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ON GEOMETRY OF CONVEX IDEAL POLYHEDRA IN HYPERBOLIC 3-SPACE

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1. INTRODUCTION

LET P be a convex polyhedron in hyperbolic 3-space H^3, such that all the vertices of P are lying on the sphere at infinity S^2_∞. We will call such polyhedra **ideal**. The following questions arise naturally:

1. What are the possible values of dihedral angles of convex ideal polyhedra?
2. Is an ideal polyhedron uniquely determined by its dihedral angles?
3. What are the possible combinatorial structures of ideal polyhedra?

In his 1971 paper [2], E. M. Andreev answered the first question, but with the additional condition that the dihedral angles be non-obtuse (that is, the dihedral angles are at most $\pi/2$). This condition is very restrictive and has been removed in the present work.

DEFINITION. *The* **dual polyhedron** P^* *of a convex ideal polyhedron* P *is the Poincaré dual of* P. *Furthermore, each edge* e^* *of* P^* *is assigned the weight* $w(e^*)$ *equal to the exterior dihedral angle at the corresponding edge* e *of* P.

Notation. We will denote the set of edges of a polyhedron Q by $E(Q)$, the vertices of Q by $V(Q)$ and the faces of Q by $F(Q)$.

THEOREM 1. *The dual polyhedron* P^* *of a convex ideal polyhedron* P *satisfies the following conditions*:

CONDITION 1. $0 < w(e^*) < \pi$ *for all* $e^* \in E(P^*)$.

CONDITION 2. *If the edges* $e_1^*, e_2^*, \ldots, e_k^*$ *form the boundary of a face of* P^*, *then* $w(e_1^*) + w(e_2^*) + \ldots + w(e_k^*) = 2\pi$.

CONDITION 3. *If* $e_1^*, e_2^*, \ldots, e_k^*$ *form a simple circuit which does not bound a face of* P^*, *then* $w(e_1^*) + w(e_2^*) + \ldots w(e_k^*) > 2\pi$.

Note. Recently, the author has succeeded in showing that the conditions of Theorem 1. are *sufficient*, that is any polyhedron P^* with weighted edges satisfying the conditions of the Main theorem is the Poincaré dual of a convex ideal polyhedron P with the exterior dihedral angles equal to the weights.

The following also holds:

THEOREM 2. *Two combinatorially equivalent ideal polyhedra P_1 and P_2 with pair-wise equal dihedral angles and corresponding faces of P_1 and P_2 are congruent.*

Note: The author recently showed that it is unnecessary to assume that the corresponding faces are congruent.

2. PROOF OF THEOREM 1

Proof of Condition 1. This follows trivially from convexity of P—the dihedral angles of a convex polyhedron are contained in the open interval $(0, \pi)$, and so the exterior angles are also. □

Proof of Condition 2. Condition 2 is easily seen to be equivalent to the statement that the sum of the exterior dihedral angles incident to any vertex v of P is equal to 2π. This is well known, and follows from considering a small horosphere H_v tangent to the sphere at infinity S^2_∞ at v. The intrinsic metric of H_v is that of the Euclidean plane E^2. The intersection of P with H_v is a convex Euclidean polygon p_v, and the dihedral angles at edges incident to v are equal to the plane angles of p_v, and likewise the exterior angles. The sum of exterior angles of a Euclidean polygon is well known to be 2π. □

To prove Condition 3, we will need a few subsidiary results.

DEFINITION. *Let γ be a closed polygonal curve in H^3, with vertices $p_1, p_2, \ldots, p_k$, $p_k = p_1$. Then define the* **turning** *$\tau_i(\gamma)$ of γ at the vertex p_i be the exterior angle of the triangle $S_i = p_{i-1}p_ip_{i+1}$ at p_i, and define the* **total turning** *$\tau(\gamma)$ of γ to be $\sum_{i=1}^k \tau_i(\gamma)$.*

THEOREM 3. (Hyperbolic Fenchel's Theorem) *The total turning of a closed polygonal curve γ is greater than 2π, unless all of the vertices are collinear.*

Proof. (All notation is as in the definition above.) Consider the collection of triangles $T_i = p_1p_ip_{i+1}$, where $1 < i < k - 1$, which "span" γ—that is, their union is an immersed disc, whose boundary is γ. Now, for each vertex p_j consider all of the T_i that are incident to p_j—there are two such T_i, except at p_1. Let the (interior) angle of T_i at p_j be α_i^j. Now consider the intersection of a small sphere around p_j with all of the incident T_i and also S_j. By the spherical triangle inequality, it follows that

$$\sum_i \alpha_i^j \geq \pi - \tau_j(\gamma),$$

and summing the above inequalities over all i, we obtain

$$\sum_{i=2}^{k-2} \text{angle sum of } T_i > (k-1)\pi - \tau(\gamma).$$

Since the T_i are hyperbolic triangles, their angle sums are all not greater than π, and some are smaller (unless all of the vertices of γ are collinear). Therefore,

$$(k-3)\pi > (k-1)\pi - \tau(\gamma),$$

and hence $\tau(\gamma) > 2\pi$. □

LEMMA 4. *Let ABC be a spherical triangle such that $|AB| + |BC| = \pi$. Then $|AC| \leq \angle B$, with equality if and only if $|AB| = |BC| = \pi/2$.*

Proof. The spherical law of cosines states that

$$\cos|AC| = \sin|AB|\sin|BC|\cos \angle B - \cos|AB|\cos|BC|.$$

Setting $|AB| = \pi/2 - x$, $|BC| = \pi/2 + x$, the law of cosines can be rewritten as

$$\cos|AC| = \cos^2 x \cos \angle B + \sin^2 x.$$

Since by the double angle formula for cos, $\cos\alpha = 1 - 2\sin^2\alpha$, we can rewrite the above equation as $1 - 2\sin^2(|AC|/2) = \cos^2 x(1 - 2\sin^2(\angle B/2)) + \sin^2 x$, which simplifies to:

$$\frac{\sin^2 \frac{|AC|}{2}}{\sin^2 \frac{\angle B}{2}} = \cos^2 x.$$

Since sin is monotonic on $[0, \pi/2]$, and $\cos^2 x < 1$, unless $x = 0$, the result follows. □

COROLLARY 5. *Let H_1 and H_2 be two geodesic half-planes meeting at a dihedral angle α, and let $p_1 \in H_1$ and $p_2 \in H_2$ be two points, joined by a geodesic γ (geodesic in the intrinsic metric of $H_1 \cup H_2$), so that $\gamma \cap (H_1 \cap H_2) = p$. Then the turning angle $\tau_p(\gamma)$ is smaller than the exterior dihedral angle at $H_1 \cap H_2$.*

Proof. By combination of Snell's law with Lemma 4. That is, in the intrinsic metric of $H_1 \cup H_2$ γ and $\rho = H_1 \cap H_2$ are two (hyperbolic) straight lines. Now consider α_1, the angle between the segment $p_1 p$ and ρ and α_2, the angle between the segment $p_2 p$ and ρ. It is easy to see that $\alpha_1 + \alpha_2 = \pi$. Now, back in H^3, consider a small sphere S centered at p, the segments $p_1 p$, $p_2 p$ and the "left" part of ρ intersect S in a triangle $t_1 t_2 t_3$, such that $|t_1 t_3| = \alpha_1$, $|t_2 t_3| = \alpha_2$, $\pi - |t_1 t_2|$ is the turning of γ at p and the angle at t_3 is the (interior) dihedral angle at $H_1 \cap H_2$. □

Proof of Condition 3. A circuit in P^* corresponds to a chain of faces $f_1, f_2, \ldots, f_k$ of P, $(f_{k+1} = f_1)$ such that $f_i \cap f_{i+1} = e_i$. The edges e_i do not all share a common ideal vertex. $F = \cup_{i=1}^k f_i$ is a hyperbolic surface, with a number of boundary components and cusps, which can be completed (by extending geodesically past the boundary components) to a complete hyperbolic surface $\tilde{F}$, immersed in H^3. $\tilde{F}$ is topologically equivalent to an infinite cylinder, and both its ends are of infinite volume. (The first part is clear, since each completed face $\tilde{f}_i$ is a topologically an infinite strip. Geometrically, the "left" (orienting $\tilde{F}$ in a consistent fashion) end of $\tilde{f}_i$ and has finite volume if and only if $\tilde{f}_i$ is incident to an ideal vertex v_i of P on the left. For the left end $\tilde{F}$ to have finite volume, *all* of $\tilde{f}_i$ must be incident to such a v_i on the left, and since $v_i = e_i \cap e_{i+1}$, $v_j = v_k$ for all j, k. Hence $v_i = v$ for all i, and hence all of the f_i share the ideal vertex v contrary to the assumption.)

This means that there exists a unique closed geodesic γ homotopic to the meridian of $\tilde{F}$. γ is embedded in H^3 as a polygonal curve, with turning at intersections with e_i. The total turning of γ is greater than 2π, by Theorem 3, and by Corollary 5, the sum of the dihedral angles at e_i is no smaller than the total turning of γ. □

3. PROOF OF THEOREM 2

To show Theorem 2 it is necessary to understand the intrinsic geometry of an ideal polyhedron P. Let us assume that all of the faces of P are triangles (if that is not the case, they can always be triangulated). All ideal hyperbolic triangles are congruent, and so the intrinsic metric of P is completely determined by the gluing of adjacent pairs of triangles—there is a one-parameter family of gluings, corresponding to sliding the "left" triangle t_l with respect to the "right" triangle t_r. If v is an ideal vertex common to t_l and t_r, such that $t_l \cap t_r = e$ consider the intersections of a horosphere h_v with t_l and with t_r. These will be (Euclidean) segments s_l and s_r and the sliding parameter will be equal to $S(e) = \log(s_l/s_r)$. $S(e)$ is independent of the particular horosphere h_v, and also of the choice of the vortex of e.

DEFINITION. *The* **link** $l(v)$ *of a vertex* v *of an ideal polyhedron* P *is the intersection of a small horosphere centered at* v *with* P *(only determined up to homothety).* $l(v)$ *is a convex Euclidean polygon.*

The actual argument is modelled on that used by A. Cauchy in the proof of his celebrated rigidity theorem for convex polyhedra in E^3.

The following lemma of A. D. Aleksandrov (see [1] and [3] for proof and other implications) will prove necessary:

LEMMA 6. *Let* C_1 *and* C_2 *be two convex polygons in* E^2, *such that neither can be placed inside the other by a parallel translation. Let* $s_1^1, s_2^1, \ldots, s_k^1 = s_1^1$ *and* $s_1^2, s_2^2, \ldots, s_k^2 = s_1^2$ *be the corresponding sequences of (lengths of) parallel sides of* C_1 *and* C_2 *(if there is no actual side* s_i^2 *parallel to a side* s_i^1, *then* s_i^2 *is considered to exist but be of length* 0), *then the sequence* $\operatorname{sgn}(s_1^1 - s_1^2), \operatorname{sgn}(s_2^1 - s_2^2), \ldots, \operatorname{sgn}(s_k^1 - s_k^2)$ *has at least four sign changes.*

This statement is actually a little more general than necessary, since this lemma will be used for pairs of polygons whose sides are pairwise parallel.

The following corollary is easily seen to hold:

COROLLARY 7. *Let* C_i, $i = 1, 2$ *be as in Lemma 6. To a vertex* v_j *of* C_i *assign the quantity* $l_j^i = \log(s_j^i)/\log(s_{j+1}^i)$. *Then under the assumptions of Lemma 6, the sequence* $\operatorname{sgn}(l_1^1 - l_1^2), \operatorname{sgn}(l_2^1 - l_2^2), \ldots, \operatorname{sgn}(l_k^1 - l_k^2)$ *has at least four sign changes.*

Proof. Each sign change in the sequence of Lemma 6 gives rise to one in the sequence of the Corollary. □

The following lemma of Cauchy will also be necessary; its proof can be found in references [1] and [3].

THEOREM 8. (Cauchy's Lemma) *Let* G *be a graph on the surface of a sphere, such that each edge* e *of* G *is assigned a weight* $w(e)$ *of* $+$, $-$ *or* 0 *in such a way that if* $e_1(v), e_2(v), \ldots, e_k(v) = e_1(v)$ *are edges incident to a vertex* v *of* G *(in clockwise order), then the sequence* $w(e_1(v)), w(e_2(v)), \ldots, w(e_k(v))$ *has either no sign changes or at least four, for any* v. *Then all the edges actually have the same sign (or* 0) *assigned to them.*

Now let P_1 and P_2 be two ideal polyhedra with triangular faces with the same combinatorial structure and assignment of dihedral angles.

It is clear that if P_1 and P_2 also have the same sliding parameters, than they are congruent (since there is exactly one way to glue faces together along an edge, if both the sliding parameter and the dihedral angle are prescribed).

Now consider the links of corresponding vertices of P_1 and P_2. These will be Euclidean polygons, and for the corresponding vertices v of P_1 and v of P_2 the links $l(v)$ and $l(v')$ can be chosen to have the same area. By Corollary 7, $l(v)$ and $l(v')$ are either congruent, or there are four sign changes in the sign sequence for v. Therefore, by Cauchy's Lemma, all labels are actually 0, and so P_1 is congruent to P_2. □

Note. Actually the above argument goes through *in toto* to show that two ideal polyhedra with the same dihedral angles and pairwise isometric faces are congruent.

4. INSCRIPTION OF POLYHEDRA AND SOME COUNTEREXAMPLES

The Klein (projective) model of H^3 represents H^3 as the interior of the unit ball $B^3 \in E^3$. Hyperbolic lines are represented by Euclidean lines, and in general hyperbolic k-flats are represented by intersections of Euclidean *k—flats* with B^3. Consequently ideal polyhedra are represented by Euclidean polyhedra inscribed in the sphere, and Theorem 1 can be used to produce examples of polyhedra, not combinatorially equivalent to a polyhedron inscribed in the sphere (in fact the conditions of Theorem 1 can be shown to be "efficient", that is, it can be decided in polynomial time whether a polyhedral graph admits a weighting that satisfies them, as was observed by Warren D. Smith. Together with the fact that the conditions of Theorem 1 are sufficient, as recently proved by the author, this answers a question asked by Jakob Steiner in 1832).

We will describe some simple examples below.

Consider a convex polygon p, with vertices $v_1, v_2, \ldots, v_k = v_1$. Now add a vertex v_0 in the interior of P, and triangulate from v_0 (that is, replace P by the union of triangles $v_0 v_i v_{i+1}$ for $i = 1, 2, \ldots, k-1$. This process is called **stellation**. A stellation of a polyhedron P consists of stellating each of the faces of P.

THEOREM 9. *The stellation of a polyhedron P such that the number of vertices $V(P)$ is no greater than the number of faces $F(P)$ cannot be inscribed in the sphere.*

Proof. This follows from conditions 1 and 2 of Theorem 1. Let $s(P)$ be the stellation of P. The vertices of $s(P)$ fall into two classes:

V_P The original vertices of P.

V_S The new vertices added in the process of stellation.

Each edge incident to a vertex in V_S is also incident to a vertex in V_P. Suppose now that $s(P)$ can be inscribed. That means that to each edge we can associate a dihedral angle $w(e)$ of the corresponding ideal polyhedron. Furthermore, to each vertex v of $s(P)$ we associate the total weight $w(v)$ of the edges incident to it. It is easy to see that $\sum_{v \in V_S} w(v) < \sum v \in V_P w(v)$ (since every edge counted in the first sum is counted in the second, but not *vice versa*, and $w(e) > 0$ always. On the other hand, $\sum_{v \in V_S} w(v) = 2\pi F(P)$, whereas $\sum_{v \in V_P} w(v) = 2\pi V(P)$, and since $F(P) \geq V(P)$ by the assumption of the theorem, we have arrived at a contradiction. □

In fact, it is an easy consequence of Euler's formula that the hypothesis of the theorem holds for any polyhedron with triangular faces, so there is the following corollary:

COROLLARY 10. *The stellation of any simplicial polyhedron can not be inscribed.*

There are also examples of polyhedra failing condition 3, found in a computer search by M. Dillencourt, the author and W. Smith, that apparently haven't been known before. The family of examples of Theorem 9 was known to E. Steinitz.

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