^\infty_{\mathbf{F}}([0,T])\times\mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ is a solution of the BSDE (2.2) with $\xi\in L^\infty(\mathcal{F}_T)$, and g satisfies (2.11). Then $Z\in BMO$, and the following estimate holds:*

$$\|Z\|^2_{\mathrm{BMO}}\le(1+T)\mathrm{e}^{8k\|Y\|_\infty}. \quad \blacksquare$$

3. Quadratic $\mathcal{F}$-expectations

In this section we introduce the notion of "quadratic $\mathcal{F}$-consistent nonlinear expectation". To begin with, we recall from [14] that an $\mathcal{F}$-consistent nonlinear expectation is a family of operators, denoted by $\{\mathcal{E}_t\}_{t\ge0}$, such that for each $t\in[0,T]$, $\mathcal{E}_t : L^0(\mathcal{F}_T)\mapsto L^0(\mathcal{F}_t)$, and that the following axioms are fulfilled:

(A1) *monotonicity*: $\mathcal{E}_t[\xi]\ge\mathcal{E}_t[\eta]$, P-a.s., if $\xi\ge\eta$, P-a.s.;
(A2) *constant-preserving*: $\mathcal{E}_t[\xi]=\xi$, P-a.s., $\forall\xi\in L^0(\mathcal{F}_t)$;
(A3) *time-consistency*: $\mathcal{E}_s[\mathcal{E}_t[\xi]]=\mathcal{E}_s[\xi]$, P-a.s., $\forall s\in[0,t]$;
(A4) *"zero–one law"*: $\mathcal{E}_t[\mathbf{1}_A\xi]=\mathbf{1}_A\mathcal{E}_t[\xi]$, P-a.s., $\forall A\in\mathcal{F}_t$.

The operator $\mathcal{E}_t[\cdot]$ has been called the "nonlinear conditional expectation", and denoted by $\mathcal{E}\{\cdot|\mathcal{F}_t\}$ for obvious reasons. It was worth noting that in all the previous cases the natural "domain" of the nonlinear expectation is the space $L^2(\mathcal{F}_T)$; thus a nonlinear expectation can be related to the solution to the BSDEs using the "classical" theory.

In the quadratic case, however, the situation is quite different. In particular, if the main concern is the representation theorem where the quadratic BSDE is inevitable, then the domain of the nonlinear expectation will become a fundamental issue. For example, due to the limitation of the well-posedness of a quadratic BSDE, a quadratic nonlinear expectation would naturally be restricted to the space $L^\infty(\mathcal{F}_T)$. But on the other hand, in the light of the previous works (see, e.g., [6,14]), we see that technically the domain of $\mathcal{E}$ should also include the following set:

$$\mathscr{L}^\infty_T\stackrel{\triangle}{=}\{\xi=\xi_0+zB_T:\xi_0\in L^\infty(\mathcal{F}_T),z\in\mathbb{R}^d\}. \tag{3.1}$$

Here B is the driving Brownian motion. A simple observation of the Axioms (A3) and (A4) clearly indicates that $\mathcal{E}$ cannot be defined simply as a mapping from $\mathscr{L}^\infty_T$ to $\mathscr{L}^\infty_t$. For example, in general the random variable $\mathbf{1}_A\xi$ will not even be an element of $\mathscr{L}^\infty_T$(!), thus (A4) will not make sense.

To overcome this difficulty let us now find a larger subset $\Lambda\subseteq L^0(\mathcal{F}_T)$ that contains $\mathscr{L}^\infty_T$ and can serve as a possible domain of a nonlinear expectation. First, we observe that such a set must satisfy the following property in order that Axioms (A1)–(A4) can be well defined.

Definition 3.1. Let $\mathscr{D}(\mathcal{F}_T)$ denote the totality of all subsets Λ in $L^0(\mathcal{F}_T)$ satisfying: for all $t \in [0, T]$, the set $\Lambda_t \stackrel{\triangle}{=} \Lambda \cap L^0(\mathcal{F}_t)$ is closed under the multiplication with $\mathcal{F}_t$ indicator functions. That is, if $\xi \in \Lambda_t$ and $A \in \mathcal{F}_t$, then $\mathbf{1}_A \xi \in \Lambda_t$. ■

It is easy to see that $L^\infty(\mathcal{F}_T) \in \mathscr{D}(\mathcal{F}_T)$ and $\mathscr{D}(\mathcal{F}_T)$ is closed under intersections and unions. Thus for any $S \subset L^0(\mathcal{F}_T)$, we can define the smallest element in $\mathscr{D}(\mathcal{F}_T)$ that contains S as usual by $\Lambda(S) \stackrel{\triangle}{=} \bigcap_{\Lambda \in \mathscr{D}(\mathcal{F}_T), S \subset \Lambda} \Lambda$. We are now ready to define the quadratic $\mathcal{F}$-consistent nonlinear expectations.

Definition 3.2. An $\mathcal{F}$-consistent nonlinear expectation with domain Λ is a pair $(\mathcal{E}, \Lambda)$, where $\Lambda \in \mathscr{D}(\mathcal{F}_T)$, and $\mathcal{E} = \{\mathcal{E}_t\}_{t \geq 0}$ is a family of operators $\mathcal{E}_t : \Lambda \mapsto \Lambda_t$, $t \in [0, T]$, satisfying Axioms (A1)–(A4).

Moreover, $\mathcal{E}$ is called "translation invariant" if $\Lambda + L_T^\infty \subset \Lambda$ and (2.5) holds for any $\xi \in \Lambda$, any $t \in [0, T]$ and any $\eta \in L^\infty(\mathcal{F}_t)$. ■

Again, we shall define $\mathcal{E}_t[\cdot] = \mathcal{E}[\cdot|\mathcal{F}_t]$ as usual, and we define $\Lambda = \mathrm{Dom}(\mathcal{E})$ to be the domain of $\mathcal{E}$. To simplify notation, in what follows when we say an $\mathcal{F}$-consistent nonlinear expectation $\mathcal{E}$, we always mean the pair $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$. Note that a standard g-expectation and the $\mathcal{F}$-consistent nonlinear expectation studied in [6] and [14] all have domain $\Lambda = L^2(\mathcal{F}_T)$, and they are translation invariant if g is independent of y. The quadratic g-expectation studied in [11] is one with domain $\Lambda = L^\infty(\mathcal{F}_T)$.

We now turn to the notion of "quadratic" $\mathcal{F}$-consistent nonlinear expectations.

Definition 3.3. An $\mathcal{F}$-consistent nonlinear expectation $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is called upper (resp. lower) semi-quadratic if there exists a quadratic g-expectation $(\mathcal{E}^g, \mathrm{Dom}(\mathcal{E}^g))$ with $\mathrm{Dom}(\mathcal{E}^g) \subseteq \mathrm{Dom}(\mathcal{E})$ such that for any $t \in [0, T]$ and any $\xi \in \mathrm{Dom}(\mathcal{E}^g)$, it holds that

$$\mathcal{E}[\xi|\mathcal{F}_t] \leq (\text{resp. } \geq) \mathcal{E}^g[\xi|\mathcal{F}_t], \quad P\text{-a.s.} \tag{3.2}$$

Moreover, $\mathcal{E}$ is called quadratic if there exist two quadratic g-expectations $\mathcal{E}^{g_1}$ and $\mathcal{E}^{g_2}$ with $\mathrm{Dom}(\mathcal{E}^{g_1}) \cap \mathrm{Dom}(\mathcal{E}^{g_2}) \subseteq \mathrm{Dom}(\mathcal{E})$ such that for any $t \in [0, T]$ and any $\xi \in \mathrm{Dom}(\mathcal{E}^{g_1}) \cap \mathrm{Dom}(\mathcal{E}^{g_2})$, it holds that

$$\mathcal{E}^{g_1}[\xi|\mathcal{F}_t] \leq \mathcal{E}[\xi|\mathcal{F}_t] \leq \mathcal{E}^{g_2}[\xi|\mathcal{F}_t], \quad P\text{-a.s.} \tag{3.3}$$

In what follows, we shall call an $\mathcal{F}$-consistent nonlinear expectation an "$\mathcal{F}$-expectation" for simplicity. Note that a quadratic g-expectation $(\mathcal{E}^g, L^\infty(\mathcal{F}_T))$ would be a trivial example of quadratic $\mathcal{F}$-expectations. The following example is a little more subtle.

Example 3.4. Consider the BSDE (2.2) in which the generator g is Lipschitz in y and has quadratic growth in z. Furthermore, assume that g is convex in (t, y, z). Then, by a recent result of Briand and Hu [5], for any $\xi \in L^0(\mathcal{F}_T)$ such that it has exponential moments of all orders (i.e. $E\{e^{\lambda|\xi|}\} < \infty, \forall \lambda > 0$), the BSDE (2.2) admits a unique solution (Y, Z). In particular, if we assume further that g satisfies $g|_{z=0} = 0$, then it is easy to check that the g-expectation $\mathcal{E}^g(\xi) = Y_0$ defines an $\mathcal{F}$-expectation with domain $\mathrm{Dom}(\mathcal{E}^g) \stackrel{\triangle}{=} \{\xi \in \mathcal{F}_T : E[e^{\lambda|\xi|}] < \infty, \forall \lambda > 0\}$. We should note that in this case the domain does indeed contain the set $\mathscr{L}_T^\infty$ defined in (3.1)! ■

Since we are only interested in the quadratic g-expectations whose domain contains at least the set $\mathscr{L}_T^\infty$, we now introduce the following notion.

Definition 3.5. A quadratic g-expectation $\mathcal{E}^g$ is called "regular" if

$$\{\xi + zB_\tau : \xi \in L^\infty(\mathcal{F}_T), z \in \mathbb{R}^d, \tau \in \mathcal{M}_{0,T}\} \subseteq \mathrm{Dom}(\mathcal{E}^g).$$

Correspondingly, a (semi-)quadratic $\mathcal{F}$-expectation is called "regular" if it is dominated by regular quadratic g-expectation in the sense of Definition 3.3.

Example 3.4 shows the existence of the regular quadratic g-expectations. But it is worth pointing out that because of special form of the set $\mathscr{L}_T^\infty$, the class of regular quadratic g-expectations is much larger. To see this, let us consider any quadratic BSDE with g satisfying (H1) and (H2),

$$Y_t = \xi + zB_\tau + \int_t^T g(s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0, T], \tag{3.4}$$

where $\xi \in L^\infty(\mathcal{F}_T)$, $z \in \mathbb{R}^d$, and $\tau \in \mathcal{M}_{0,T}$. Now, if we set $\widetilde{Y}_t = Y_t - zB_{t\wedge\tau}$, $\widetilde{Z}_t = Z_t - z\mathbf{1}_{\{t\le\tau\}}$, then (3.4) becomes

$$\widetilde{Y}_t = \xi + \int_t^T g(s, \widetilde{Z}_s + z\mathbf{1}_{\{s\le\tau\}})\mathrm{d}s - \int_t^T \widetilde{Z}_s \mathrm{d}B_s, \quad \forall t \in [0, T]. \tag{3.5}$$

Since $\xi \in L^\infty(\mathcal{F}_T)$, the BSDE (3.5) is uniquely solvable whenever g satisfies (H1) and (H2). In other words, any g satisfying (H1) and (H2) can generate a regular g-expectation!

Remark 3.6. For any generator g satisfying (H1) and (H2), one can deduce like for (3.4) and (3.5) that

$$\tilde{\mathscr{L}}_T^\infty \triangleq \left\{\xi + \int_0^T \zeta_s \mathrm{d}B_s : \xi \in L^\infty(\mathcal{F}_T), \zeta \in L_{\mathbf{F}}^\infty([0, T]; \mathbb{R}^d)\right\} \subset \mathrm{Dom}(\mathcal{E}^g). \tag{3.6}$$

Therefore, it follows from Definition 3.3 that $\tilde{\mathscr{L}}_T^\infty \subset \mathrm{Dom}(\mathcal{E}^{g_1}) \cap \mathrm{Dom}(\mathcal{E}^{g_2}) \subset \mathrm{Dom}(\mathcal{E})$, as both g_1 and g_2 satisfy (H1) and (H2). The set $\tilde{\mathscr{L}}_T^\infty$ is very important for the proof of the representation theorem in the last section. ■

3.1. Domination of quadratic $\mathcal{F}$-expectations

In the theory of nonlinear expectations, especially in the proofs of decomposition and representation theorems (cf. [6] and [14]), the notion of "domination" for the difference of two values of $\mathcal{F}$-expectations plays a central role. To be more precise, it was assumed that the following property holds for an $\mathcal{F}$-expectation $\mathcal{E}$: for some g-expectation $\mathcal{E}^g$, it holds for any $X, Y \in L^2(\mathcal{F}_T)$ that

$$\mathcal{E}(X + Y) - \mathcal{E}(X) \le \mathcal{E}^g(Y). \tag{3.7}$$

In the case when g is Lipschitz, this definition of domination is very natural (especially when $g = g(z) = \mu|z|$, $\mu > 0$). However, this notion becomes very ill-posed in the quadratic case. We explain this in the following simple example.

Example 3.7. Consider the simplest quadratic case: $g = g(z) = \frac{1}{2}|z|^2$, and take $\mathcal{E} = \mathcal{E}^g$. We show that even such a simple quadratic g-expectation cannot find a domination in the sense of

(3.7). Indeed, note that

$$\mathcal{E}^g(X+Y) = X + Y + \frac{1}{2}\int_0^T |Z_s^{(1)}|^2 \mathrm{d}s - \int_0^T Z_s^{(1)} \mathrm{d}B_s;$$
$$\mathcal{E}^g(X) = X + \frac{1}{2}\int_0^T |Z_s^{(2)}|^2 \mathrm{d}s - \int_0^T Z_s^{(2)} \mathrm{d}B_s.$$

Defining $Z = Z^{(1)} - Z^{(2)}$ we have

$$\mathcal{E}^g(X+Y) - \mathcal{E}^g(X) = Y + \frac{1}{2}\int_0^T (|Z_s^{(2)} + Z_s|^2 - |Z_s^{(2)}|^2)\mathrm{d}s - \int_0^T Z_s \mathrm{d}B_s.$$

But in the above the drift $\frac{1}{2}(|Z_s^{(2)} + Z_s|^2 - |Z_s^{(2)}|^2) \le |Z_s|^2 + \frac{1}{2}|Z_s^{(2)}|^2$ cannot be dominated by any g satisfying (H1) and (H2). ■

Since finding a general domination rule in the quadratic case is a formidable task, we are now trying to find a reasonable replacement that can serve our purpose. It turns out that the following definition of domination is sufficient for our purpose.

Definition 3.8. (1) A regular quadratic $\mathcal{F}$-expectation $\mathcal{E}$ is said to satisfy the "L^p-domination" if for any $K, R > 0$, there exist constants $p = p(K, R) > 0$ and $C = C_R > 0$ such that for any two stopping times $0 \le \tau_2 \le \tau_1 \le T$, any $\xi_i \in L^\infty_{\tau_i}$ with $\|\xi_i\|_\infty \le K$, $i = 1, 2$, and any $z \in \mathbb{R}^d$ with $|z| \le R$, it holds for each $t \in [0, T]$ that

$$\begin{aligned} &\|(\mathcal{E}\{\xi_1 + zB_{\tau_1}|\mathcal{F}_t\} - zB_{t\wedge\tau_1}) - (\mathcal{E}\{\xi_2 + zB_{\tau_2}|\mathcal{F}_t\} - zB_{t\wedge\tau_2})\|_p \\ &\quad \le 3\|\xi_1 - \xi_2\|_p + C_R\|\tau_1 - \tau_2\|_p. \end{aligned} \tag{3.8}$$

(2) A regular quadratic $\mathcal{F}$-expectation $\mathcal{E}$ is said to satisfy the "L^∞-domination" if for any stopping time $\tau \in \mathcal{M}_{0,T}$, any $\xi_i \in L^\infty(\mathcal{F}_T)$, $i = 1, 2$, and any $z \in \mathbb{R}^d$, the process $\{\mathcal{E}\{\xi_i + zB_\tau|\mathcal{F}_t\} - zB_{t\wedge\tau}, t \in [0, T]\} \in L^\infty_{\mathbf{F}}([0, T])$, $i = 1, 2$, and

$$\|\mathcal{E}\{\xi_1 + zB_\tau|\mathcal{F}_t\} - \mathcal{E}\{\xi_2 + zB_\tau|\mathcal{F}_t\}\|_\infty \le \|\xi_1 - \xi_2\|_\infty, \quad \forall t \in [0, T]. \tag{3.9}$$

(3) A regular quadratic $\mathcal{F}$-expectation $\mathcal{E}$ is said to satisfy the "one-sided g-domination" if for any $K, R > 0$, there are constants $J = J(K, R) > 0$ and $\alpha = \alpha(K, R) > 0$ such that for any stopping time $\tau \in \mathcal{M}_{0,T}$, $\xi \in L^\infty(\mathcal{F}_T)$ with $\|\xi\|_\infty \le K$, and any $z \in \mathbb{R}^d$ with $|z| \le R$, there is a $\gamma \in \mathrm{BMO}$ with $\|\gamma\|^2_{\mathrm{BMO}} \le J(K, R)$ and a function $g_\alpha(z) \overset{\triangle}{=} \alpha(K, R)|z|^2$, $z \in \mathbb{R}^d$, such that for any $\eta \in L^\infty(\mathcal{F}_T)$, it holds that

$$\mathcal{E}[\eta + \xi + zB_\tau|\mathcal{F}_t] - \mathcal{E}[\xi + zB_\tau|\mathcal{F}_t] \le \mathcal{E}^{g_\alpha}_\gamma[\eta|\mathcal{F}_t], \quad \forall t \in [0, T],\ P^\gamma\text{-a.s.} \tag{3.10}$$

Here, P^γ is defined by $\mathrm{d}P^\gamma/\mathrm{d}P = \mathscr{E}(\gamma \bullet B)_T$, and $\mathcal{E}^{g_\alpha}_\gamma$ is the g_α-martingale on the probability space $(\Omega, \mathcal{F}, P^\gamma)$, and with Brownian motion B^γ. ■

The following theorem more or less justifies the ideas of these "dominations".

Theorem 3.9. *Assume that g is a random field satisfying* (H1) *and* (H2)*, and that it satisfies $g|_{z=0} = 0$. Then the quadratic g-expectation $\mathcal{E}^g$ satisfies both L^p- and L^∞-dominations* (3.8) *and* (3.9).

Furthermore, if g also satisfies that $|\frac{\partial^2 g}{\partial z^2}| \le \ell'$ for some $\ell' > 0$, then $\mathcal{E}^g$ also satisfies the one-sided g-domination (3.10) *with $\alpha(K, R) \equiv \ell'/2$.*

Proof. (1) We first show that the L^p-domination holds. Let (Y^i, Z^i), $i = 1, 2$, be the unique solutions of BSDE (3.4) for $\xi_i + zB_{\tau_i}$, $i = 1, 2$, respectively. Define $U_t^i \stackrel{\triangle}{=} Y_t^i - zB_{t\wedge\tau_i}$, $V_t^i = Z_t^i - z\mathbf{1}_{\{t\le\tau_i\}}$, $\Delta U_t = U^1 - U^2$, and $\Delta V = V^1 - V^2$. Then, in the light of (3.4) and (3.5) one can easily check that

$$\Delta U_t = \xi_1 - \xi_2 + \int_{t\vee\tau_2}^{t\vee\tau_1} g(s, z)\mathrm{d}s + \int_t^T \langle\gamma_s, \Delta V_s\rangle\mathrm{d}s - \int_t^T \Delta V_s \mathrm{d}B_s, \quad \forall t \in [0, T], \tag{3.11}$$

where $\gamma_t \stackrel{\triangle}{=} \mathbf{1}_{\{t\le\tau_1\}} \int_0^1 \frac{\partial g}{\partial z}(t, V_t^2 + \theta\Delta V_t + z)\mathrm{d}\theta$. In what follows we shall denote all the constants depending only on T and ℓ in (H2) by a generic one $C > 0$, which may vary from line to line. Applying Proposition 2.3 and Corollary 2.2 of [10] we see that both V^1 and V^2 are BMO with

$$\|V^i\|_{\mathrm{BMO}}^2 \le C\exp\{C(1 + |z|^2)[1 + |z|^2 + \|\xi_i\|_\infty]\}.$$

Thus, by definition of γ we have, for any $K, R > 0$, with $\|\xi^1\|_\infty \vee \|\xi^2\|_\infty \le K$ and $|z| \le R$,

$$\begin{aligned} \|\gamma\|_{\mathrm{BMO}}^2 &\le C[1 + |z|^2 + \|V^1\|_{\mathrm{BMO}}^2 + \|V^2\|_{\mathrm{BMO}}^2] \\ &\le C(1 + |z|^2) + C\exp\{C(1 + |z|^2)[\|\xi_1\|_\infty \vee \|\xi_2\|_\infty + 1 + |z|^2]\} \\ &\le C(1 + R^2) + C\exp\{C(1 + R^2)[1 + K + R^2]\} \stackrel{\triangle}{=} J(K, R). \end{aligned} \tag{3.12}$$

Let us now define $\mathscr{E}(\gamma)_s^t \stackrel{\triangle}{=} \frac{\mathscr{E}(\gamma\bullet B)_t}{\mathscr{E}(\gamma\bullet B)_s} = \exp\{\int_s^t \gamma_r \mathrm{d}B_r - \frac{1}{2}\int_s^t |\gamma_s|^2\mathrm{d}s\}$, for $0 \le s \le t$, and define a new probability measure P^γ by $\mathrm{d}P^\gamma/\mathrm{d}P \stackrel{\triangle}{=} \mathscr{E}(\gamma)_0^T$. Since γ is BMO, applying the Girsanov theorem we derive from (3.11) that

$$\begin{aligned} \Delta U_t &= E^\gamma\left\{\xi_1 - \xi_2 + \int_{t\vee\tau_2}^{t\vee\tau_1} g(s, z)\mathrm{d}s|\mathcal{F}_t\right\} \\ &= E\left\{\left(\xi_1 - \xi_2 + \int_{t\vee\tau_2}^{t\vee\tau_1} g(s, z)\mathrm{d}s\right)\mathscr{E}(\gamma)_t^T|\mathcal{F}_t\right\}, \end{aligned} \tag{3.13}$$

for all $t \in [0, T]$. Since g satisfies (H2), applying the Hölder inequality we have, for any $p, q > 1$ with $1/p + 1/q = 1$,

$$|\Delta U_t|^p \le E\{[|\xi_1 - \xi_2| + \ell(1 + |z|^2)|\tau_1 - \tau_2|]^p|\mathcal{F}_t\}E\{[\mathscr{E}(\gamma)_t^T]^q|\mathcal{F}_t\}^{p/q}.$$

Now recall the function ϕ_α defined by (2.9). Let $\alpha = 3$ and $q = q(K, R) > 1$ so that $\phi_3(q) = J(K, R)$. Applying the reverse Hölder inequality (2.10) we obtain, for $p = p(K, R) = q/(q - 1)$,

$$|\Delta U_t|^p \le 3^p E\{[|\xi_1 - \xi_2| + \ell(1 + |z|^2)|\tau_1 - \tau_2|]^p|\mathcal{F}_t\}.$$

Taking the expectation, defining $C_R = 3\ell(1 + R^2)$, and recalling the definition of U, we have

$$\begin{aligned} &\|(\mathcal{E}^g[\xi_1 + zB_{\tau_1}|\mathcal{F}_t] - zB_{t\wedge\tau_1}) - (\mathcal{E}^g[\xi_2 + zB_{\tau_2}|\mathcal{F}_t] - zB_{t\wedge\tau_2})\|_p \\ &\quad \le 3\|\xi_1 - \xi_2\|_p + C_R\|\tau_1 - \tau_2\|_p, \end{aligned}$$

for all $t \in [0, T]$, proving (3.8).

(2) The proof of "L^∞-domination" (3.9) is similar but much easier. Again we let (Y^i, Z^i) be the solution of (3.4) for $\xi_i + zB_\tau$, $i = 1, 2$, respectively. Defining $\Delta Y = Y^1 - Y^2$ and

$\Delta Z = Z^1 - Z^2$, we have

$$\Delta Y_t = \Delta\xi + \int_t^T \langle \gamma_s, \Delta Z_s\rangle \mathrm{d}s - \int_t^T \Delta Z_s \mathrm{d}B_s, \quad \forall t \in [0, T],$$

where $\gamma_t \stackrel{\triangle}{=} \int_0^1 \frac{\partial g}{\partial z}(t, \lambda Z_t^1 + (1-\lambda)Z_t^2)\mathrm{d}\lambda \in \mathrm{BMO}$. Applying Girsanov's theorem again we obtain that, under some equivalent probability measure P^γ, it holds that

$$\Delta Y_t = E^\gamma[\Delta\xi|\mathcal{F}_t], \quad \forall t \in [0, T], P\text{-a.s.}$$

The estimate (3.9) then follows immediately.

(3) We now prove the one-sided g-domination (3.10). This time we let (Y^1, Z^1) and (Y^2, Z^2) be the solutions of BSDE (3.4) with terminal conditions $\eta + \xi + zB_\tau$ and $\xi + zB_\tau$, respectively. Then (3.5) implies that, for all $t \in [0, T]$,

$$\begin{aligned}\Delta Y_t &= \Delta\widetilde{Y}_t = \eta + \int_t^T (g(s, \widetilde{Z}_s^1 + z\mathbf{1}_{\{s\le\tau\}}) - g(s, \widetilde{Z}_s^2 + z\mathbf{1}_{\{s\le\tau\}}))\mathrm{d}s - \int_t^T \Delta\widetilde{Z}_s \mathrm{d}B_s \\ &= \eta + \int_t^T \left\langle \int_0^1 \frac{\partial g}{\partial z}(s, \lambda\Delta\widetilde{Z}_s + \widetilde{Z}_s^2 + z\mathbf{1}_{\{s\le\tau\}})\mathrm{d}\lambda, \Delta\widetilde{Z}_s \right\rangle \mathrm{d}s - \int_t^T \Delta\widetilde{Z}_s \mathrm{d}B_s,\end{aligned}$$

where $\widetilde{Y}_t^i \stackrel{\triangle}{=} Y_t^i - zB_{t\wedge\tau}$ and $\widetilde{Z}_t^i \stackrel{\triangle}{=} Z_t^i - z\mathbf{1}_{\{t\le\tau\}}$, $i = 1, 2$. Since $\widetilde{Z}^i \in \mathrm{BMO}$, $i = 1, 2$, thanks to Proposition 2.3, it is easy to check that $\gamma_\cdot \stackrel{\triangle}{=} \frac{\partial g}{\partial z}(\cdot, Z_\cdot^2) \in \mathrm{BMO}$ as well, and the estimate (3.12) remains true. It is worth noting that γ is independent of η since Z^2 is so. By Girsanov's theorem,

$$\begin{aligned}\Delta Y_t = \eta + \int_t^T \Bigg\langle \int_0^1 \Bigg(&\frac{\partial g}{\partial z}(s, \lambda\Delta\widetilde{Z}_s + \widetilde{Z}_s^2 + z\mathbf{1}_{\{s\le\tau\}}) \\ &- \frac{\partial g}{\partial z}(s, \widetilde{Z}_s^2 + z\mathbf{1}_{\{s\le\tau\}})\Bigg) \mathrm{d}\lambda, \Delta\widetilde{Z}_s \Bigg\rangle \mathrm{d}s - \int_t^T \Delta\widetilde{Z}_s \mathrm{d}B_s^\gamma, \quad \forall t \in [0, T],\end{aligned}$$

where P^γ is the equivalent probability measure as before. Now with the extra assumption on the boundedness of $\frac{\partial^2 g}{\partial z^2}$ we conclude that, with $\alpha(K, R) \equiv \ell'/2$,

$$\begin{aligned}&\left|\left\langle \int_0^1 \left(\frac{\partial g}{\partial z}(s, \lambda\Delta\widetilde{Z}_s + \widetilde{Z}_s^2 + z\mathbf{1}_{\{s\le\tau\}}) - \frac{\partial g}{\partial z}(s, \widetilde{Z}_s^2 + z\mathbf{1}_{\{s\le\tau\}})\right) \mathrm{d}\lambda, \Delta\widetilde{Z}_s \right\rangle\right| \\ &\quad \le \alpha(K, R)|\Delta\widetilde{Z}_s|^2.\end{aligned}$$

The comparison theorem of quadratic BSDEs (cf. [10, Theorem 2.6]) then leads to

$$\mathcal{E}^g[\eta + \xi + zB_\tau|\mathcal{F}_t] - \mathcal{E}^g[\xi + zB_\tau|\mathcal{F}_t] \le \mathcal{E}_\gamma^{g_\alpha}[\eta|\mathcal{F}_t], \quad \forall t \in [0, T],$$

proving (3.10), whence the theorem. ■

4. Properties of quadratic $\mathcal{F}$-expectations

In this section, we assume that $\mathcal{E}$ is a translation invariant semi-quadratic $\mathcal{F}$-expectation dominated by a quadratic g-expectation $\mathcal{E}^g$ with g satisfying (H1) and (H2). Clearly $\mathcal{E}$ is regular. We also assume that $\mathcal{E}$ satisfies both the L^p-domination (3.8) and the L^∞-domination (3.9).

We first give a path regularity result for $\mathcal{E}$-martingales, which will be very useful in our future discussion.

Proposition 4.1. *For any $\tau \in \mathcal{M}_{0,T}$, $\xi \in L^\infty(\mathcal{F}_\tau)$, and $z \in \mathbb{R}^d$, the process $\mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]$, $t \in [0, T]$ admits a càdlàg modification.*

Proof. We first assume that $\mathcal{E}$ is an upper semi-quadratic $\mathcal{F}$-expectation first. By the L^∞-domination, $X_\cdot \stackrel{\triangle}{=} \mathcal{E}[\xi + zB_\tau|\mathcal{F}_\cdot] - zB_{\cdot\wedge\tau} \in L^\infty_{\mathbf{F}}([0, T])$, which implies that $|X_t| \le \|X\|_\infty$, P-a.s. for any $t \in [0, T]$ except a null set $\mathcal{T}$. We may assume that there is a dense set $\mathcal{D}$ of $[0, T] \setminus \mathcal{T}$ such that $|X_t| \le \|X\|_\infty, \forall t \in \mathcal{D}$, P-a.s. Now we define a new generator

$$\hat{g}(t, \omega, \zeta) \stackrel{\triangle}{=} g(t, \omega, \zeta + \mathbf{1}_{\{t\le\tau\}}z) - g(t, \omega, \mathbf{1}_{\{t\le\tau\}}z), \quad \forall (t, \omega, \zeta) \in [0, T] \times \Omega \times \mathbb{R}^d. \tag{4.1}$$

For any $0 \le s \le t \le T$ and any $\eta \in L^\infty(\mathcal{F}_t)$, it is easy to check that P-a.s.

$$\mathcal{E}^g[\eta + zB_{t\wedge\tau}|\mathcal{F}_s] - zB_{s\wedge\tau} + \int_0^s g(r, \mathbf{1}_{\{r\le\tau\}}z)\mathrm{d}r = \mathcal{E}^{\hat{g}}\left[\eta + \int_0^t g(r, \mathbf{1}_{\{r\le\tau\}}z)\mathrm{d}r|\mathcal{F}_s\right]. \tag{4.2}$$

In particular, by the definition and the properties of upper semi-quadratic $\mathcal{F}$-expectation, letting $\eta = X_t$ in (4.2) shows that P-a.s.

$$\begin{aligned}\mathcal{E}[\xi + zB_\tau|\mathcal{F}_s] &= \mathcal{E}[\mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[X_t + zB_{t\wedge\tau}|\mathcal{F}_s] \le \mathcal{E}^g[X_t + zB_{t\wedge\tau}|\mathcal{F}_s] \\ &= \mathcal{E}^{\hat{g}}\left\{X_t + \int_0^t g(r, \mathbf{1}_{\{r\le\tau\}}z)\mathrm{d}r|\mathcal{F}_s\right\} + zB_{s\wedge\tau} - \int_0^s g(r, \mathbf{1}_{\{r\le\tau\}}z)\mathrm{d}r.\end{aligned}$$

In other words, the process $t \mapsto X_t + \int_0^t g(r, \mathbf{1}_{\{r\le\tau\}}z)\mathrm{d}r$ is in fact a $\hat{g}$-submartingale. Thus by Theorem 2.1 we can define a càdlàg process

$$Y_t \stackrel{\triangle}{=} \lim_{r\searrow t, r\in\mathcal{D}} X_r, \quad \forall t \in [0, T) \quad \text{and} \quad Y_T \stackrel{\triangle}{=} X_T = \xi.$$

Clearly, $Y \in \mathbb{D}^\infty_{\mathbf{F}}([0, T])$. Moreover, the constant-preserving property of $\mathcal{E}$ and "zero–one law" imply that

$$\mathcal{E}[\xi'|\mathcal{F}_t] \in \mathcal{F}_{t\wedge\tau}, \quad \forall \xi' \in \Lambda_\tau, \forall t \in [0, T]. \tag{4.3}$$

To see this, one needs only to note that for any $s \in [0, t)$,

$$\mathbf{1}_{\{t\wedge\tau\le s\}}\mathcal{E}[\xi'|\mathcal{F}_t] = \mathbf{1}_{\{\tau\le s\}}\mathcal{E}[\xi'|\mathcal{F}_t] = \mathcal{E}[\mathbf{1}_{\{\tau\le s\}}\xi'|\mathcal{F}_t] = \mathbf{1}_{\{\tau\le s\}}\xi' \in \mathcal{F}_s.$$

Thus $X_t \in \mathcal{F}_{t\wedge\tau}, \forall t \in [0, T]$, and so is Y by the right-continuity of the filtration $\mathbf{F}$. Now, for any $t \in [0, T)$ and $r \in (t, T] \cap \mathcal{D}$, we write

$$X_t - Y_t = \mathcal{E}[\xi + zB_\tau|\mathcal{F}_t] - zB_{t\wedge\tau} - Y_t = \mathcal{E}[X_r + zB_{r\wedge\tau}|\mathcal{F}_t] - \mathcal{E}[Y_t + zB_{t\wedge\tau}|\mathcal{F}_t].$$

Then applying (3.8) with $K = \|X\|_\infty$ and $R = |z|$ we can find a $p = p(K, R)$ such that

$$\|X_t - Y_t\|_p \le 3\|X_r - Y_t\|_p + C_R\|r\wedge\tau - t\wedge\tau\|_p \le 3\|X_r - Y_t\|_p + C_R(r - t).$$

Letting $r \searrow t$ in the above, the bounded convergence theorem then implies that $X_t = Y_t$, P-a.s. To wit, the process $Y_t + zB_{t\wedge\tau}$, $t \in [0, T]$ is a càdlàg modification of $\mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]$, $t \in [0, T]$.

The case when $\mathcal{E}$ is lower semi-quadratic can be argued similarly. The proof is complete. ■

Next, we prove the "optional sampling theorem" for the quadratic $\mathcal{F}$-expectation. To begin with, we recall that the nonlinear conditional expectation $\mathcal{E}[\cdot|\mathcal{F}_\sigma]$ is defined as follows. If $\xi \in \mathrm{Dom}(\mathcal{E})$, define $Y_t \stackrel{\triangle}{=} \mathcal{E}[\xi|\mathcal{F}_t]$, $t \in [0, T]$; then for any $\sigma \in \mathcal{M}_{0,T}$, we define

$$\mathcal{E}[\xi|\mathcal{F}_\sigma] \stackrel{\triangle}{=} Y_\sigma, \quad P\text{-a.s.} \tag{4.4}$$

The following properties of $\mathcal{E}[\cdot|\mathcal{F}_\sigma]$ are important.

Proposition 4.2. *For any $\tau, \sigma \in \mathcal{M}_{0,T}$, $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$, and $z \in \mathbb{R}^d$, it holds that*

(i) $\mathcal{E}[\xi + zB_\tau|\mathcal{F}_\sigma] \le \mathcal{E}[\eta + zB_\tau|\mathcal{F}_\sigma]$, P-a.s., *if* $\xi \le \eta$ P-a.s.*;*
(ii) $\mathcal{E}[\xi + zB_\tau|\mathcal{F}_\tau] = \xi + zB_\tau$, P-a.s.*;*
(iii) $\mathbf{1}_A\mathcal{E}[\xi + zB_\tau|\mathcal{F}_\sigma] = \mathbf{1}_A\mathcal{E}[\mathbf{1}_A\xi + zB_\tau|\mathcal{F}_\sigma]$, P-a.s., $\forall A \in \mathcal{F}_{\tau\wedge\sigma}$*;*
(iv) *if further $\eta \in L^\infty(\mathcal{F}_{\tau\wedge\sigma})$, the following "translation invariance" property holds:*

$$\mathcal{E}[\xi + zB_\tau + \eta|\mathcal{F}_\sigma] = \mathcal{E}[\xi + zB_\tau|\mathcal{F}_\sigma] + \eta, \quad P\text{-a.s.}$$

Proof. (i) is a direct consequence of the monotonicity of $\mathcal{E}$ and Proposition 4.1.

To see (ii), we first assume that τ takes values in a finite set: $0 \le t_1 < \cdots < t_n \le T$. Actually, for any $\xi' \in \Lambda_\tau$, the constant-preserving of $\mathcal{E}$ and "zero–one law" imply that

$$\mathcal{E}[\xi'|\mathcal{F}_\tau] = \sum_{j=1}^n \mathbf{1}_{\{\tau=t_j\}}\mathcal{E}[\xi'|\mathcal{F}_{t_j}] = \sum_{j=1}^n \mathcal{E}[\mathbf{1}_{\{\tau=t_j\}}\xi'|\mathcal{F}_{t_j}] = \sum_{j=1}^n \mathbf{1}_{\{\tau=t_j\}}\xi' = \xi', \quad P\text{-a.s.}$$

For general stopping time τ, we first choose a sequence of finite-valued stopping times $\{\tau_n\}$ such that $\tau_n \searrow \tau$, P-a.s. Since for each n it holds that

$$\mathcal{E}[\xi + zB_\tau|\mathcal{F}_{\tau_n}] = \xi + zB_\tau, \quad P\text{-a.s.}, n = 1, 2, \ldots,$$

letting $n \to \infty$ and applying Proposition 4.1 we obtain that $\mathcal{E}[\xi + zB_\tau|\mathcal{F}_\tau] = \xi + zB_\tau$, P-a.s., proving (ii).

We now prove (iii). Again, we assume first that σ takes finite values in $0 \le t_1 < \cdots < t_n \le T$. For any $A \in \mathcal{F}_{\tau\wedge\sigma}$, let $A_j = A \cap \{\sigma = t_j\} \in \mathcal{F}_{t_j}$, $1 \le j \le n$. Then it holds P-a.s. that

$$\begin{aligned}\mathbf{1}_A\mathcal{E}[\mathbf{1}_A\xi + zB_\tau|\mathcal{F}_\sigma] &= \sum_{j=1}^n \mathbf{1}_{A_j}\mathcal{E}[\mathbf{1}_A\xi + zB_\tau|\mathcal{F}_{t_j}] = \sum_{j=1}^n \mathcal{E}[\mathbf{1}_{A_j}\xi + \mathbf{1}_{A_j}zB_\tau|\mathcal{F}_{t_j}]\\ &= \sum_{j=1}^n \mathbf{1}_{A_j}\mathcal{E}[\xi + zB_\tau|\mathcal{F}_{t_j}] = \mathbf{1}_A\mathcal{E}[\xi + zB_\tau|\mathcal{F}_\sigma].\end{aligned}$$

For general stopping time σ, we again approximate σ from above by a sequence of finite-valued stopping times $\{\sigma_n\}_{n\ge 0}$. Then for any $A \in \mathcal{F}_{\tau\wedge\sigma} \subset \mathcal{F}_{\tau\wedge\sigma_n}$, $\forall n \in \mathbb{N}$, we have

$$\mathbf{1}_A\mathcal{E}[\xi + zB_\tau|\mathcal{F}_{\sigma_n}] = \mathbf{1}_A\mathcal{E}[\mathbf{1}_A\xi + zB_\tau|\mathcal{F}_{\sigma_n}], \quad P\text{-a.s.}, \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ and applying Proposition 4.1 again we can prove (iii).

(iv) The proof is quite similar; thus we shall only consider the case where σ takes values in a finite set $0 \le t_1 < \cdots < t_n \le T$. In this case we have

$$\mathcal{E}[\xi + zB_\tau + \eta|\mathcal{F}_\sigma] = \sum_{j=1}^n \mathbf{1}_{\{\sigma=t_j\}}\mathcal{E}[\xi + zB_\tau + \eta|\mathcal{F}_{t_j}]$$

$$= \sum_{j=1}^{n} \mathcal{E}[\mathbf{1}_{\{\sigma=t_j\}}(\xi + zB_\tau) + \mathbf{1}_{\{\sigma=t_j\}}\eta|\mathcal{F}_{t_j}]$$
$$= \sum_{j=1}^{n} \{\mathcal{E}[\mathbf{1}_{\{\sigma=t_j\}}(\xi + zB_\tau)|\mathcal{F}_{t_j}] + \mathbf{1}_{\{\sigma=t_j\}}\eta\}$$
$$= \sum_{j=1}^{n} \mathbf{1}_{\{\sigma=t_j\}}\mathcal{E}[\xi + zB_\tau|\mathcal{F}_{t_j}] + \sum_{j=1}^{n} \mathbf{1}_{\{\sigma=t_j\}}\eta = \mathcal{E}[\xi + zB_\tau|\mathcal{F}_\sigma] + \eta.$$

The third equality is due to the "translation invariance" of $\mathcal{E}$ and $\mathbf{1}_{\{\sigma=t_j\}}\eta \in L^\infty(\mathcal{F}_{t_j})$. The rest of the proof can be carried out in a similar way to other cases; we leave it to the interested reader. The proof is complete. ∎

We now prove an important property of $\mathcal{E}\{\cdot|\mathcal{F}_t\}$, which we shall refer to as the "*optional sampling theorem*" in the future.

Theorem 4.3. *For any $X \in L^\infty_{\mathbf{F}}([0,T])$ and $z \in \mathbb{R}^d$ such that $t \mapsto X_t + zB_t$ is a right-continuous $\mathcal{E}$-submartingale (resp. $\mathcal{E}$-supermartingale or $\mathcal{E}$-martingale), then for any stopping times $\tau, \sigma \in [0,T]$, it holds that*

$$\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_\sigma] \geq (\textit{resp.} \leq \textit{or} =) X_{\tau\wedge\sigma} + zB_{\tau\wedge\sigma}, \quad P\text{-a.s.}$$

Proof. We shall consider only the $\mathcal{E}$-submartingale case, as the other cases can be deduced easily by standard arguments. To begin with, we assume that $\sigma \equiv t \in [0,T]$ and assume that τ takes finite values in $0 \leq t_1 < \cdots < t_N \leq T$. Note that if $t \geq t_N$, then $X_\tau + zB_\tau \in \mathcal{F}_t$ and $\tau \wedge t = \tau$; thus

$$\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t] = X_\tau + zB_\tau = X_{\tau\wedge t} + zB_{\tau\wedge t}, \quad P\text{-a.s.},$$

thanks to the constant-preserving property of $\mathcal{E}$. We can then argue inductively to show that the statement holds for $t \geq t_m$, for all $1 \leq m \leq N$. In fact, assume that for $m \in \{2, \ldots N\}$

$$\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t] \geq X_{\tau\wedge t} + zB_{\tau\wedge t}, \quad P\text{-a.s. } \forall t \geq t_m. \tag{4.5}$$

Then, again using the translatability and the "zero–one" law, one shows that for any $t \in [t_{m-1}, t_m)$, it holds P-a.s. that

$$\begin{aligned}\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t] &= \mathcal{E}[\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_{t_m}]|\mathcal{F}_t] \geq \mathcal{E}[X_{\tau\wedge t_m} + zB_{\tau\wedge t_m}|\mathcal{F}_t] \\ &= \mathcal{E}[\mathbf{1}_{\{\tau\leq t_{m-1}\}}(X_{\tau\wedge t} + zB_{\tau\wedge t}) + \mathbf{1}_{\{\tau\geq t_m\}}(X_{t_m} + zB_{t_m})|\mathcal{F}_t] \\ &= \mathbf{1}_{\{\tau\leq t_{m-1}\}}(X_{\tau\wedge t} + zB_{\tau\wedge t}) + \mathbf{1}_{\{\tau\geq t_m\}}\mathcal{E}[X_{t_m} + zB_{t_m}|\mathcal{F}_t] \\ &\geq \mathbf{1}_{\{\tau\leq t_{m-1}\}}(X_{\tau\wedge t} + zB_{\tau\wedge t}) + \mathbf{1}_{\{\tau\geq t_m\}}(X_t + zB_t) \\ &= X_{\tau\wedge t} + zB_{\tau\wedge t}.\end{aligned}$$

Namely (4.5) also holds for any $t \geq t_{m-1}$. This completes the inductive step. Thus (4.5) holds for all finite-valued stopping times.

Now let τ be a general stopping time; we still choose $\{\tau_n\}$ to be a sequence of finite-valued stopping times such that $\tau_n \searrow \tau$, P-a.s. Then (4.5) holds for all τ_n's. Now let $K = \|X\|_\infty$, $R = |z|$, and $p = p(K,R)$. Applying the L^p-domination (3.8) for $\mathcal{E}$ we see that for any $n \in \mathbb{N}$,

$$\begin{aligned}&\|\mathcal{E}[X_{\tau_n} + zB_{\tau_n}|\mathcal{F}_t] - \mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t]\|_p \\ &\quad \leq \|(\mathcal{E}[X_{\tau_n} + zB_{\tau_n}|\mathcal{F}_t] - zB_{\tau_n\wedge t}) - (\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t] - zB_{\tau\wedge t})\|_p\end{aligned}$$

$$
\begin{aligned}
&+ R\|B_{\tau_n\wedge t} - B_{\tau\wedge t}\|_p \\
&\le 3\|X_{\tau_n} - X_\tau\|_p + C_R\|\tau_n - \tau\|_p + R\|B_{\tau_n\wedge t} - B_{\tau\wedge t}\|_p. \qquad (4.6)
\end{aligned}
$$

Since X is a bounded càdlàg process, we can then apply the bounded convergence theorem to conclude that the first and second terms on the right hand side of (4.6) tend to 0, as $n \to \infty$. Furthermore, applying the Burkholder–Davis–Gundy inequality and bounded convergence theorem, we conclude that the last term on the right hand side of (4.6) also goes to 0. Thus, possibly along a subsequence, we see that for any $t \in [0, T]$

$$
\begin{aligned}
\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t] &= \lim_{n\to\infty} \mathcal{E}[X_{\tau_n} + zB_{\tau_n}|\mathcal{F}_t] \ge \lim_{n\to\infty} (X_{\tau_n\wedge t} + zB_{\tau_n\wedge t}) \\
&= X_{\tau\wedge t} + zB_{\tau\wedge t}, \quad P\text{-a.s.}
\end{aligned}
$$

Thus we obtain (4.5) again.

Finally, let us consider the case when σ is also a general stopping time. Following the previous argument, with the help of Proposition 4.1, we have, P-a.s.

$$
\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_t] \ge X_{\tau\wedge t} + zB_{\tau\wedge t}, \quad \forall t \in [0, T].
$$

Consequently, we obtain that $\mathcal{E}[X_\tau + zB_\tau|\mathcal{F}_\sigma] \ge X_{\tau\wedge\sigma} + zB_{\tau\wedge\sigma}$, P-a.s., proving the theorem. ■

To end this section we consider a special BSDE involving the quadratic $\mathcal{F}$-expectation $\mathcal{E}$, which will be very useful in the rest of the paper:

$$
Y_t + zB_t + \int_0^t f(s, Y_s)\mathrm{d}s = \mathcal{E}\left\{\xi + zB_T + \int_0^T f(s, Y_s)\mathrm{d}s|\mathcal{F}_t\right\}, \quad \forall t \in [0, T], \qquad (4.7)
$$

where $f : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that it satisfies the following assumption:

(H3) The function f is uniformly Lipschitz in y with Lipschitz constant $\kappa > 0$, uniform in (t, ω), such that $\int_0^T |f(t, \cdot, 0)|\mathrm{d}t \in L^\infty(\mathcal{F}_T)$.

We have the following existence and uniqueness result for the BSDE (4.7).

Proposition 4.4. *Assume* (H3). *Then for any $\xi \in L^\infty(\mathcal{F}_T)$ and any $z \in \mathbb{R}^d$, the BSDE* (4.7) *admits a unique solution in $\mathbb{D}^\infty_{\mathbf{F}}([0, T])$.*

Proof. We first consider the case when $T \le 1/2\kappa$, where κ is the Lipschitz constant of f in (H3). For any $Y \in \mathbb{D}^\infty_{\mathbf{F}}([0, T])$, and $t \in [0, T]$, using (H3) we have

$$
\left\|\int_0^t f(s, Y_s)\mathrm{d}s\right\|_\infty \le \left\|\int_0^T |f(s, 0)|\mathrm{d}s\right\|_\infty + \kappa t\|Y\|_\infty < \infty.
$$

In particular, we have $\xi + \int_0^T f(s, Y_s)\mathrm{d}s \in L^\infty(\mathcal{F}_T)$ so that $\mathcal{E}\{\xi + zB_T + \int_0^T f(s, Y_s)\mathrm{d}s|\mathcal{F}_t\}$ is well defined, and we can define a mapping $\Phi : \mathbb{D}^\infty_{\mathbf{F}}([0, T]) \mapsto \mathbb{D}^\infty_{\mathbf{F}}([0, T])$ by

$$
\Phi_t(Y) \triangleq \mathcal{E}\left\{\xi + zB_T + \int_0^T f(s, Y_s)\mathrm{d}s|\mathcal{F}_t\right\} - zB_t - \int_0^t f(s, Y_s)\mathrm{d}s, \quad t \in [0, T]. \qquad (4.8)
$$

We claim that Φ is a contraction. Indeed, since $\mathcal{E}$ satisfies the L^∞-domination, for any $Y, \hat{Y} \in \mathbb{D}^\infty_{\mathbf{F}}([0,T])$, (3.9) implies that for any $t \in [0,T]$, it holds P-a.s. that

$$\begin{aligned}|\Phi_t(Y) - \Phi_t(\hat{Y})| &= \left|\mathcal{E}\left[\xi + zB_T + \int_t^T f(s,Y_s)\mathrm{d}s|\mathcal{F}_t\right] - \mathcal{E}\left[\xi + zB_T\right.\right.\\ &\quad \left.\left.+ \int_t^T f(s,\hat{Y}_s)\mathrm{d}s|\mathcal{F}_t\right]\right| \le \left\|\int_t^T (f(s,Y_s) - f(s,\hat{Y}_s))\mathrm{d}s\right\|_\infty \\ &\le \kappa(T-t)\|Y-\hat{Y}\|_\infty \le \frac{1}{2}\|Y-\hat{Y}\|_\infty. \end{aligned} \tag{4.9}$$

Since the process $t \mapsto \Phi_t(Y)$ is càdlàg , thanks to Proposition 4.1, we conclude that $\|\Phi(Y) - \Phi(\hat{Y})\|_\infty \le \frac{1}{2}\|Y - \hat{Y}\|_\infty$. Thus Φ is a contraction, and the lemma holds in this case.

The general case can now be argued using a standard "patching-up" method. Namely we take a partition of $[0,T]$: $0 = t_0 < t_1 < \cdots < t_N = T$, such that $\max|t_n - t_{n-1}| < 1/2\kappa$. We first solve the BSDE (4.7) on $[t_{N-1}, t_N]$ to get a solution Y^N. We then solve (4.7) on $[t_{N-2}, t_{N-1}]$ to get Y^{N-1}, satisfying the terminal condition $Y^{N-1}_{t_{N-1}} = Y^N_{t_{N-1}}$, and so on, thanks to the result proved in the first part. Denoting the solution on $[t_{n-1}, t_n]$ by Y^n, we can then define a new process by $Y_t \triangleq Y^n_t$, $t \in [t_{n-1}, t_n]$, $n = 1, \ldots, N$, and prove that Y solves (4.7) over $[0,T]$ by induction.

To see this, we first note that $Y \in \mathbb{D}^\infty_{\mathbf{F}}([0,T])$. Now assuming that Y solves (4.7) on $[t_n, T]$, we show that it solves (4.7) on $[t_{n-1}, T]$ as well. Indeed, for any $t \in [t_{n-1}, t_n]$, we have

$$\begin{aligned} Y_t + zB_t + \int_0^t f(s,Y_s)\mathrm{d}s &= Y^n_t + zB_t + \int_{t_{n-1}}^t f(s,Y^n_s)\mathrm{d}s + \int_0^{t_{n-1}} f(s,Y_s)\mathrm{d}s \\ &= \mathcal{E}\left\{Y^n_{t_n} + zB_{t_n} + \int_{t_{n-1}}^{t_n} f(s,Y^n_s)\mathrm{d}s|\mathcal{F}_t\right\} + \int_0^{t_{n-1}} f(s,Y_s)\mathrm{d}s \\ &= \mathcal{E}\left\{Y_{t_n} + zB_{t_n} + \int_0^{t_n} f(s,Y_s)\mathrm{d}s|\mathcal{F}_t\right\} \\ &= \mathcal{E}\left\{\mathcal{E}\left\{\xi + zB_T + \int_0^T f(s,Y_s)\mathrm{d}s|\mathcal{F}_{t_n}\right\}|\mathcal{F}_t\right\} \\ &= \mathcal{E}\left\{\xi + zB_T + \int_0^T f(s,Y_s)\mathrm{d}s|\mathcal{F}_t\right\}. \end{aligned}$$

In the above the second equality is due to the fact that Y^n solves (4.7) on $[t_{n-1}, t_n]$; the third equality is due to the "translation invariance" of $\mathcal{E}\{\cdot|\mathcal{F}_t\}$; the fourth equality is because of the inductional hypothesis that Y solves (4.7) on $[t_n, T]$; and the last equality is the "time-consistency" property of $\mathcal{E}\{\cdot|\mathcal{F}_t\}$. This shows that Y solves (4.7) on $[t_{n-1}, T]$, whence the existence.

The uniqueness can be argued in a similar way. First note that the BSDE (4.7) can be written in a "local" form: for $n = 1, 2, \ldots, N$,

$$Y_t + zB_t = \mathcal{E}\left\{Y_{t_n} + zB_{t_n} + \int_t^{t_n} f(s,Y_s)\mathrm{d}s|\mathcal{F}_t\right\}, \quad t \in [t_{n-1}, t_n], \tag{4.10}$$

thanks to the translation invariance property of $\mathcal{E}\{\cdot|\mathcal{F}_t\}$. Assume that $\hat{Y} \in \mathbb{D}^\infty_{\mathbf{F}}([0,T])$ is another solution of (4.7). Then it must satisfy (4.10) on $[t_{N-1}, T]$. The fixed point argument in the first part then shows that $Y = \hat{Y}$ in $\mathbb{D}^\infty_{\mathbf{F}}([t_{N-1}, T])$; thus $Y_{t_{N-1}} = \hat{Y}_{t_{N-1}}$, P-a.s. We can repeat the same argument for $[t_{N-2}, t_{N-1}]$, and so on to conclude after finitely many steps that Y and $\hat{Y}$ are indistinguishable over the whole interval $[0, T]$. The proof is now complete. ■

5. Doob–Meyer decomposition of quadratic $\mathcal{F}$-martingales

In this section we prove a Doob–Meyer type decomposition theorem for quadratic $\mathcal{F}$-martingales. We shall assume that $\mathcal{E}$ is a translation invariant quadratic $\mathcal{F}$-expectation dominated by two quadratic g-expectations $\mathcal{E}^{g_1}$ and $\mathcal{E}^{g_2}$ from below and above, and both g_1 and g_2 satisfy (H1) and (H2) with the same $\ell > 0$. We also assume that $\mathcal{E}$ satisfies both the L^p-domination (3.8) and the L^∞-domination (3.9).

The following proposition will play an essential role in the rest of this paper.

Proposition 5.1. *For any $\tau \in \mathcal{M}_{0,T}$, $\xi \in L^\infty(\mathcal{F}_\tau)$, and $z \in \mathbb{R}^d$, define $Y_t \stackrel{\triangle}{=} \mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]$, $t \in [0, T]$. Then there exists a unique pair $(h, Z) \in L^1_{\mathbf{F}}([0, T]) \times \mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that*

$$-\ell(|Z_t| + |Z_t|^2) \le g_1(t, Z_t) \le h_t \le g_2(t, Z_t) \le \ell(|Z_t| + |Z_t|^2), \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.}, \tag{5.1}$$

and (Y, Z) satisfies the BSDE

$$Y_t = Y_T + \int_t^T h_s \mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad \forall t \in [0, T]. \tag{5.2}$$

Moreover, if we assume that $\mathcal{E}$ also satisfies the one-sided g-domination (3.10), with $K \ge \|\xi\|_\infty$, $R \ge |z|$, $\alpha = \alpha(K, R)$, $J = J(K, R)$ and $\|\gamma\|^2_{\mathrm{BMO}} \le J$, then for any $\eta \in L^\infty(\mathcal{F}_\tau)$, the pair $(\hat{h}, \hat{Z})$ corresponding to the process $\mathcal{E}\{\eta + zB_\tau|\mathcal{F}_t\}$, $t \in [0, T]$, satisfies

$$\hat{h}_t - h_t \le \alpha|\hat{Z}_t - Z_t|^2 + \langle \gamma_t, \hat{Z}_t - Z_t \rangle, \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.} \tag{5.3}$$

Proof. For each $z \in \mathbb{R}^d$, define a process $\widetilde{Y}_t \stackrel{\triangle}{=} Y_t - zB_{t\wedge\tau}$, $t \in [0, T]$ and a new generator

$$g_i^z(t, \omega, \zeta) \stackrel{\triangle}{=} g_i(t, \omega, \zeta + \mathbf{1}_{\{t\le\tau\}}z), \quad \forall (t, \omega, \zeta) \in [0, T] \times \Omega \times \mathbb{R}^d, i = 1, 2.$$

By the definition of the L^∞-domination (see Definition 3.8(2)) and the fact (4.3) we see that $\widetilde{Y} \in L^\infty_{\mathbf{F}}([0, T])$ and $\widetilde{Y}_t \in \mathcal{F}_{t\wedge\tau}$, $\forall t \in [0, T]$. It is easy to check that for $0 \le s \le t \le T$ and any $\eta \in L^\infty(\mathcal{F}_t)$,

$$\mathcal{E}^{g_i}[\eta + zB_{t\wedge\tau}|\mathcal{F}_s] = \mathcal{E}^{g_i^z}[\eta|\mathcal{F}_s] + zB_{s\wedge\tau}, \quad P\text{-a.s. } i = 1, 2.$$

Thus the upper domination of $\mathcal{E}$ by $\mathcal{E}^{g_1}$ and the time-consistency of $\mathcal{E}$ imply that, P-a.s.,

$$\begin{aligned}\mathcal{E}^{g_1^z}[\widetilde{Y}_t|\mathcal{F}_s] &= \mathcal{E}^{g_1}[\widetilde{Y}_t + zB_{t\wedge\tau}|\mathcal{F}_s] - zB_{s\wedge\tau} = \mathcal{E}^{g_1}[\mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]|\mathcal{F}_s] - zB_{s\wedge\tau} \\ &\le \mathcal{E}[\mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]|\mathcal{F}_s] - zB_{s\wedge\tau} = \mathcal{E}[\xi + zB_\tau|\mathcal{F}_s] - zB_{s\wedge\tau} = \widetilde{Y}_s.\end{aligned}$$

Namely, $\widetilde{Y}$ is both a g_1^z-supermartingale and a g_2^z-submartingale. Applying Theorem 2.1 we obtain two increasing processes A^1 and A^2 (we may assume that both are càdlàg and null at 0) and two processes $\widetilde{Z}^1, \widetilde{Z}^2 \in \mathcal{H}^2_{\mathcal{F}}(\mathbb{R}^d)$, such that

$$\widetilde{Y}_t = \widetilde{Y}_T + \int_t^T g_i^z(s, \widetilde{Z}_s^i)\mathrm{d}s + (-1)^i(A_t^i - A_T^i) - \int_t^T \widetilde{Z}_s^i \mathrm{d}B_s, \quad t \in [0, T], i = 1, 2.$$

Letting $Z_t^i = \widetilde{Z}_t^i + \mathbf{1}_{\{t\le\tau\}}z$ we have, for $i = 1, 2$,

$$Y_t = Y_T + \int_t^T g_i(s, Z_s^i)\mathrm{d}s + (-1)^i(A_t^i - A_T^i) - \int_t^T Z_s^i \mathrm{d}B_s, \quad \forall t \in [0, T]. \tag{5.4}$$

By comparing the martingale parts and bounded variation parts of two BSDEs in (5.4), one has

$$Z_t^1 \equiv Z_t^2, \quad \text{and} \quad -g_1(t, Z_t^1)\mathrm{d}t - \mathrm{d}A_t^1 \equiv -g_2(t, Z_t^2)\mathrm{d}t + \mathrm{d}A_t^2, \quad t \in [0, T],\ P\text{-a.s.}$$

Consequently, we have that $\mathrm{d}A_t^1 + \mathrm{d}A_t^2 \equiv (g_2(t, Z_t^1) - g_1(t, Z_t^1))\mathrm{d}t$, which implies that both A^1 and A^2 are absolutely continuous and $\mathrm{d}A_t^i = a_t^i \mathrm{d}t$ with $a_t^i \geq 0$, $i = 1, 2$. The conclusion follows by setting $Z_t \stackrel{\triangle}{=} Z_t^1$ and $h_t \stackrel{\triangle}{=} g_1(t, Z_t) + a_t^1$.

Moreover, if $\mathcal{E}$ also satisfies the one-sided g-domination (3.10), then for any $\eta \in L^\infty(\mathcal{F}_\tau)$, we can set $\hat{Y}_t \stackrel{\triangle}{=} \mathcal{E}[\eta + zB_\tau|\mathcal{F}_t]$, $\forall t \in [0, T]$ and let $(\hat{h}, \hat{Z})$ be the corresponding pair. Applying the L^∞-domination (3.9) for $\mathcal{E}$, we see that $\hat{Y} - Y \in L^\infty_{\mathbf{F}}([0, T])$ under P, whence under P^γ. In fact, $\hat{Y} - Y$ is a g_α-submartingale under P^γ: for $0 \leq s \leq t \leq T$,

$$\begin{aligned}
\hat{Y}_s - Y_s &= \mathcal{E}[\hat{Y}_t|\mathcal{F}_s] - \mathcal{E}[\xi + zB_\tau|\mathcal{F}_s] \\
&= \mathcal{E}[\hat{Y}_t - Y_t + \mathcal{E}[\xi + zB_\tau|\mathcal{F}_t]|\mathcal{F}_s] - \mathcal{E}[\xi + zB_\tau|\mathcal{F}_s] \\
&= \mathcal{E}[\hat{Y}_t - Y_t + \xi + zB_\tau|\mathcal{F}_s] - \mathcal{E}[\xi + zB_\tau|\mathcal{F}_s] \leq \mathcal{E}_\gamma^{g_\alpha}[\hat{Y}_t - Y_t|\mathcal{F}_s], \quad P^\gamma\text{-a.s.}
\end{aligned}$$

Applying Theorem 2.1 again, we can find an increasing càdlàg process A null at 0 and a process $\bar{Z} \in \mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that

$$\hat{Y}_t - Y_t = \eta - \xi + \int_t^T \alpha|\bar{Z}_s|^2 \mathrm{d}s - A_T + A_t - \int_t^T \bar{Z}_s \mathrm{d}B_s^\gamma, \quad \forall t \in [0, T],\ P^\gamma\text{-a.s.},$$

which, in the light of the Girsanov theorem, is equivalent to

$$\begin{aligned}
\hat{Y}_t - Y_t &= \eta - \xi + \int_t^T (\alpha|\bar{Z}_s|^2 + \langle \gamma_s, \bar{Z}_s \rangle)\mathrm{d}s - A_T + A_t \\
&\quad - \int_t^T \bar{Z}_s \mathrm{d}B_s, \quad \forall t \in [0, T],\ P\text{-a.s.}
\end{aligned}$$

On the other hand, we also have

$$\hat{Y}_t - Y_t = \eta - \xi + \int_t^T (\hat{h}_s - h_s)\mathrm{d}s - \int_t^T (\hat{Z}_s - Z_s)\mathrm{d}B_s, \quad \forall t \in [0, T],\ P\text{-a.s.}$$

Thus by comparing the martingale parts and the bounded variation parts, one has

$$\hat{Z}_t - Z_t \equiv \bar{Z}_t \quad \text{and} \quad (\hat{h}_t - h_t)\mathrm{d}t \equiv (\alpha|\bar{Z}_t|^2 + \langle \gamma_t, \bar{Z}_t \rangle)\mathrm{d}t - \mathrm{d}A_t,$$

which implies that A is absolutely continuous and $\mathrm{d}A_t = a_t \mathrm{d}t$ with $a_t \geq 0$. Consequently,

$$\begin{aligned}
\hat{h}_t - h_t &= \alpha|\hat{Z}_t - Z_t|^2 + \langle \gamma_t, \hat{Z}_t - Z_t \rangle - a_t \\
&\leq \alpha|\hat{Z}_t - Z_t|^2 + \langle \gamma_t, \hat{Z}_t - Z_t \rangle, \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.}
\end{aligned}$$

This proves the proposition. ∎

We remark that one of the consequences of Proposition 5.1, especially the representation (5.2), is that the "càdlàg modification" that we found in Proposition 4.1 is actually continuous. In other words, the unique solution of BSDE (4.7) should belong to $\mathbb{C}^\infty_{\mathbf{F}}([0, T])$.

We now turn our attention to a comparison theorem for the solutions to the BSDE (4.7). To begin with, let us note that if f satisfies (H3), then for any $\phi \in L^\infty_{\mathbf{F}}([0, T])$, the function

$f^{\phi}(t,\omega,y) \triangleq f(t,\omega,y)+\phi(t,\omega)$, $\forall (t,\omega,y) \in [0,T]\times\Omega\times\mathbb{R}$, also satisfies (H3). Thus for any $\xi' \in L^\infty(\mathcal{F}_T)$ and $z \in \mathbb{R}^d$, the BSDE

$$Y_t + zB_t + \int_0^t [f(s,Y_s)+\phi_s]\mathrm{d}s = \mathcal{E}\left\{\xi' + zB_T + \int_0^T [f(s,Y_s)+\phi_s]\mathrm{d}s|\mathcal{F}_t\right\},$$
$$t \in [0,T], \tag{5.5}$$

admits a unique solution in $\mathbb{C}^\infty_{\mathbf{F}}([0,T])$. We shall denote this solution by Y'.

Theorem 5.2 (*Comparison Theorem*). *Assume that f satisfies* (H3). *For fixed $z \in \mathbb{R}^d$, let Y, $Y' \in \mathbb{C}^\infty_{\mathbf{F}}([0,T])$ be the unique solutions of* (4.7) *and* (5.5) *respectively. Suppose that*

$$\xi' \geq \xi, \quad P\text{-a.s.} \quad and \quad \phi \geq 0, \quad \mathrm{d}t\times\mathrm{d}P\text{-a.s.};$$

then it holds P-a.s. that $Y'_t \geq Y_t$, $\forall t \in [0,T]$.

Proof. We first assume $\phi_t \equiv 0$. For any $\delta \in \mathbb{Q}^+$, define two stopping times

$$\sigma^\delta \triangleq \inf\{t \in [0,T)|Y'_t \leq Y_t - \delta\} \quad \text{and} \quad \tau^\delta \triangleq \inf\{t \in [\sigma^\delta,T]|Y'_t \geq Y_t\}.$$

Here we use the convention that $\inf\,\emptyset \triangleq T$. Since $Y'_T = \xi' \geq \xi = Y_T$, P-a.s., we must have $\sigma^\delta \leq \tau^\delta \leq T$, P-a.s. Further, since both Y and Y' have continuous paths, we know that on $G^\delta \triangleq \{\sigma^\delta < T\}$, it holds that

$$Y'_{\sigma^\delta} = Y_{\sigma^\delta} - \delta, \qquad Y'_{\tau^\delta} = Y_{\tau^\delta}, \quad P\text{-a.s.} \tag{5.6}$$

Next, for a given $t \in [0,T]$, we define a stopping time $\hat{t} \triangleq t \vee \sigma^\delta \wedge \tau^\delta$. Then, applying Theorem 4.3 and Proposition 4.2(iv) we have, P-a.s.,

$$Y_{\hat{t}} + zB_{\hat{t}} + \int_0^{\hat{t}} f(s,Y_s)\mathrm{d}s = \mathcal{E}\left\{Y_{\tau^\delta} + zB_{\tau^\delta} + \int_{\hat{t}}^{\tau^\delta} f(s,Y_s)\mathrm{d}s|\mathcal{F}_{\hat{t}}\right\} + \int_0^{\hat{t}} f(s,Y_s)\mathrm{d}s, \quad P\text{-a.s.}$$

Moreover, since $G^\delta \in \mathcal{F}_{\sigma^\delta} \subset \mathcal{F}_{\hat{t}}$, we can deduce from Proposition 4.2(iii) that

$$\begin{aligned}
&\mathbf{1}_{G^\delta}\mathcal{E}\left\{\mathbf{1}_{G^\delta}Y_{\tau^\delta} + zB_{\tau^\delta} + \int_{\hat{t}}^{\tau^\delta} \mathbf{1}_{G^\delta} f(s,\mathbf{1}_{G^\delta}Y_{\hat{s}})\mathrm{d}s|\mathcal{F}_{\hat{t}}\right\}\\
&\quad= \mathbf{1}_{G^\delta}\mathcal{E}\left\{\mathbf{1}_{G^\delta}Y_{\tau^\delta} + zB_{\tau^\delta} + \int_{\hat{t}}^{\tau^\delta} \mathbf{1}_{G^\delta} f(s,Y_s)\mathrm{d}s|\mathcal{F}_{\hat{t}}\right\}\\
&\quad= \mathbf{1}_{G^\delta}\mathcal{E}\left\{Y_{\tau^\delta} + zB_{\tau^\delta} + \int_{\hat{t}}^{\tau^\delta} f(s,Y_s)\mathrm{d}s|\mathcal{F}_{\hat{t}}\right\} = \mathbf{1}_{G^\delta}Y_{\hat{t}} + \mathbf{1}_{G^\delta}zB_{\hat{t}}.
\end{aligned} \tag{5.7}$$

By using the L^∞-domination (3.9) for $\mathcal{E}$ and Proposition 4.1 one shows that P-a.s.

$$\left|\mathcal{E}\left\{\mathbf{1}_{G^\delta}Y'_{\tau^\delta} + zB_{\tau^\delta} + \int_{\hat{t}}^{\tau^\delta} \mathbf{1}_{G^\delta} f(s,\mathbf{1}_{G^\delta}Y'_{\hat{s}})\mathrm{d}s|\mathcal{F}_r\right\}\right.$$

$$
\begin{aligned}
&- \mathcal{E}\left\{\mathbf{1}_{G^\delta} Y_{\tau^\delta} + zB_{\tau^\delta} + \int_{\hat t}^{\tau^\delta} \mathbf{1}_{G^\delta} f(s, \mathbf{1}_{G^\delta} Y_{\hat s}) \mathrm{d}s | \mathcal{F}_r\right\}\Bigg| \\
&\le \left\| \int_{\hat t}^{\tau^\delta} \mathbf{1}_{G^\delta}[f(s, \mathbf{1}_{G^\delta} Y'_{\hat s}) - f(s, \mathbf{1}_{G^\delta} Y_{\hat s})] \mathrm{d}s \right\|_\infty \\
&\le \kappa \int_t^T \|\mathbf{1}_{G^\delta} Y'_{\hat s} - \mathbf{1}_{G^\delta} Y_{\hat s}\|_\infty \mathrm{d}s, \quad \forall r \in [0, T].
\end{aligned}
$$

Setting $r = \hat t$ in the above and using (5.7) we obtain that

$$
\|\mathbf{1}_{G^\delta} Y'_{\hat t} - \mathbf{1}_{G^\delta} Y_{\hat t}\|_\infty \le \kappa \int_t^T \|\mathbf{1}_{G^\delta} Y'_{\hat s} - \mathbf{1}_{G^\delta} Y_{\hat s}\|_\infty \mathrm{d}s.
$$

The Gronwall inequality then leads to that $\|\mathbf{1}_{G^\delta} Y'_{\hat t} - \mathbf{1}_{G^\delta} Y_{\hat t}\|_\infty = 0$ for any $t \in [0, T]$. In particular, for $t = 0$, we obtain that $\mathbf{1}_{G^\delta} Y'_{\sigma^\delta} = \mathbf{1}_{G^\delta} Y_{\sigma^\delta}$, P-a.s., which, together with (i), shows that $G^\delta = \{\sigma^\delta < T\}$ is a null set. Since $Y'_T \ge Y_T$, P-a.s., and $\{Y'_t \ge Y_t, \forall t \in [0, T)\}^c \subset \bigcup_{\delta \in \mathbb{Q}^+} \{\sigma^\delta < T\}$, we conclude that

$$
Y'_t \ge Y_t, \quad \forall t \in [0, T], \quad P\text{-a.s.} \tag{5.8}
$$

We now consider the case when $\phi_t \ge 0$, $\mathrm{d}t \times \mathrm{d}P$-a.s. We proceed as follows. For any $n \in \mathbb{N}$, let $t_j^n \triangleq \frac{j}{n} T$, $j = 0, 1, \ldots, n$, be a partition of $[0, T]$, and define recursively a sequence of BSDEs:

$$
\begin{aligned}
&Y_t^{j,n} + zB_t + \int_0^t f(s, Y_s^{j,n}) \mathrm{d}s = \mathcal{E}\left\{X_j^n + \int_{t_{j-1}^n}^{t_j^n} \phi_s \mathrm{d}s + zB_{t_j^n} + \int_0^{t_j^n} f(s, Y_s^{j,n}) \mathrm{d}s | \mathcal{F}_t\right\}, \\
&\quad t \in [0, t_j^n],
\end{aligned}
$$

where $\{X_j^n\}_{j \ge 0}$ are defined recursively by $X_n^n = \xi'$, and $X_{j-1}^n \triangleq Y_{t_{j-1}^n}^{j,n}$, for $j = n, \ldots, 1$. Now, applying the result for $\phi = 0$ (similar to (5.8)) with $\xi_j^n \triangleq X_j^n + \int_{t_{j-1}^n}^{t_j^n} \phi_s \mathrm{d}s$, we can then show by induction that for each $1 \le j \le n$, it holds that $Y_t^{j,n} \ge Y_t$, $t \in [0, t_j^n]$, P-a.s. We now define a new process by $Y_t^n \triangleq Y_t^{j,n}$, $t \in [t_{j-1}^n, t_j^n]$, $j = 1, \ldots, n$. It is easy to check that for any $j = 1, \ldots, n$ and any $t \in [t_{j-1}^n, t_j^n)$,

$$
Y_t^n + zB_t = \mathcal{E}\left\{\xi' + \int_{t_{j-1}^n}^T \phi_s \mathrm{d}s + zB_T + \int_t^T f(s, Y_s^n) \mathrm{d}s | \mathcal{F}_t\right\}, \quad P\text{-a.s.}
$$

Applying L^∞-domination (3.9) for $\mathcal{E}$ we see that for any $j = 1, \ldots, n$ and any $t \in [t_{j-1}^n, t_j^n)$

$$
\begin{aligned}
\|Y_t^n - Y'_t\|_\infty &= \Bigg\| \mathcal{E}\left\{\xi' + \int_{t_{j-1}^n}^T \phi_s \mathrm{d}s + zB_T + \int_t^T f(s, Y_s^n) \mathrm{d}s | \mathcal{F}_t\right\} \\
&\qquad - \mathcal{E}\left\{\xi' + zB_T + \int_t^T [f(s, Y'_s) + \phi_s] \mathrm{d}s | \mathcal{F}_t\right\} \Bigg\|_\infty \\
&\le \left\| \int_{t_{j-1}^n}^t \phi_s \mathrm{d}s + \int_t^T (f(s, Y_s^n) - f(s, Y'_s)) \mathrm{d}s \right\|_\infty
\end{aligned}
$$

$$\leq \frac{T}{n}\|\phi\|_\infty + \kappa \int_t^T \|Y_s^n - Y_s'\|_\infty \mathrm{d}s.$$

First applying Gronwall's inequality and then letting $n \to \infty$ we see that Y_t^n converges to Y_t' in $L^\infty(\mathcal{F}_t)$, for each $t \in [0, T]$. Since both Y and Y' are continuous, we conclude that $Y_t' \geq Y_t$, $\forall t \in [0, T]$, P-a.s. The proof is now complete. ■

We can now follow the scheme of [6,14] to derive the Doob–Meyer decomposition. For any $Y \in \mathbb{D}^\infty_{\mathbf{F}}([0, T])$ and $z \in \mathbb{R}^d$, we define

$$f^n(t, \omega, y) \triangleq n(Y(t, \omega) - y), \quad \forall (t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}, \forall n \in \mathbb{N}.$$

It is easy to check that each f^n satisfies (H3); thus the BSDE

$$y_t^n + zB_t + \int_0^t f^n(s, y_s^n)\mathrm{d}s = \mathcal{E}\left\{Y_T + zB_T + \int_0^T f^n(s, y_s^n)\mathrm{d}s|\mathcal{F}_t\right\}, \quad \forall t \in [0, T], \tag{5.9}$$

admits a unique solution $y^n \in \mathbb{C}^\infty_{\mathbf{F}}([0, T])$. We have the following lemma.

Lemma 5.3. *Assume* (H3)*, and let y^n be the solution of* (5.9)*,* $n \geq 1$. *Suppose that for a given $Y \in \mathbb{D}^\infty_{\mathbf{F}}([0, T])$ and $z \in \mathbb{R}^d$, the process $Y_t + zB_t$, $t \in [0, T]$, is an $\mathcal{E}$-submartingale (resp. $\mathcal{E}$-supermartingale); then it holds that*

$$y_t^n \geq (\mathit{resp.} \leq) y_t^{n+1} \geq (\mathit{resp.} \leq) Y_t, \quad t \in [0, T], n \in \mathbb{N}, P\text{-a.s.}$$

Proof. We shall prove only the submartingale case; the supermartingale case is similar. For any $n \in \mathbb{N}$ and any $\delta \in \mathbb{Q}^+$, let us define two stopping times

$$\sigma^{n,\delta} \triangleq \inf\{t \in [0, T)| y_t^n \leq Y_t - \delta\} \quad \text{and} \quad \tau^{n,\delta} \triangleq \inf\{t \in [\sigma^{n,\delta}, T]| y_t^n \geq Y_t\}.$$

It is easy to see that $\sigma^{n,\delta} \leq \tau^{n,\delta} \leq T$, P-a.s. Then the right-continuity of y^n and Y leads to that

$$y^n_{\sigma^{n,\delta}} \leq Y_{\sigma^{n,\delta}} - \delta, \quad P\text{-a.s. on } \{\sigma^{n,\delta} < T\}, \quad \text{and} \quad y^n_{\tau^{n,\delta}} \geq Y_{\tau^{n,\delta}}, \quad P\text{-a.s.} \tag{5.10}$$

Applying Proposition 4.2(iv) and Theorem 4.3, one has

$$y^n_{\sigma^{n,\delta}} + zB_{\sigma^{n,\delta}} = \mathcal{E}\left[y^n_{\tau^{n,\delta}} + zB_{\tau^{n,\delta}} + \int_{\sigma^{n,\delta}}^{\tau^{n,\delta}} n(Y_s - y_s^n)\mathrm{d}s|\mathcal{F}_{\sigma^{n,\delta}}\right], \quad P\text{-a.s.}$$

Using (5.10) we deduce that $\int_{\sigma^{n,\delta}}^{\tau^{n,\delta}} n(Y_s - y_s^n)\mathrm{d}s \geq 0$, P-a.s., and combining this with Proposition 4.2(i) and Theorem 4.3, we obtain that

$$y^n_{\sigma^{n,\delta}} + zB_{\sigma^{n,\delta}} \geq \mathcal{E}[Y_{\tau^{n,\delta}} + zB_{\tau^{n,\delta}}|\mathcal{F}_{\sigma^{n,\delta}}] \geq Y_{\sigma^{n,\delta}} + zB_{\sigma^{n,\delta}}.$$

This implies that $\{y^n_{\sigma^{n,\delta}} \leq Y_{\sigma^{n,\delta}} - \delta\}$ is a null set; thus so is $\{\sigma^{n,\delta} < T\}$. Furthermore, since

$$\{y_t^n \geq Y_t, t \in [0, T), n \in \mathbb{N}\}^c \subset \bigcup_{n\in\mathbb{N}} \bigcup_{\delta\in\mathbb{Q}^+} \{\sigma^{n,\delta} < T\} \quad \text{and} \quad y_T^n \geq Y_T, \quad n \in \mathbb{N},$$

it holds that $P\{y_t^n \geq Y_t, t \in [0, T], n \in \mathbb{N}\} = 1$. Consequently, we have that P-a.s.

$$f^n(t, y_t^n) = n(Y_t - y_t^n) \geq (n + 1)(Y_t - y_t^n) = f^{n+1}(t, y_t^n), \quad \forall t \in [0, T], \forall n \in \mathbb{N}.$$

It then follows from Theorem 5.2 that P-a.s. $y_t^n \geq y_t^{n+1} \geq Y_t$, for all $t \in [0, T]$ and $n \in \mathbb{N}$. This completes the proof. ■

We should note that Lemma 5.3 indicates that if $Y_\cdot + zB_\cdot$ is an $\mathcal{E}$-submartingale, then all the processes $A_t^n = \int_0^t n(y_s^n - Y_s)\mathrm{d}s$, $t \geq 0$, are increasing (or decreasing if Y is a $\mathcal{E}$-supermartingale), $\|y^n\|_\infty \leq \|Y\|_\infty \vee \|y^1\|_\infty$, and $y_t^n - A_t^n + zB_t$, $t \geq 0$ is an $\mathcal{E}$-martingale. Thus, Proposition 5.1 implies that there is a unique pair $(h^n, Z^n) \in L^1_{\mathbf{F}}([0,T]) \times \mathcal{H}^2_{\mathbf{F}}([0,T]; \mathbb{R}^d)$ such that

$$y_t^n - A_t^n + zB_t = y_T^n - A_T^n + zB_T + \int_t^T h_s^n \mathrm{d}s - \int_t^T Z_s^n \mathrm{d}B_s, \quad t \in [0, T], \tag{5.11}$$

and the following estimates hold:

$$-\ell(|Z_t^n| + |Z_t^n|^2) \leq g_1(t, Z_t^n) \leq h_t^n \leq g_2(t, Z_t^n) \leq \ell(|Z_t^n| + |Z_t^n|^2), \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.} \tag{5.12}$$

We shall prove that both $\{Z^n\}_{n\in\mathbb{N}}$ and $\{A_T^n\}_{n\in\mathbb{N}}$ are bounded in a very strong sense.

Lemma 5.4. *Let the process $Y_t + zB_t$, $t \in [0, T]$, be either an $\mathcal{E}$-submartingale or an $\mathcal{E}$-supermartingale like those in Lemma 5.3, and let $\{A^n\}$ and $\{Z^n\}$ be processes defined in (5.11). Then, for any $p > 0$, $\{Z^n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}^p(\mathbb{R}^d)$ and $\{A_T^n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\mathcal{F}_T)$.*

Proof. We shall only prove the submartingale case. That is, we assume that A^n is increasing. From BSDE (5.11) we see that

$$A_T^n = y_T^n - y_0^n + \int_0^T h_s^n \mathrm{d}s - \int_0^T (Z_s^n - z)\mathrm{d}B_s, \quad P\text{-a.s.}$$

Let $M \triangleq \|Y\|_\infty \vee \|y^1\|_\infty$ and use the domination (5.12) of h^n; we have

$$|A_T^n| \leq 2M + \ell T + 2\ell \int_0^T |Z_s^n|^2 \mathrm{d}s + \sup_{0\leq t\leq T} \left| \int_0^t (Z_s^n - z)\mathrm{d}B_s \right|, \quad P\text{-a.s.} \tag{5.13}$$

In what follows for each $p > 0$ we define $C_p > 0$ to be a generic constant depending only on p, as well as $\ell, T, M, |z|$, which may vary from line to line. Using (5.13) and the Burkholder–Davis–Gundy inequality one shows that

$$\begin{aligned} E|A_T^n|^p &\leq C_p \left\{ 1 + E\left[\int_0^T |Z_s^n|^2 \mathrm{d}s \right]^p + E\left[\int_0^T |Z_s^n - z|^2 \mathrm{d}s \right]^{p/2} \right\} \\ &\leq C_p \left\{ 1 + E\left[\int_0^T |Z_s^n - z|^2 \mathrm{d}s \right]^p \right\}. \end{aligned}$$

Thus it suffices to show that $\sup_{n\in\mathbb{N}} E(\int_0^T |Z_s^n - z|^2 \mathrm{d}s)^p < \infty$. For any $\alpha > 0$, we apply Itô's formula to $\mathrm{e}^{\alpha y_t^n}$ to get

$$\begin{aligned} &\mathrm{e}^{\alpha y_0^n} + \frac{\alpha^2}{2} \int_0^T \mathrm{e}^{\alpha y_s^n} |Z_s^n - z|^2 \mathrm{d}s \\ &\quad = \mathrm{e}^{\alpha y_T^n} + \alpha \left[\int_0^T \mathrm{e}^{\alpha y_s^n} h_s^n \mathrm{d}s - \int_0^T \mathrm{e}^{\alpha y_s^n} \mathrm{d}A_s^n - \int_0^T \mathrm{e}^{\alpha y_s^n} (Z_s^n - z)\mathrm{d}B_s \right] \end{aligned}$$

$$\leq \mathrm{e}^{\alpha y_T^n} + \alpha\ell \int_0^T \mathrm{e}^{\alpha y_s^n}\mathrm{d}s + 4\alpha\ell \int_0^T \mathrm{e}^{\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s + 4\alpha\ell \int_0^T \mathrm{e}^{\alpha y_s^n}|z|^2\mathrm{d}s$$
$$- \alpha \int_0^T \mathrm{e}^{\alpha y_s^n}(Z_s^n - z)\mathrm{d}B_s. \tag{5.14}$$

Note that the last inequality is due to the fact that A^n is increasing. It then follows that

$$\left(\frac{\alpha^2}{2} - 4\alpha\ell\right)\int_0^T \mathrm{e}^{\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s \leq C_p + \alpha \sup_{0\leq t\leq T}\left|\int_0^t \mathrm{e}^{\alpha y_s^n}(Z_s^n - z)\mathrm{d}B_s\right|.$$

Choose $\alpha > 8\ell$; applying the Burkholder–Davis–Gundy inequality again we obtain that

$$\begin{aligned} E\left(\int_0^T \mathrm{e}^{\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s\right)^p &\leq C_p + C_p E\left(\int_0^T \mathrm{e}^{2\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s\right)^{p/2} \\ &\leq C_p + C_p \mathrm{e}^{pM\alpha/2} E\left(\int_0^T \mathrm{e}^{\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s\right)^{p/2} \\ &\leq C_p + \frac{1}{2}E\left(\int_0^T \mathrm{e}^{\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s\right)^p, \end{aligned}$$

which implies that $E(\int_0^T \mathrm{e}^{\alpha y_s^n}|Z_s^n - z|^2\mathrm{d}s)^p$ is dominated by a constant independent of n. This proves the lemma in the submartingale case. The supermartingale case can be proved in the same way except that in (5.14) Itô's formula should be applied to $\mathrm{e}^{-\alpha y_t^n}$. The proof is now complete. ■

We are now ready to prove the Doob–Meyer decomposition theorem.

Theorem 5.5. *Assume that $\mathcal{E}$ is a regular quadratic $\mathcal{F}$-expectation satisfying the one-sided g-domination (3.10). For any $Y \in \mathbb{C}_{\mathbf{F}}^\infty([0,T])$ and any $z \in \mathbb{R}^d$, if the process $Y_t + zB_t$, $t \in [0,T]$, is an $\mathcal{E}$-submartingale (resp. $\mathcal{E}$-supermartingale), then there exists a continuous increasing (resp. decreasing) process A null at 0 such that $Y_t - A_t + zB_t$, $t \geq 0$, is a local $\mathcal{E}$-martingale. Furthermore, if A is bounded, then $Y_t - A_t + zB_t$, $t \geq 0$, is an $\mathcal{E}$-martingale.*

Proof. We again prove only the submartingale case, as the submartingale case is similar. To begin with, let y^n be the solutions to (5.9), $n = 1, 2, \ldots$, and still have $M \stackrel{\triangle}{=} \|Y\|_\infty \vee \|y^1\|_\infty$. Since $y^n \geq Y$, by the definition of processes A_n and Lemma 5.3, we see that

$$E\int_0^T |y_s^n - Y_s|\mathrm{d}s = \frac{1}{n}E[|A_T^n|] \leq \frac{1}{n}\sup_{n\in\mathbb{N}}\|A_T^n\|_1 \to 0,$$

as $n \to \infty$. Moreover, since the y^n's converge decreasingly to Y, and Y is continuous, we can further conclude, in the light of Dini's theorem, that P-a.s.

$$\lim_{n\to\infty}\sup_{t\in[0,T]}(y_t^n - Y_t) = 0, \quad \text{thus} \quad \lim_{m,n\to\infty}\sup_{t\in[0,T]}|y_t^m - y_t^n| = 0. \tag{5.15}$$

We first show that there exists a subsequence of $\{A^n\}$, still denoted by $\{A^n\}$, such that the sequence $\{A_T^n\}_{n\in\mathbb{N}}$ is uniformly integrable. To see this, we claim that the processes Z^n converge to some process Z in $\mathcal{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d)$, as $n \to \infty$. In fact, applying Itô's formula to $|y_t^m - y_t^n|^2$

on $[0, T]$ we obtain

$$
\begin{aligned}
&|y_0^m - y_0^n|^2 + \int_0^T |Z_s^m - Z_s^n|^2 \mathrm{d}s \\
&\quad = |y_T^m - y_T^n|^2 + 2\int_0^T (y_s^m - y_s^n)[(h_s^m - h_s^n)\mathrm{d}s - (\mathrm{d}A_s^m - \mathrm{d}A_s^n) - (Z_s^m - Z_s^n)\mathrm{d}B_s] \\
&\quad \le |y_T^m - y_T^n|^2 + 2\sup_{s\in[0,T]} |y_s^m - y_s^n| \left\{ \int_0^T 2\ell(1 + |Z_s^m|^2 + |Z_s^n|^2)\mathrm{d}s + A_T^m + A_T^n \right\} \\
&\qquad - 2\int_0^T (y_s^m - y_s^n)(Z_s^m - Z_s^n)\mathrm{d}B_s.
\end{aligned} \tag{5.16}
$$

Taking the expectation on both sides of (5.16) and applying Hölder's inequality one has

$$
\begin{aligned}
&E\left\{\int_0^T |Z_s^m - Z_s^n|^2 \mathrm{d}s\right\} \le E\left\{\sup_{s\in[0,T]} |y_s^m - y_s^n|^2\right\} \\
&\quad + 2\left\{ E\left[\sup_{s\in[0,T]} |y_s^m - y_s^n|^2\right] E\left[\int_0^T 2\ell(1 + |Z_s^m|^2 + |Z_s^n|^2)\mathrm{d}s + A_T^m + A_T^n\right]^2 \right\}^{1/2} \\
&\quad \le E\left\{\sup_{s\in[0,T]} |y_s^m - y_s^n|^2\right\} + C\left\{E\left[\sup_{s\in[0,T]} |y_s^m - y_s^n|^2\right]\right\}^{1/2} \\
&\qquad \times \left[1 + \sup_{k\in\mathbb{N}} \|Z^k\|_{\mathcal{M}^4}^2 + \sup_{k\in\mathbb{N}} \|A_T^k\|_{L^2(\mathcal{F}_T)}\right],
\end{aligned}
$$

where $C > 0$ is a constant depending only on ℓ and T. This, together with Lemma 5.4, implies that $\{Z^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_{\mathbf{F}}^2([0, T]; \mathbb{R}^d)$, and hence has a limit $Z \in \mathcal{H}_{\mathbf{F}}^2([0, T]; \mathbb{R}^d)$. A simple application of the Burkholder–Davis–Gundy inequality leads to that

$$
\sup_{t\in[0,T]} \left|\int_0^t (Z_s^n - Z_s)\mathrm{d}B_s\right| \to 0 \quad \text{in } L^2(\mathcal{F}_T), \text{ as } n \to \infty. \tag{5.17}
$$

Applying [10, Lemma 2.5] we can find a subsequence of $\{Z^n\}_{n\in\mathbb{N}}$, still denoted by $\{Z^n\}_{n\in\mathbb{N}}$, such that $\sup_n |Z^n| \in \mathcal{H}_{\mathbf{F}}^2([0, T]; \mathbb{R}^d)$ and that $\sup_n |\int_0^T (Z_s^n - z)\mathrm{d}B_s| \in L^2(\mathcal{F}_T)$. Then in the light of (5.12) and (5.11), it holds P-a.s. that for any $n \in \mathbb{N}$

$$
\begin{aligned}
A_T^n &= y_T^n - y_0^n + \int_0^T h_s^n \mathrm{d}s - \int_0^T (Z_s^n - z)\mathrm{d}B_s \\
&\le 2M + \ell T + 2\ell \int_0^T \sup_n |Z_s^n|^2 \mathrm{d}s + \sup_n \left|\int_0^T (Z_s^n - z)\mathrm{d}B_s\right| \in L^1(\mathcal{F}_T).
\end{aligned}
$$

We can then deduce that $\sup_{n\in\mathbb{N}} A_T^n \in L^1(\mathcal{F}_T)$, which implies that, P-almost surely, $A_t^n \le E[\sup_{n\in\mathbb{N}} A_T^n | \mathcal{F}_t]$, for all $t \in [0, T]$, $n \in \mathbb{N}$. Now let us define a sequence of stopping times

$$
\tau_k \triangleq \inf\left\{t \in [0, T] : E\left[\sup_{n\in\mathbb{N}} A_T^n | \mathcal{F}_t\right] > k\right\} \wedge T, \quad k \in \mathbb{N}. \tag{5.18}
$$

Clearly, $\tau_k \nearrow T$, P-a.s., as $k \to \infty$. Furthermore, let us define $p_k \triangleq p(k + M, |z|)$, $J_k \triangleq J(k+M, |z|)$ and $\alpha_k \triangleq \alpha(k+M, |z|)$, and define $Y_t^k \triangleq Y_{t\wedge\tau_k}$, $y_t^{n,k} \triangleq y_{t\wedge\tau_k}^n$, $A_t^{n,k} \triangleq A_{t\wedge\tau_k}^n$,

$t \in [0, T]$. We will show that for any $k \in \mathbb{N}$, there exists a subsequence of $\{A^n\}_{n\in\mathbb{N}}$, denoted again by $\{A^n\}_{n\in\mathbb{N}}$ itself, such that for all $k \in \mathbb{N}$, it holds that $\lim_{n\to\infty} A_t^{n,k} = \tilde{A}_t^k$, $t \in [0, T]$, P-a.s. for some continuous, increasing process $\tilde{A}^k$.

To see this, let us first fix $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, applying Theorem 4.3 and Proposition 4.1 we have

$$y_t^{n,k} - A_t^{n,k} + zB_{t\wedge\tau_k} = \mathcal{E}[y_{\tau_k}^n - A_{\tau_k}^n + zB_{\tau_k}|\mathcal{F}_t], \quad \forall t \in [0, T].$$

Applying Proposition 5.1, we can find a unique pair $(h^{n,k}, Z^{n,k}) \in L^1_{\mathbf{F}}([0, T]) \times \mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that

$$y_t^{n,k} - A_t^{n,k} = y_T^{n,k} - A_T^{n,k} + \int_t^T h_s^{n,k}\mathrm{d}s - \int_t^T (Z_s^{n,k} - \mathbf{1}_{\{s\le\tau_k\}}z)\mathrm{d}B_s, \quad \forall t \in [0, T]. \tag{5.19}$$

On the other hand, by (5.11) we have

$$y_t^{n,k} - A_t^{n,k} = y_T^{n,k} - A_T^{n,k} + \int_t^T \mathbf{1}_{\{s\le\tau_k\}}h_s^n\mathrm{d}s - \int_t^T \mathbf{1}_{\{s\le\tau_k\}}(Z_s^n - z)\mathrm{d}B_s, \quad \forall t \in [0, T]. \tag{5.20}$$

Thus by comparing the martingale parts and the bounded variation parts of (5.19) and (5.20), one has $h_t^{n,k} \equiv \mathbf{1}_{\{t\le\tau_k\}}h_t^n$ and $Z_t^{n,k} \equiv \mathbf{1}_{\{t\le\tau_k\}}Z_t^n$. Moreover, it also follows from Proposition 5.1 that there is a BMO process $\gamma^{n,k}$ with $\|\gamma^{n,k}\|^2_{\mathrm{BMO}} \le J_k$ such that

$$\begin{aligned}-\alpha_k|Z_t^{m,k} - Z_t^{n,k}|^2 + \langle\gamma_t^{m,k}, Z_t^{m,k} - Z_t^{n,k}\rangle &\le h_t^{m,k} - h_t^{n,k}\\ &\le \alpha_k|Z_t^{m,k} - Z_t^{n,k}|^2 + \langle\gamma_t^{n,k}, Z_t^{m,k} - Z_t^{n,k}\rangle, \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.}\end{aligned} \tag{5.21}$$

Note that (5.21) implies that for any $m, n \in \mathbb{N}$,

$$\begin{aligned}E\int_0^{\tau_k} |h_s^m - h_s^n|\mathrm{d}s &\le E\int_0^{\tau_k} [\alpha_k|Z_s^m - Z_s^n|^2 + (|\gamma_s^{m,k}| \vee |\gamma_s^{n,k}|)|Z_s^m - Z_s^n|]\mathrm{d}s\\ &\le \alpha_k E\int_0^T |Z_s^m - Z_s^n|^2\mathrm{d}s + \left\{E\int_0^T (|\gamma_s^{m,k}|^2 + |\gamma_s^{n,k}|^2)\mathrm{d}s\, E\int_0^T |Z_s^m - Z_s^n|^2\mathrm{d}s\right\}^{\frac{1}{2}}.\end{aligned}$$

Hence, one can deduce from the convergence of Z^n in $\mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ that $\{\mathbf{1}_{\{\cdot\wedge\tau_k\}}h^n_\cdot\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1_{\mathbf{F}}([0, T])$. Let $\tilde{h}^k$ be its limit in $L^1_{\mathbf{F}}([0, T])$; it then follows that

$$\sup_{t\in[0,T]}\left|\int_0^{t\wedge\tau_k} (h_s^n - \tilde{h}_s^k)\mathrm{d}s\right| \to 0 \quad \text{in } L^2(\mathcal{F}_{\tau_k}), \text{ as } n \to \infty. \tag{5.22}$$

Now let us define $\tilde{A}_t^k \triangleq Y_t^k - Y_0^k + \int_0^{t\wedge\tau_k} \tilde{h}_s^k\mathrm{d}s - \int_0^{t\wedge\tau_k}(Z_s - z)\mathrm{d}B_s$, $t \in [0, T]$. Clearly, $\tilde{A}^k$ is continuous. Furthermore, since

$$A_t^{n,k} = y_t^{n,k} - y_0^{n,k} + \int_0^{t\wedge\tau_k} h_s^n\mathrm{d}s - \int_0^{t\wedge\tau_k}(Z_s^n - z)\mathrm{d}B_s, \quad \forall t \in [0, T], \forall n \in \mathbb{N},$$

applying the bounded convergence theorem as well as (5.15), (5.17) and (5.22), one shows that $\sup_{t\in[0,T]}|A_t^{n,k} - \tilde{A}_t^k|$ converges to 0 in $L^1(\mathcal{F}_{\tau_k})$, as $n \to \infty$. Therefore, we can find a subsequence of $\{A^n\}_{n\in\mathbb{N}}$, still denoted by $\{A^n\}_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} A_t^{n,k} = \tilde{A}_t^k, \quad \forall t \in [0, T], P\text{-a.s.} \tag{5.23}$$

We note that (5.23) indicates that $\tilde{A}^k$ is an increasing process. Furthermore, applying the Helly selection theorem if necessary, we can assume that the convergence in (5.23) holds true for all $k \in \mathbb{N}$ for this subsequence.

We can now complete the proof. By the definition of τ_k (5.18) and the continuity of A^n, one can deduce that for any $k, n \in \mathbb{N}$, $A^n_{\tau_k} \le k$, P-a.s.

Hence for any $k \in \mathbb{N}$, (5.23) implies that P-a.s.

$$|\tilde{A}^k_t| \le k, \quad \forall t \in [0, T] \quad \text{and} \quad \tilde{A}^k_t \equiv \tilde{A}^k_{\tau_k}, \quad \forall t \in [\tau_k, T].$$

Note that $\tilde{A}^k_t = \lim_{n\to\infty} A^{n,k}_t = \lim_{n\to\infty} A^{n,k+1}_{t\wedge\tau_k} = A^{k+1}_{t\wedge\tau_k}$, $t \in [0, T]$, P-a.s., we can define a continuous, increasing process $A_t \stackrel{\triangle}{=} \tilde{A}^k_t$, $t \in [0, \tau_k]$, $k \in \mathbb{N}$. Clearly, A is null at 0. For fixed $k \in \mathbb{N}$, and $t \in [0, T]$, applying the L^{p_k}-domination (3.8) of $\mathcal{E}$ yields that

$$\begin{aligned}
&\|\mathcal{E}[y^n_{\tau_k} - A^n_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t] - \mathcal{E}[Y_{\tau_k} - A_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t]\|_{p_k}\\
&\quad \le 3\|y^n_{\tau_k} - Y_{\tau_k}\|_{p_k} + 3\|A^n_{\tau_k} - A_{\tau_k}\|_{p_k}.
\end{aligned}$$

By considering a subsequence, we have, P-a.s.,

$$\begin{aligned}
Y_{t\wedge\tau_k} - A_{t\wedge\tau_k} + zB_{t\wedge\tau_k} &= \lim_{n\to\infty} (y^{n,k}_{t\wedge\tau_k} - A^{n,k}_{t\wedge\tau_k} + zB_{t\wedge\tau_k})\\
&= \lim_{n\to\infty} \mathcal{E}[y^{n,k}_{\tau_k} - A^{n,k}_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t] = \mathcal{E}[Y_{\tau_k} - A_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t].
\end{aligned}$$

Then, Proposition 4.1, together with the continuity of Y and A, implies that P-a.s.

$$Y_{t\wedge\tau_k} - A_{t\wedge\tau_k} + zB_{t\wedge\tau_k} = \mathcal{E}[Y_{\tau_k} - A_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t], \quad \forall t \in [0, T]. \tag{5.24}$$

In other words, $Y_t - A_t + zB_t$, $t \ge 0$ is a local $\mathcal{E}$-martingale, proving the first part of the theorem.

To see the last part of the theorem, we assume further that A is bounded. Let $K = \|Y\|_\infty + \|A\|_\infty$, $R = |z|$ and $p = p(K, R)$. Fix a $t \in [0, T]$; applying L^p-domination (3.8) again we obtain that for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\|\mathcal{E}[Y_{\tau_k} - A_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t] - \mathcal{E}[Y_T - A_T + zB_T|\mathcal{F}_t]\|_p\\
&\quad \le R\|B_{t\wedge\tau_k} - B_t\|_p + 3\|Y_{\tau_k} - Y_T\|_p + 3\|A_{\tau_k} - A_T\|_p + C_R\|T - \tau_k\|_p.
\end{aligned}$$

Clearly, the right hand side above converges to 0 as $k \to \infty$, thanks to the Burkholder–Davis–Gundy inequality and the bounded convergence theorem. Thus, taking a subsequence if necessary, we may assume that $\mathcal{E}[Y_{\tau_k} - A_{\tau_k} + zB_{\tau_k}|\mathcal{F}_t]$ converges P-a.s. to $\mathcal{E}[Y_T - A_T + zB_T|\mathcal{F}_t]$. Letting $k \to \infty$ in (5.24), the continuity of Y and A imply that

$$Y_t - A_t + zB_t = \mathcal{E}[Y_T - A_T + zB_T|\mathcal{F}_t], \quad P\text{-a.s.}$$

Eventually, applying Proposition 4.1 and using the continuity of Y and A again we have P-a.s.

$$Y_t - A_t + zB_t = \mathcal{E}[Y_T - A_T + zB_T|\mathcal{F}_t], \quad \forall t \in [0, T],$$

which means that $Y_t - A_t + zB_t$, $t \ge 0$, is an $\mathcal{E}$-martingale. The proof is now complete. ∎

6. Representation theorem of quadratic $\mathcal{F}$-expectations

In this section we prove the representation theorem for quadratic $\mathcal{F}$-expectations. We assume that $\mathcal{E}$ is a translation invariant quadratic $\mathcal{F}$-expectation dominated by two quadratic g-expectations $\mathcal{E}^{g_1}$ and $\mathcal{E}^{g_2}$ from below and above, and both g_1 and g_2 satisfy (H1) and (H2) with the same constant $\ell > 0$. We also assume that $\mathcal{E}$ satisfies the L^p-domination (3.8), the L^∞-domination (3.9), and the one-sided g-domination (3.10).

We begin our discussion by considering the following special semi-martingale:

$$Y_t^z \stackrel{\triangle}{=} \ell(|z| + |z|^2)t + zB_t, \quad \forall t \in [0, T], z \in \mathbb{R}^d. \tag{6.1}$$

By the comparison theorem of BSDEs, it is easy to see that Y^z is an $\mathcal{E}^{g_1}$-submartingale, whence an $\mathcal{E}$-submartingale. Then, by the Doob–Meyer decomposition (Theorem 5.5) there exists a continuous, increasing process A^z null at 0 such that $Y^z - A^z$ is a local $\mathcal{E}$-martingale. We claim that $A_T^z \in L^\infty(\Omega)$, and hence $Y^z - A^z$ is a true $\mathcal{E}$-martingale. Indeed, let $\{\tau_n^z\}_{n\ge 1}$ be a sequence of "reducing" stopping times, that is, $\tau_n^z \nearrow T$, P-a.s., such that

$$Y_t^{z,n} - A_t^{z,n} = \mathcal{E}[Y_T^{z,n} - A_T^{z,n}|\mathcal{F}_t], \quad \forall t \in [0, T], P\text{-a.s.}, \tag{6.2}$$

where $Y_t^{z,n} \stackrel{\triangle}{=} Y^z_{t\wedge\tau_n^z}$, $A_t^{z,n} \stackrel{\triangle}{=} A^z_{t\wedge\tau_n^z}$, $\forall t \in [0, T]$. For any $n \in \mathbb{N}$, we know from Proposition 5.1 that there is a unique pair $(h^{z,n}, Z^{z,n}) \in L^1_{\mathbf{F}}([0, T]) \times \mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that

$$Y_t^{z,n} = Y_T^{z,n} - A_T^{z,n} + A_t^{z,n} + \int_t^T h_s^{z,n}\mathrm{d}s - \int_t^T Z_s^{z,n}\mathrm{d}B_s, \quad \forall t \in [0, T], \tag{6.3}$$

such that the generator h satisfies the following estimate:

$$\begin{aligned} -\ell(|Z_t^{z,n}| + |Z_t^{z,n}|^2) &\le g_1(t, Z_t^{z,n}) \le h_t^{z,n} \le g_2(t, Z_t^{z,n}) \\ &\le \ell(|Z_t^{z,n}| + |Z_t^{z,n}|^2), \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.} \end{aligned} \tag{6.4}$$

Comparing (6.1) and (6.3) we see that

$$\mathrm{d}A_t^{z,n} - h_t^{z,n}\mathrm{d}t \equiv \mathbf{1}_{\{t\le\tau_n^z\}}\ell(|z| + |z|^2)\mathrm{d}t \quad \text{and} \quad Z_t^{z,n} \equiv \mathbf{1}_{\{t\le\tau_n^z\}}z. \tag{6.5}$$

This, together with (6.4), implies that P-a.s.

$$A_T^{z,n} = \int_0^T h_t^{z,n}\mathrm{d}t + \int_0^T \mathbf{1}_{\{t\le\tau_n^z\}}\ell(|z| + |z|^2)\mathrm{d}t \le 2\ell(|z| + |z|^2)T.$$

Letting $n \to \infty$ we obtain that A_T^z is bounded by $2\ell(|z| + |z|^2)T$, proving the claim.

Now, in the light of Proposition 5.1, we can assume that there exists a unique pair $(h^z, Z^z) \in L^1_{\mathbf{F}}([0, T]) \times \mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that (6.3)–(6.5) hold. In other words, defining

$$g(t, \omega, z) \stackrel{\triangle}{=} h_t^z(\omega), \quad (t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d, \tag{6.6}$$

it holds that

$$Y_t^z - A_t^z = Y_T^z - A_T^z + \int_t^T g(s, z)\mathrm{d}s - \int_t^T z\mathrm{d}B_s, \quad t \in [0, T], \tag{6.7}$$

$$-\ell(|z| + |z|^2) \le g_1(t, z) \le g(t, z) \le g_2(t, z) \le \ell(|z| + |z|^2), \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.}, \tag{6.8}$$

$$\mathrm{d}A_t^z = g(t, z)\mathrm{d}t + \ell(|z| + |z|^2)\mathrm{d}t, \quad t \in [0, T]. \tag{6.9}$$

We shall show that g is the desired representation generator of the quadratic $\mathcal{F}$-expectation $\mathcal{E}$.

To begin with, let us define, for any $z, z' \in \mathbb{R}^d$, a function

$$g_\ell^{z,z'}(v) \stackrel{\triangle}{=} \ell(1 + |z| + |z'|)|v|, \quad \forall v \in \mathbb{R}^d, \tag{6.10}$$

and denote the corresponding $g_\ell^{z,z'}$-expectation by $\mathcal{E}_\ell^{z,z'}(\cdot)$. It is worth noting that $\mathcal{E}_\ell^{z,z'}(\cdot)$ is a Lipschitz g-expectation studied in [3,14]. We should note here that if g is a quadratic generator satisfying (H1) and (H2), then it must satisfy a "local Lipschitz property" which can be written as

$$|g(t,z)-g(t,z')| \le \ell(1+|z|+|z'|)|z-z'| = g_\ell^{z,z'}(|z-z'|), \quad \forall z,z' \in \mathbb{R}^d. \tag{6.11}$$

Now let g be a given deterministic quadratic generator satisfying (H1) and (H2). For fixed $z \in \mathbb{R}^d$, consider the process $Y_t^{g,z} \triangleq \mathcal{E}^g\{zB_T|\mathcal{F}_t\}$, $t \ge 0$. Since $zB_T \in \mathscr{L}_T^\infty$, we know that (recall the BSDEs (3.4) and (3.5)) $Y_t^{g,z}$ must have the following explicit expression:

$$Y_t^{g,z} = zB_t + \int_t^T g(s,z)\mathrm{d}s, \quad t \in [0,T]. \tag{6.12}$$

Let us fix $z,z' \in \mathbb{R}^d$, and define $\hat{\mathcal{E}}_t^{z,z'} \triangleq Y_t^{g,z} - Y_t^{g,z'} = (z-z')B_t + \int_t^T (g(s,z)-g(s,z'))\mathrm{d}s$, $t \ge 0$. We have the following lemma.

Lemma 6.1. *Assume that g is a deterministic function satisfying* (H1) *and* (H2). *Then the process $\xi_t \triangleq \hat{\mathcal{E}}_t^{z,z'}$, $t \ge 0$, is an $\mathcal{E}_\ell^{z,z'}$-submartingale.*

Proof. For any $s \le t$, define

$$\begin{aligned}\tilde{Y}_s \triangleq \mathcal{E}_\ell^{z,z'}\{\hat{\mathcal{E}}_t^{z,z'}|\mathcal{F}_s\} &= \left[(z-z')B_t + \int_t^T (g(r,z)-g(r,z'))\mathrm{d}r\right] \\ &\quad + \int_s^t \mu(1+|z|+|z'|)|\tilde{Z}_r|\mathrm{d}r - \int_s^t \tilde{Z}_r \mathrm{d}B_r.\end{aligned} \tag{6.13}$$

Since g is deterministic, the BSDE (6.13) has a unique solution $(\hat{Y},\hat{Z})$, where

$$\hat{Y}_s \triangleq (z-z')B_s + \int_t^T (g(r,z)-g(r,z'))\mathrm{d}r + \int_s^t \mu(1+|z|+|z'|)|z-z'|\mathrm{d}r,$$

and $\hat{Z} \equiv z-z'$. Thus, defining $\delta g(r) \triangleq g(r,z)-g(r,z')$, we have

$$\begin{aligned}\tilde{Y}_s = \hat{Y}_s &= (z-z')B_s + \int_t^T \delta g(r)\mathrm{d}r + \int_s^t \mu(1+|z|+|z'|)|z-z'|\mathrm{d}r \\ &= (z-z')B_s + \int_s^T \delta g(r)\mathrm{d}r + \int_s^t \{\mu(1+|z|+|z'|)|z-z'| - \delta g(r)\}\mathrm{d}r \\ &\ge (z-z')B_s + \int_s^T \delta g(r)\mathrm{d}r.\end{aligned}$$

But by definition of $\hat{\mathcal{E}}^{z,z'}$ we see that the right hand side above is exactly $\hat{\mathcal{E}}_s^{z,z'} = \xi_s$. This, combined with (6.13), shows that $\xi = \hat{\mathcal{E}}^{z,z'}$ is an $\mathcal{E}_\ell^{z,z'}$-submartingale. ■

We now introduce some extra assumptions on the quadratic $\mathcal{F}$-expectation $\mathcal{E}$, which will be useful in the study of the representation theorem. The first one is motivated by Lemma 6.1.

(H4) There exists a constant $\mu > 0$ such that for any fixed z, z', it holds that

$$\mathcal{E}\{zB_T|\mathcal{F}_t\} - \mathcal{E}\{z'B_T|\mathcal{F}_t\} \le \mathcal{E}_\mu^{z,z'}\{(z-z')B_T|\mathcal{F}_t\}. \tag{6.14}$$

The next assumption extends the "translation invariance" of the nonlinear expectation $\mathcal{E}$.

(H5) For any $z \in \mathbb{R}^d$, $\tau \in \mathcal{M}_{0,T}$, $0 \leq t < \tilde{t} \leq T$, and $\xi \in L^\infty(\mathcal{F}_{\tilde{t}\wedge\tau})$, it holds that

$$\mathcal{E}[\xi + zB_{\tilde{t}\wedge\tau} - zB_{t\wedge\tau}|\mathcal{F}_t] = \mathcal{E}[\xi + zB_{\tilde{t}\wedge\tau}|\mathcal{F}_t] - zB_{t\wedge\tau}, \quad P\text{-a.s.} \tag{6.15}$$

We note that the assumption (H5) is not a consequence of Proposition 4.2(iv), since the random variable zB_t is not bounded(!). However, the left hand side of (6.15) is well defined, since $\xi + zB_{\tilde{t}\wedge\tau} - zB_{t\wedge\tau} = \xi + \int_t^{\tilde{t}} z\mathbf{1}_{\{s\leq\tau\}}\mathrm{d}B_s \in \tilde{\mathscr{L}}_T^\infty \subset \mathrm{Dom}(\mathcal{E})$ (see Remark 3.6).

Finally, we give an assumption that essentially states that the process $\{zB_t\}_{t\geq 0}$ has the "independent increments" property under the nonlinear expectation $\mathcal{E}$.

(H6) For any $z \in \mathbb{R}^d$, and any $0 \leq s \leq t \leq T$, it holds that

$$\mathcal{E}[z(B_t - B_s)|\mathcal{F}_s] = \mathcal{E}[z(B_t - B_s)], \quad P\text{-a.s.} \tag{6.16}$$

The following lemma is more or less motivated by the assumption (H6), and it will play an important role in the proof of the representation theorem.

Lemma 6.2. *Assume that $\mathcal{E}$ is a regular quadratic $\mathcal{F}$-expectation satisfying* (H6). *Then the random function g defined in (6.6) is deterministic, and it holds that*

$$g(t,z) = \lim_{h\to 0} \frac{\mathcal{E}\{z(B_{t+h} - B_t)\}}{h}, \quad P\text{-a.s.}, \forall (t,z) \in [0,T] \times \mathbb{R}^d. \tag{6.17}$$

Moreover, if in addition $\mathcal{E}$ satisfies (H4), *then g is local Lipschitz continuous.*

Proof. We first show that g is deterministic. To this end, we fix $z \in \mathbb{R}^d$. For any $0 \leq t < t+h \leq T$, one can deduce from (6.7) that

$$z(B_{t+h} - zB_t) - \int_t^{t+h} g(s,z)\mathrm{d}s = Y_{t+h}^z - A_{t+h}^z - (Y_t^z - A_t^z), \quad P\text{-a.s.}$$

Since $Y_t^z - A_t^z - zB_t \in L^\infty(\mathcal{F}_t)$, using the assumption (H5) one can check that

$$\begin{aligned}\mathcal{E}\left\{z(B_{t+h} - B_t) - \int_t^{t+h} g(s,z)\mathrm{d}s|\mathcal{F}_t\right\} &= \mathcal{E}\{Y_{t+h}^z - A_{t+h}^z|\mathcal{F}_t\} \\ &\quad - (Y_t^z - A_t^z) = 0, \quad P\text{-a.s.}\end{aligned} \tag{6.18}$$

Therefore, applying (6.16) we have

$$\begin{aligned}hg(t,z) &= \mathcal{E}\left\{z(B_{t+h} - B_t) - \int_t^{t+h}(g(s,z) - g(t,z))\mathrm{d}s|\mathcal{F}_t\right\} \\ &= \mathcal{E}[z(B_{t+h} - B_t)|\mathcal{F}_t] + v(t,h) = \mathcal{E}[z(B_{t+h} - B_t)] + v(t,h),\end{aligned}$$

where

$$\begin{aligned}v(t,h) &\triangleq \mathcal{E}\left\{z(B_{t+h} - B_t) - \int_t^{t+h}(g(s,z) - g(t,z))\mathrm{d}s|\mathcal{F}_t\right\} - \mathcal{E}[z(B_{t+h} - B_t)|\mathcal{F}_t] \\ &= \mathcal{E}\left\{zB_{t+h} - \int_t^{t+h}(g(s,z) - g(t,z))\mathrm{d}s|\mathcal{F}_t\right\} - \mathcal{E}[zB_{t+h}|\mathcal{F}_t].\end{aligned}$$

Now, applying L^p-domination (3.8) for the $\mathcal{F}$-expectation $\mathcal{E}$ with $p = p(2\| \int_0^T |g(s,z)|\mathrm{d}s\|_\infty, |z|)$, we obtain that

$$E\left\{\frac{1}{h^p}|v(t,h)|^p\right\} = \|v(t,h)\|_p^p \leq 3^p \left\| \int_t^{t+h} |g(s,z)-g(t,z)|\mathrm{d}s \right\|_p^p \leq 3^p E\left\{\frac{1}{h}\int_t^{t+h} |g(s,z)-g(t,z)|\mathrm{d}s\right\}^p.$$

Since $z \in \mathbb{R}^d$ is fixed, thus by the Lebesgue differentiation theorem, P-almost surely one has

$$\frac{1}{h}\int_t^{t+h} |g(s,z)-g(t,z)|\mathrm{d}s \to 0, \quad \text{for a.e. } t \in [0,T].$$

The dominated convergence theorem then implies that

$$E\left\{\int_0^T \left[\frac{1}{h}|v(t,h)|\right]^p \mathrm{d}t\right\} \leq 3^p E\left\{\int_0^T \left[\frac{1}{h}\int_t^{t+h} |g(s,z)-g(t,z)|\mathrm{d}s\right]^p \mathrm{d}t\right\} \to 0.$$

In other words, we have proved that $v(t,h) = o(h)$ in $\mathcal{H}_{\mathbf{F}}^p([0,T])$. Thus

$$g(t,z) = \lim_{h\to 0} \frac{\mathcal{E}[z(B_{t+h}-B_t)]}{h}, \quad P\text{-a.s.},$$

and it follows that g is deterministic.

Now assume that $\mathcal{E}$ also satisfies (H4); we show that g is local Lipschitz continuous. To see this, taking $t+h = T$ in (6.18) and applying (H5) with $\tilde{t} = \tau = T$ we obtain that $\mathcal{E}\{zB_T - \int_t^T g(s,z)\mathrm{d}s|\mathcal{F}_t\} = zB_t$, P-a.s. Since g is deterministic, this implies that $\int_t^T g(s,z)\mathrm{d}s = \mathcal{E}\{zB_T|\mathcal{F}_t\} - zB_t$. Similarly, one has $\int_t^T g(s,z')\mathrm{d}s = \mathcal{E}\{z'B_T|\mathcal{F}_t\} - z'B_t$. Combining, we have

$$\int_t^T [g(s,z')-g(s,z)]\mathrm{d}s = \mathcal{E}\{zB_T|\mathcal{F}_t\} - \mathcal{E}\{z'B_T|\mathcal{F}_t\} - (z-z')B_t \leq \mathcal{E}_\mu^{z,z'}\{(z-z')B_T|\mathcal{F}_t\} - (z-z')B_t.$$

Note that for $g_\mu^{z,z'}(v) \stackrel{\triangle}{=} \mu(1+|z|+|z'|)|v|$, one has

$$\mathcal{E}_\mu^{z,z'}\{(z-z')B_T|\mathcal{F}_t\} = (z-z')B_t + \int_t^T \mu(1+|z|+|z'|)|z-z'|\mathrm{d}s.$$

We deduce that

$$\int_t^T [g(s,z')-g(s,z)]\mathrm{d}s \leq \int_t^T \mu(1+|z|+|z'|)|z-z'|\mathrm{d}s.$$

Replacing T by an arbitrary $t' \in (0,T]$ in the above, we can then deduce that for any $t' \in (0,T]$, it holds that

$$|g(t',z)-g(t',z')| \leq \mu(1+|z|+|z'|)|z-z'|,$$

proving the local Lipschitz property of g. ■

The main result of this paper is the following representation theorem.

Theorem 6.3. *Assume that $\mathcal{E}$ is a regular quadratic $\mathcal{F}$-expectation that satisfies* (H4)–(H6). *Then, there exists a local Lipschitz continuous function $g(t,z):[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}$ such that for any $z\in\mathbb{R}^d$,*

$$g_1(t,\omega,z)\le g(t,z)\le g_2(t,\omega,z),\quad \mathrm{d}t\times\mathrm{d}P\text{-a.s.},\tag{6.19}$$

and that for any $\xi\in L^\infty(\mathcal{F}_T)$, it holds P-a.s. that

$$\mathcal{E}[\xi|\mathcal{F}_t]=\mathcal{E}^g[\xi|\mathcal{F}_t],\quad \forall t\in[0,T].$$

Proof. Let g be the random field defined in (6.6). We know from Lemma 6.2 that g is deterministic and local Lipschitz continuous. Then (6.19) follows from (6.8) and we see that $g|_{z=0}=0$. For any $\xi\in L^\infty(\mathcal{F}_T)$, we can apply the result of [10, Theorem 2.3] to conclude that BSDE(ξ,g) admits a solution $(\hat{Y},\hat{Z})\in\mathbb{C}^\infty_{\mathbf{F}}([0,T])\times\mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Furthermore, by virtue of (6.11), it follows from [12] (or [8]) that the solution is unique. (We remark that the result of [10] cannot be applied here since g is not necessarily differentiable.) Let $\{\Psi^n\}_{n\in\mathbb{N}}$ be a sequence of simple processes that approximates $\hat{Z}$ in $\mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Then it holds that $\sup_{t\in[0,T]}|\int_0^t(\Psi^n_s-\hat{Z}_s)\mathrm{d}B_s|\to 0$ in $L^2(\mathcal{F}_T)$, thanks to the Burkholder–Davis–Gundy inequality. Applying [10, Lemma 2.5] we can find a subsequence of $\{\Psi^n\}_{n\in\mathbb{N}}$, still denoted by $\{\Psi^n\}_{n\in\mathbb{N}}$, such that

$$\Psi^n_t\to\hat{Z}_t,\quad \mathrm{d}t\times\mathrm{d}P\text{-a.s.}\quad\text{and}\quad \sup_{t\in[0,T]}\left|\int_0^t(\Psi^n_s-\hat{Z}_s)\mathrm{d}B_s\right|\to 0,\quad P\text{-a.s.}\tag{6.20}$$

with $\sup_{n\in\mathbb{N}}|\Psi^n_t|\in\mathcal{H}^2_{\mathbf{F}}([0,T])$ and $\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}|\int_0^t(\Psi^n_s-\hat{Z}_s)\mathrm{d}B_s|\in L^2(\mathcal{F}_T)$. We define stopping times

$$\sigma_k\stackrel{\triangle}{=}\inf\left\{t\in[0,T]:\int_0^t\sup_{n\in\mathbb{N}}|\Psi^n_s|^2\mathrm{d}s+\sup_{n\in\mathbb{N}}\sup_{s\in[0,t]}\left|\int_0^s\Psi^n_r\mathrm{d}B_r\right|>k\right\}\wedge T,\quad\forall k\in\mathbb{N}.\tag{6.21}$$

It is easy to see that $\sigma_k\nearrow T$, P-a.s.

For any $z\in\mathbb{R}^d$, $0\le t<\tilde{t}\le T$ and $\tau\in\mathcal{M}_{0,T}$, it follows from (6.18) and (H6) that

$$\mathcal{E}\left\{\int_t^{\tilde{t}}\mathbf{1}_{\{s\le\tau\}}[-g(s,z)\mathrm{d}s+z\mathrm{d}B_s]|\mathcal{F}_t\right\}=0,\quad P\text{-a.s.}\tag{6.22}$$

Let Ψ be any member of $\{\Psi^n\}_{n\in\mathbb{N}}$. Without loss of generality we assume that Ψ is in the form of

$$\Psi_t(t,\omega)=\sum_{i=0}^m\sum_{j=1}^{n_i}z^i_j\mathbf{1}_{[s_i,s_{i+1})\times E^i_j}(t,w),\quad\forall(t,\omega)\in[0,T]\times\Omega,$$

where $0=s_0<s_1<\cdots<s_m<s_{m+1}=T$, $\{E^i_j\}_{j=1}^{n_i}$ is an $\mathcal{F}_{s_i}$-measurable partition of Ω for $i=0,1\ldots,m$, and each $z^i_j\in\mathbb{R}^d$.

Now fix $k\in\mathbb{N}$; for any $t\in[0,T]$, there exist $\alpha\in\{0,1,\ldots,m\}$ such that $t\in[s_\alpha,s_{\alpha+1})$. By refining the partition if necessary we may assume that $t=s_\alpha$. Since the quadratic $\mathcal{F}$-expectation $\mathcal{E}$ is "translation invariant" and satisfies the "zero–one law", using (6.22) one can show that P-a.s.

$$\mathcal{E}\left\{\int_t^T\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,\Psi_s)\mathrm{d}s+\Psi_s\mathrm{d}B_s]|\mathcal{F}_t\right\}$$

$$
\begin{aligned}
&= \mathcal{E}\left\{\sum_{i=\alpha}^{m}\sum_{j=1}^{n_i}\mathbf{1}_{E^i_j}\int_{s_i}^{s_{i+1}}\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,z^i_j)\mathrm{d}s+z^i_j\mathrm{d}B_s]|\mathcal{F}_t\right\}\\
&= \mathcal{E}\left\{\sum_{i=\alpha}^{m-1}\sum_{j=1}^{n_i}\mathbf{1}_{E^i_j}\int_{s_i}^{s_{i+1}}\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,z^i_j)\mathrm{d}s+z^i_j\mathrm{d}B_s]\right.\\
&\quad\left.+\sum_{j=1}^{n_m}\mathbf{1}_{E^m_j}\mathcal{E}\left[\int_{s_m}^{T}\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,z^m_j)\mathrm{d}s+z^m_j\mathrm{d}B_s]|\mathcal{F}_{s_m}\right]\middle|\mathcal{F}_t\right\}\\
&= \mathcal{E}\left\{\sum_{i=\alpha}^{m-1}\sum_{j=1}^{n_i}\mathbf{1}_{E^i_j}\int_{s_i}^{s_{i+1}}\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,z^i_j)\mathrm{d}s+z^i_j\mathrm{d}B_s]|\mathcal{F}_t\right\}\\
&\ldots\\
&= \mathcal{E}\left\{\sum_{j=1}^{n_\alpha}\mathbf{1}_{E^\alpha_j}\int_{t}^{s_{\alpha+1}}\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,z^\alpha_j)\mathrm{d}s+z^\alpha_j\mathrm{d}B_s]|\mathcal{F}_t\right\}\\
&= \sum_{j=1}^{n_\alpha}\mathbf{1}_{E^\alpha_j}\mathcal{E}\left\{\int_{t}^{s_{\alpha+1}}\mathbf{1}_{\{s\le\sigma_k\}}[-g(s,z^\alpha_j)\mathrm{d}s+z^\alpha_j\mathrm{d}B_s]|\mathcal{F}_t\right\}=0. \qquad (6.23)
\end{aligned}
$$

For any $k\in\mathbb{N}$, since g is continuous and has quadratic growth in z, using (6.20) and applying the dominated convergence theorem we deduce that $\int_t^T \mathbf{1}_{\{s\le\sigma_k\}}[-g(s,\Psi^n_s)\mathrm{d}s+\Psi^n_s\mathrm{d}B_s]$ converges to $\int_t^T \mathbf{1}_{\{s\le\sigma_k\}}[-g(s,\hat{Z}_s)\mathrm{d}s+\hat{Z}_s\mathrm{d}B_s]$ almost surely. We also see from the definition of σ_k (6.21) that

$$
\left|\int_t^T \mathbf{1}_{\{s\le\sigma_k\}}[-g(s,\Psi^n_s)\mathrm{d}s+\Psi^n_s\mathrm{d}B_s]\right|\le \ell T+2(1+\ell)k,\quad P\text{-a.s., }\forall n\in\mathbb{N}.
$$

Let $K=\ell T+2(1+\ell)k$ and $p\stackrel{\triangle}{=}p(K,0)$; applying L^p-domination of $\mathcal{E}$ and using (6.23) for each Ψ^n one can then deduce that $\mathcal{E}\{\int_t^T \mathbf{1}_{\{s\le\sigma_k\}}[-g(s,\hat{Z}_s)\mathrm{d}s+\hat{Z}_s\mathrm{d}B_s]|\mathcal{F}_t\}=0$, P-a.s. The "translation invariance" of $\mathcal{E}$ then implies that

$$
\mathcal{E}[\hat{Y}_{\sigma_k}|\mathcal{F}_t]=\mathcal{E}\left\{\hat{Y}_{t\wedge\sigma_k}+\int_t^T \mathbf{1}_{\{s\le\sigma_k\}}[-g(s,\hat{Z}_s)\mathrm{d}s+\hat{Z}_s\mathrm{d}B_s]|\mathcal{F}_t\right\}=\hat{Y}_{t\wedge\sigma_k},\quad P\text{-a.s.}
$$

Letting $p\stackrel{\triangle}{=}p(\|\hat{Y}\|_\infty,0)$ and applying Theorem 4.3 as well as L^p-domination for $\mathcal{E}$ again, we obtain that

$$
\|\hat{Y}_{t\wedge\sigma_k}-\mathcal{E}[\xi|\mathcal{F}_t]\|_p=\|\mathcal{E}[\hat{Y}_{\sigma_k}|\mathcal{F}_t]-\mathcal{E}[\xi|\mathcal{F}_t]\|_p\le 3\|\hat{Y}_{\sigma_k}-\xi\|_p.
$$

Since $\sigma_k\nearrow T$, P-a.s. and $\hat{Y}$ is continuous, $\hat{Y}_{\sigma_k}$ converges P-a.s. to ξ and $\hat{Y}_{t\wedge\sigma_k}$ converges P-a.s. to $\hat{Y}_t$. These two convergences are even in the L^p sense, thanks to the Lebesgue dominated convergence theorem. Thus $\mathcal{E}[\xi|\mathcal{F}_t]=\hat{Y}_t=\mathcal{E}^g[\xi|\mathcal{F}_t]$, P-a.s. The conclusion then follows from Proposition 4.1 and the continuity of $\hat{Y}$. ■

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