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The topological completion of a bilinear form

Solomon Jekel, Neal Macmillan

Mathematics Department, Northeastern University, Boston, MA 02115, USA

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Abstract

Let $M = M_{n,m}$ be the Euclidean space $\mathbb{R}^p$ equipped with a symmetric bilinear form B_M of rank $p = n + m$ and signature $n - m$. We compactify M so that M_c is homogeneous and has as its group of isometries a Lie group whose dimension is the dimension of M plus $2p + 1$. We observe that M_c is in two ways the total space of a non-trivial sphere bundle with base space real projective space. The compactification is well understood in the classical case when M is Minkowski space. The contribution here is to observe that the construction works generally and that it admits a natural bundle description. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $M = M_{n,m}$ be the Euclidean space $\mathbb{R}^p$ equipped with a symmetric bilinear form B_M of rank $p = n + m$ and signature $n - m$. The quadratic form associated to B_M has the form

$$B_M(x_1, \ldots, x_p) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - \cdots - x_{n+m}^2.$$

As suggested by the classical situation, $M = M_{1,3} =$ Minkowski space, we refer to the first n variables as time coordinates and the last m as space coordinates.

We will complete M naturally into a compact space, M_c, which is homogeneous and has as its group of isometries a Lie group whose dimension is the dimension of M plus $2p + 1$. We will then observe that M_c is in two ways the total space of a non-trivial sphere bundle with base space real projective space. There is a bundle which we call the *space over time bundle* (its base has time coordinates and its fiber has space coordinates), whose base space

E-mail address: jekel@neu.edu (S. Jekel).

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is the real projective space $\mathbb{RP}^n$, whose fiber is the m-sphere S^m, and whose structure group $\mathbb{Z}_2$ acts on the fiber by the antipodal map. We write

$$E_{s/t} = \mathbb{RP}^n \times_{\mathbb{Z}_2} S^m.$$

Also there is a *time over space bundle* with base space the real projective space $\mathbb{RP}^m$, fiber the n-sphere S^n, and structure group $\mathbb{Z}_2$ acting on the fiber by the antipodal map,

$$E_{t/s} = \mathbb{RP}^m \times_{\mathbb{Z}_2} S^n.$$

The compactification is well understood in the classical dimension. Our contribution is to observe that the construction works generally and that it admits a natural bundle description.

The idea of compactifying Minkowski space in order to obtain additional symmetries of the metric appeared originally in the in the work of Dirac [4], and Coxeter [3]. Since then the compactification has been extensively studied, most notably by Penrose [7], as a setting for Twistors. It has been observed by Penrose and others that for Minkowski space M_c is topologically $S^1 \times S^3$. In our description it is, as a space/time bundle, an S^3 bundle over the circle $\mathbb{RP}^1$, with the group $\mathbb{Z}_2$ acting by the antipodal map, which in this dimension is an orientation preserving diffeomorphism isotopic to the identity.

By way of contrast consider the case $M = M_{1,2}$, [7]. Here the construction yields a non-orientable manifold which from our description we see is an S^2 bundle with structure group $\mathbb{Z}_2$. In this dimension the group acts by an orientation reversing diffeomorphism.

Indeed when $n = 1$ there are only two equivalence classes of S^m bundles with structure group $\mathbb{Z}_2$ corresponding to the two homomorphisms of the fundamental group $\mathbb{Z}$ of the base S^1 to $\mathbb{Z}_2$. In all cases our bundle is non-trivial, but when m is odd the bundle we describe has total space which is topologically a product, but when m is even it is not.

The bundle description of M_c also makes sense when n or m is equal to 0. When they are both equal to 0 the space/time bundle construction yields S^0 as the completion of $\mathbb{R}^0$. More generally when $n = 0$ and m is arbitrary, the completion M_c as a space/time bundle is S^m. Likewise when $m = 0$ we find that M_c is the non-trivial 2-fold cover of $\mathbb{RP}^n$ which is S^n.

We shall see that all our bundles are associated to the ones described in the above paragraph.

Our goal in this article is to present the compactification in its most elementary form. Toward this end we work in a general p-dimensional Euclidean space, we make assumptions based purely on our linear algebra setting, and we draw conclusions which are purely topological in nature.

2. The completion of $M_{n,m}$

We essentially follow the exposition in [5] except that we work in a general Euclidean space.

Given the form B_M, we can define the metric (actually the pseudometric),

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2 - dx_{n+2}^2 - \cdots - dx_{n+m}^2.$$

The set of points for which $ds^2 = 0$ is the *null-cone* N of M.

In order to complete M we embed it into $M_{n+1,m+1}$. The space we are looking for, M_c, is the projective space $\mathbb{P}N$ of lines in the light cone $N_{n+1,m+1}$ of $M_{n+1,m+1}$.

Note that M_c is a compact space and its dimension is $n+m$. It inherits a metric from $M_{n+1,m+1}$ as follows. We measure the distance between two given lines by choosing a hyperplane which cuts the lines at two points and take the corresponding distance between the points in the hyperplane. Any two cutting planes have conformally related metrics so the distance is well defined.

So we consider $M_{n+1,m+1}$ with coordinates x_i, v, and w and the metric becomes

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 + dv^2 - dw^2 - dx_{n+1}^2 - dx_{n+2}^2 - \cdots - dx_{n+m}^2.$$

The null cone of $M_{n+1,m+1}$ is given by

$$x_1^2 + x_2^2 + \cdots + x_n{}^2 + v^2 - w^2 - x_{n+1}^2 - x_{n+2}^2 - \cdots - x_{n+m}^2 = 0.$$

Consider, $\overline{M}$, the intersection of the null cone with the plane given by $v - w = 1$. We claim M and $\overline{M}$ are isometric

$$v - w = 1 \;\Rightarrow\; dv = dw \;\Rightarrow\; d^2v - d^2w = 0.$$

Substituting into the equation for the light cone gives

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2 - dx_{n+2}^2 - \cdots - dx_{n+m}^2$$

which is the metric on M.

Now let us find the points of M_c which are not in M. These are elements of $\mathbb{P}N$, that is lines, with $w - v = 0$. Consider the intersection of the hyperplane $v - w = 0$ with the null cone in $M_{n+1,m+1}$. Solving simultaneously,

$$\begin{aligned}&v - w = 0,\\ &x_1^2 + x_2^2 + \cdots + x_n^2 + v^2 - w^2 - x_{n+1}^2 - x_{n+2}^2 - \cdots - x_{n+m}^2 = 0\end{aligned}$$

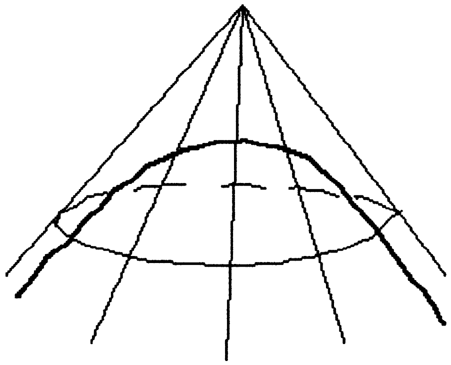
gives

$$x_1^2 + x_2^2 + \cdots + x_n{}^2 - x_{n+1}^2 - x_{n+2}^2 - \cdots - x_{n+m}^2 = 0$$

which is the equation of a light cone at the origin in $M_{n,m}$. Thus M is completed by adjoining a light cone "at infinity".

The case which can be most easily visualized is $\mathbb{R}^3$ when $n = 1$ and $m = 0$. Here we get $M_c = S^1$ as the completion of $M_{1,0} = \mathbb{R}^1$. M_c is the space of lines of the light cone $x^2 + v^2 - w^2 = 0$. By intersecting the light cone with the plane $v - w = 1$, and taking (x, w) as coordinates on this plane we see that $M_{1,0}$ embeds in the light cone as the parabola

$x^2 + 2w + 1 = 0$.



3. The topology of the completion of $M_{n,m}$

Consider the intersection of the light cone in $M_{n+1,m+1}$ with the $p+1$-sphere

$$x_1^2 + x_2^2 + \cdots + x_n^2 + v^2 + w^2 + x_{n+1}^2 + x_{n+2}^2 + \cdots + x_{n+m}^2 = 2.$$

Solving simultaneously yields,

$$x_1^2 + x_2^2 + \cdots + x_n^2 + v^2 = 1,$$

$$w^2 + x_{n+1}^2 + x_{n+2}^2 + \cdots + x_{n+m}^2 = 1.$$

This implies that the intersection of the null cone and the sphere is topologically $S^n \times S^m$.

Now, each generator of the light cone meets the $(p+1)$-sphere in two antipodal points. Thus,

$$M_c \text{ is topologicaly equivalent to } (S^n \times S^m)/\mathbb{Z}_2,$$

with the $\mathbb{Z}_2$ action induced by the product of antipodal maps $(x, y) \to (-x, -y)$.

In the above example the 3-sphere intersects the light cone in two circles which are then identified by the antipodal action.

4. The conformal group of M_c

The conformal group of $M_{n,m}$ is by definition the Lie group of transformations of $\mathbb{R}^p$ preserving the bilinear form. This is the semi-orthogonal group $O(n,m)$ which is a subgroup of dimension $p(p-1)/2$ of the orthogonal group, $p = n + m$.

The semi-orthogonal group $O(n+1, m+1)$ modulo $\pm I$ acts on the space of lines through the origin in $\mathbb{R}^{p+2}$. Furthermore M_c embeds isometrically in this space. Therefore $O(n+1, m+1)$ acts as a conformal group of transformations on the compactification M_c.

For example, when $n = 1$ and $m = 0$ the conformal group $O(2,1)/\pm I$ of M_c is the group $PSL(2,\mathbb{R})$ acting on S^1. It extends the conformal group of $\mathbb{R}$ by adding translations, dilations and inversions, each of which is 1-dimensional.

In Minkowski space, $n = 1$ and $m = 3$, the conformal group $O(2,4)/\pm I$ of M_c is a 15-dimensional Lie group which extends the 6-dimensional conformal group of Minkowski space by adding translations (4 dimensions), inversions (4 dimensions), and dilations (1 dimension). This group is isomorphic to $SU(2,2)$, see [8]. The result of adding just translations is the 10-dimensional Poincaré group, which is the semi-direct product of $O(1,3)/\pm I$, with the group of translations of $\mathbb{R}^4$.

In conclusion we note that M_c is a semi-Riemannian manifold whose action by its group of isometries is transitive. It is therefore a compact, homogeneous space [6]. In particular it is complete.

5. The bundle description of the compactification

In this section we observe that

$$M_c = \mathbb{RP}^n \times_{\mathbb{Z}_2} S^m,$$

the total space of a principle fiber bundle with base space real projective space $\mathbb{RP}^n$, fiber the m-sphere S^m, and structure group $\mathbb{Z}_2$ acting on the fiber by the antipodal map.

We first consider the case $m = 0$. The construction of the compactification yields

$$M_c = S^n \times S^0/\mathbb{Z}_2.$$

We denote this space $E_{n,0}$ and write points as equivalence classes $[(x,\pm 1)]$, where $[(x,1)] = [(-x,-1)]$.

There is a map

$$p : E_{n,0} \to \mathbb{RP}^n$$

given by

$$p : \big[(x,\pm 1)\big] = [x].$$

The group $\mathbb{Z}_2$ acts on $E_{n.0}$ by

$$-1 \cdot \big[(x,\pm 1)\big] = \big[(x,\mp 1)\big],$$

and hence restricts to an action on each fiber of p.

Now S^n is homeomorphic to $E_{n,0}$ by $h : x \to [(x,1)]$. Consider the principle $\mathbb{Z}_2$ bundle which is the 2-fold cover of $\mathbb{RP}^n$ by S^n. The homeomorphism h is compatible with the $\mathbb{Z}_2$ actions and hence induces a bundle structure of $E_{n,0}$ making it equivalent to the bundle

$$S^n \times_{\mathbb{Z}_2} S^0.$$

Now for the general case. The space $\mathbb{RP}^n \times_{\mathbb{Z}_2} S^m$ is by definition the bundle with fiber S^m associated to the principle bundle $S^n \times_{\mathbb{Z}_2} S^0$. We let $E_{n,m}$ be the S^m bundle associated to $E_{n,0}$. The homeomorphism

$$\mathbb{RP}^n \times_{\mathbb{Z}_2} S^0 \to E_{n,0}$$

induces a homeomorphism

$$\mathbb{RP}^n \times_{\mathbb{Z}_2} S^m \to E_{n,m}$$

which identifies the image $E_{n,m}$ with $S^n \times S^m/\mathbb{Z}_2$, as required.

The bundle just constructed is the bundle referred to as $E_{s/t}$ described in 1. Similar considerations show that $E_{n,m}$ is isomorphic to the bundle $E_{t/s}$.

Let us consider the space over time bundle in the case when $n = 1$,

$$E_{s/t} = S^1 \times_{\mathbb{Z}_2} S^m.$$

Here we have identified $\mathbb{RP}^1$ with the circle S^1.

It follows directly from the definition that when m is odd the bundle is orientable and when m is even it is not, for the bundle can be described by one transition function on the circle which is the antipodal map on the fiber. In the odd case the antipodal map is an orientation preserving diffeomorphism of S^m and in the even case it is an orientation reversing diffeomorphism.

Furthermore, when m is odd, again it follows from the fact that the bundle can be described by one transition, that $E_{s/t}$ is homeomorphic to the mapping torus $\mathcal{M}_a(S^m)$ of the antipodal map a of S^m. In the case when m is odd the antipodal map is isotopic to the identity so that $\mathcal{M}_a(S^m)$ is simply homeomorphic to $S^1 \times S^m$. Observe that even though the total space of the space over time bundle is homeomorphic to a product, the bundle is non-trivial; it is associated to the non-trivial $\mathbb{Z}_2$ bundle over the circle.

An analogous bundle construction in the context of Clifford modules appears in the work of Bott et al. [1,2].

6. The structure of space and time

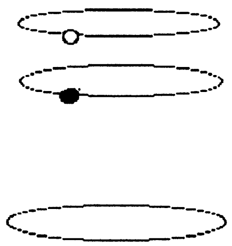
The description of $\mathbb{R}^4$ as Minkowski space provides us with what is now a familiar model of space-time. The associated compactification is a homogeneous space which incorporates and extends all the symmetries of electromagnetism and gravity. It has been thoroughly analyzed, most notably in the work of Penrose.

In this section we interpret the topological nature of the space/time bundle independently of any coordinate representation of the forces involved. What we observe, at this elementary level, is a dynamic interaction of space and time.

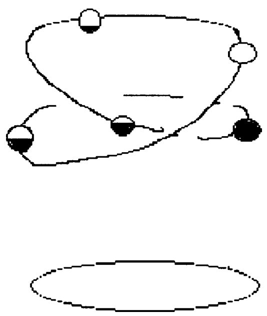
Consider a $\mathbb{Z}_2$-fiber bundle over S^1 with fiber X. Let us think of the two elements of $\mathbb{Z}_2$, 0 and 1, as standing for two distinct states of points X. The base S^1 will parametrize time, and a point in the fiber at a given time will either be in state 0 or state 1.

For the sake of concreteness let us consider X to be a container, and a point in the fiber in state 0 is occupied by the contents of X, and a point in state 1 is not, but the description that follows would apply equally well to other potentials.

There are two possible configurations of the total space of the space/time bundle. There is a trivial $\mathbb{Z}_2$-bundle which describes static states of the container; it is always either empty or full, as determined by the two components of the total space.



A non-trivial $\mathbb{Z}_2$-bundle, on the other hand, describes a dynamic process; a continuous and periodic evolution from a full to an empty state.



To illustrate this process we show above a lift of the circle to the total space of the space/time bundle.

This lifting describes a sub-bundle of the space/time bundle and forms a non-trivial double cover of the base. It passes through each fiber twice before closing up. The intersection with each fiber occurs at the same time as measured in the base space. To distinguish the intersections we have indicated two spheres in each fiber (even though there is really only one). This allows us to depict a given fiber in two complementary states.

By what mechanism does the space/time bundle change states? Our description of the compactification is as yet devoid of physics, so presents no answers. But one could ask how much can be inferred from the hierarchical structure of the constructions. For example, embedded in the compactification in the classical case are 3-manifolds which fiber over the circle. Perhaps they enrich the setting enough to analyze this process.

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