Packings and coverings of the complete directed multigraph with 3- and 4-circuits

F.E. Bennett a,*, J. Yin b

a Department of Mathematics, Mount Saint Vincent University, Halifax, Nova Scotia, Canada B3M 2J6
b Department of Mathematics, Suzhou University, Suzhou 215006, China

Received 26 August 1992; revised 6 June 1995

Abstract

Let \( \lambda DK_v \) denote the complete directed multigraph with \( v \) vertices, where any two distinct vertices \( x \) and \( y \) are joined by \( \lambda \) arcs \((x, y)\) and \( \lambda \) arcs \((y, x)\). By a \( k \)-circuit we mean a directed cycle of length \( k \). In this paper, we consider the problem of finding maximal packings and minimal coverings of \( \lambda DK_v \) with \( k \)-circuits. In particular, we completely determine the packing number and covering number for the cases \( k = 3 \) and \( 4 \).

1. Introduction

If \( G \) is a graph, let \( DG \) be the directed graph obtained by replacing each edge \( ab \) of \( G \) with the two arcs \((a, b)\) and \((b, a)\). In particular, we denote by \( \lambda DK_v \) the complete directed multigraph with \( v \) vertices where any two distinct vertices \( x \) and \( y \) are joined by \( \lambda \) arcs \((x, y)\) and \( \lambda \) arcs \((y, x)\). When \( \lambda = 1 \), we drop the notation \( \lambda \) and write \( DK_v \) for \( \lambda DK_v \).

In the last 30 years, there has been much interest in decomposing the complete multigraph \( \lambda K_v \) into edge-disjoint copies of a graph \( G \). The most popular choices for \( G \) have been a complete graph (block designs) and a cycle. This problem has also been studied in the context of directed graphs (see, for example, [2–7]).

A decomposition of \( \lambda DK_v \) into arc-disjoint \( k \)-circuits (directed cycles of length \( k \)) is, by definition, a \((v, k, \lambda)\)-Mendelsohn design (briefly \((v, k, \lambda)\)-MD). In particular, a decomposition of \( \lambda DK_v \) into \( 3 \)-circuits is equivalent to a perfect \((v, 3, \lambda)\)-MD. We refer to a decomposition of \( \lambda DK_v \) into \( k \)-circuits as a \( k \)-circuit design of \( \lambda DK_v \). A similar terminology applies to \( DK_{m,n} \), the complete bipartite directed graph with vertex set \( X \cup Y \) where \( X \) and \( Y \) are disjoint and \( |X| = m \) and \( |Y| = n \).

The following two known results can be found in [4,5].

* Corresponding author. E-mail: fran.bennett@msvu.ca.
**Lemma 1.1.** There exists a 3-circuit design of \( \lambda DK_v \) if and only if \( \lambda v(v - 1) \equiv 0 \pmod{3} \) and \( v \geq 3 \), except for \( (v, \lambda) = (6, 1) \).

**Lemma 1.2.** There exists a 4-circuit design of \( \lambda DK_v \) if and only if

\[
\begin{align*}
(1) & \quad v \equiv 0 \text{ or } 1 \pmod{4} \text{ and } v > 4 \text{ for } \lambda \geq 1, \\
(2) & \quad v \equiv 2 \text{ or } 3 \pmod{4} \text{ and } v \geq 6 \text{ or } v = 4 \text{ for } \lambda \text{ even.}
\end{align*}
\]

As a natural generalization of a circuit design, we introduce the notions of packings and coverings of \( \lambda DK_v \) by circuits. A packing (covering) of \( \lambda DK_v \) by circuits is defined to be a collection \( D \) of \( k \)-circuits of \( \lambda DK_v \) such that any two distinct vertices \( x \) and \( y \) of \( \lambda DK_v \) are linked by an arc from \( x \) to \( y \) in at most (at least) \( \lambda \) circuits of \( D \). If no other such packing (covering) has more (fewer) circuits, the packing (covering) is said to be maximum (minimum), and the number of circuits in a maximum packing (a minimum covering) is called the packing number (the covering number), denoted by \( P_{\lambda}(v, k) \) \((C_{\lambda}(v, k))\). The main problem here is to determine the values of \( P_{\lambda}(v, k) \) and \( C_{\lambda}(v, k) \) for all integers \( v \geq k \). Let

\[
S_{\lambda}(v, k) = \lfloor \lambda v(v - 1)/k \rfloor \quad \text{and} \quad T_{\lambda}(v, k) = \lceil \lambda v(v - 1)/k \rceil,
\]

where \( v \geq k \), and where \( \lfloor x \rfloor \) denotes the greatest integer \( y \) such that \( y \leq x \) and \( \lceil x \rceil \) denotes the least integer \( y \) such that \( y \geq x \). It is easy to see that the following inequalities hold:

\[
P_{\lambda}(v, k) \leq S_{\lambda}(v, k) \leq T_{\lambda}(v, k) \leq C_{\lambda}(v, k). \tag{1.1}
\]

In this paper, we shall be concerned with the cases \( k = 3 \) and \( 4 \), where we present a complete solution to the problem. Note that \( P_{\lambda}(v, k) = S_{\lambda}(v, k) \) and \( C_{\lambda}(v, k) = T_{\lambda}(v, k) \) whenever a \( k \)-circuit design of \( \lambda DK_v \) exists. Hence, in particular, we have by Lemmas 1.1 and 1.2 the following results.

**Lemma 1.3.** Let \( v \geq 3 \) and \( \lambda \geq 1 \) be integers. Then \( P_{\lambda}(v, 3) = S_{\lambda}(v, 3) \) and \( C_{\lambda}(v, 3) = T_{\lambda}(v, 3) \) if \( \lambda v(v - 1) \equiv 0 \pmod{3} \) and \( (v, \lambda) \neq (6, 1) \).

**Lemma 1.4.** Let \( v \geq 4 \) and \( \lambda \geq 1 \) be integers. Then \( P_{\lambda}(v, 4) = S_{\lambda}(v, 4) \) and \( C_{\lambda}(v, 4) = T_{\lambda}(v, 4) \) if

\[
\begin{align*}
(1) & \quad v \equiv 0 \text{ or } 1 \pmod{4} \text{ and } v > 4 \text{ for } \lambda \geq 1; \text{ and} \\
(2) & \quad v \equiv 2 \text{ or } 3 \pmod{4} \text{ and } v \geq 6 \text{ or } v = 4 \text{ for } \lambda \text{ even.}
\end{align*}
\]

For convenience, we state the main results of this paper. The following theorem is proved in Section 2 and establishes the conclusive result for both the packing number \( P_{\lambda}(v, 3) \) and the covering number \( C_{\lambda}(v, 3) \).

**Theorem 1.5.** Let \( v \geq 3 \) and \( \lambda \geq 1 \) be integers. Then

\[
\begin{align*}
(1) & \quad C_{\lambda}(v, 3) = T_{\lambda}(v, 3) + 1 \text{ if } v \equiv 2 \pmod{3} \text{ and } \lambda \equiv 1 \pmod{3} \text{ or } (v, \lambda) = (6, 1); \\
& \quad C_{\lambda}(v, 3) = T_{\lambda}(v, 3) \text{ otherwise.}
\end{align*}
\]
(2) $P_\lambda(v, 3) = S_\lambda(v, 3) - 2$ if $(v, \lambda) = (6, 1)$; $P_\lambda(v, 3) = S_\lambda(v, 3) - 1$ if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 2 \pmod{3}$; $P_\lambda(v, 3) = S_\lambda(v, 3)$ otherwise.

Our next result, which can be found in Section 3 of the paper, completely determines the packing number $P_\lambda(v, 4)$ and the covering number $C_\lambda(v, 4)$.

**Theorem 1.6.** Let $v \geq 4$ and $\lambda \geq 1$ be integers. Then

1. $C_\lambda(v, 4) = T_\lambda(v, 4)$ and $P_\lambda(v, 4) = S_\lambda(v, 4)$ except for the pair $(v, \lambda)$ where $v = 4$ and $\lambda$ is odd; and
2. $P_\lambda(4, 4) = S_\lambda(4, 4) - 1$ and $P_\lambda(4, 4) = S_\lambda(4, 4) - 1$ for each odd $\lambda$.

2. Packings and coverings of $\lambda DK_v$ by 3-circuits

In this section, we shall completely determine the packing number $P_\lambda(v, 3)$ and the covering number $C_\lambda(v, 3)$. In view of Lemma 1.3, we need to consider only cases $v = 6$ for $\lambda = 1$ and $v \equiv 2 \pmod{3}$ for $\lambda \equiv 1$ or $2 \pmod{3}$. For convenience, we shall denote a $k$-circuit by $(a_0, a_1, \ldots, a_{k-1})$ consisting of arcs $(a_i, a_{i+1})$ for $0 \leq i < k - 1$ where subscripts are reduced modulo $k$.

**Lemma 2.1.** Let $v \geq 3$ and $\lambda \geq 1$ be integers satisfying $v \equiv 2 \pmod{3}$ and $\lambda \equiv 1 \pmod{3}$. Then

1. $C_\lambda(v, 3) = T_\lambda(v, 3) + 1$, and
2. $P_\lambda(v, 3) = S_\lambda(v, 3)$.

**Proof.** For equality (1), we first observe that $C_\lambda(v, 3) \geq T_\lambda(v, 3) + 1$. In fact, if the inequality does not hold, then we have $C_\lambda(v, 3) = T_\lambda(v, 3) = \frac{1}{3}(\lambda(v - 1) + 1)$ by (1.1). This means that $\lambda DK_v$ plus some arc, say $(x, y)$, will form a decomposition into 3-circuits. We label the $v$ vertices $x, y, z, \ldots$ with $v$ different integers $f(x), f(y), f(z), \ldots$ and label each arc $(u, w)$ with the difference $f(w) - f(u)$. It is obvious that the sum of three differences in any 3-circuit is zero. Consequently, all the differences from $\lambda DK_v$ and the arc $(x, y)$ will sum to zero. However, the differences from $\lambda DK_v$ already sum to zero, and this implies that $f(y) - f(x) = 0$, which is a contradiction. It is known [1] that $DK_v$ can be decomposed into $(v(v - 1) - 2)/3$ 3-circuits and two arcs $(a, b)$ and $(b, a)$ for some pair of vertices $a$ and $b$ when $v \equiv 2 \pmod{3}$. So we can use two 3-circuits to cover the missing arcs $(a, b)$ and $(b, a)$ to get a covering of $DK_v$ by $T_1(v, 3) + 1 = \lfloor(v(v - 1))/3 \rfloor + 1$ 3-circuits. When $\lambda > 1$, we can express it in the form $\lambda = \lambda_1 + 1$ where $\lambda_1 \equiv 0 \pmod{3}$. Combining the above covering of $DK_v$ and a 3-circuit design of $\lambda_1 DK_v$ mentioned in Lemma 1.1 gives a covering of $\lambda DK_v$ with $T_\lambda(v, 3) + 1$ 3-circuits, which implies that $C_\lambda(v, 3) \leq T_\lambda(v, 3) + 1$. Therefore, the equality (1) holds.

For equality (2), noticing the fact that $DK_v$ can be decomposed into $\frac{1}{3}(v(v - 1) - 2)$ 3-circuits and two arcs $(a, b)$ and $(b, a)$ for some pair of vertices $a$ and $b$ and (1.1),
we have $P_1(v, 3) = S_1(v, 3)$. As we did above, we now combine a maximum packing of $DK_v$ and a 3-circuit design of $\lambda_1 DK_v$ to obtain a packing of $\lambda DK_v$ by $S_1(v, 3)$ 3-circuits when $\lambda = \lambda_1 + 1 > 1$. Then the equality (2) follows from (1.1) and the proof is complete.

**Lemma 2.2.** Let $v \geq 3$ and $\lambda \geq 1$ be integers satisfying $v \equiv \lambda \equiv 2 \pmod{3}$. Then

1. $C_2(v, 3) = T_2(v, 3)$, and
2. $P_2(v, 3) = S_2(v, 3) - 1$.

**Proof.** The proof is similar to that above.

For (1), we first show that there exists a covering of $\lambda DK_v$ with $T_2(v, 3)$ 3-circuits. As mentioned earlier, $DK_v$ can be decomposed into $\frac{1}{3}(v(v - 1) - 2)$ 3-circuits and two arcs $(a, b)$ and $(b, a)$ for some pair of vertices $a$ and $b$ when $v \equiv 2 \pmod{3}$. Hence, we can construct a packing $D$ of $2DK_v$ with $\frac{1}{3}(v(v - 1) - 2)$ 3-circuits which cover all the arcs of $2DK_v$ except for two pairs of arcs in the form $(a, b)$, $(b, a)$ and $(c, b), (b, c)$. This packing together with another two 3-circuits $(a, b, c)$ and $(c, b, a)$ provides a covering of $2DK_v$ with $T_2(v, 3)$ 3-circuits. If $\lambda > 2$, then we can write $\lambda = \lambda_1 + 2$ where $\lambda_1 \equiv 0 \pmod{3}$, and hence a covering of $\lambda DK_v$ with $T_2(v, 3)$ 3-circuits is obtained by taking a minimum covering of $2DK_v$ and a 3-circuit design of $\lambda_1 DK_v$ in Lemma 1.1. We then use (1.1) to establish the equality (1).

For (2), we can construct a packing of $\lambda DK_v$ with $S_2(v, 3) - 1$ 3-circuits by combining two copies of a maximum packing of $DK_v$ and one copy of a 3-circuit design of $\lambda_1 DK_v$, where $\lambda = \lambda_1 + 2$ and $\lambda_1 \equiv 0 \pmod{3}$. In a manner similar to that in Lemma 2.1, we can also show that $P_2(v, 3) \leq S_2(v, 3) - 1$. Therefore, the Eq. (2) holds and the proof is complete.

**Lemma 2.3.** (1) $P_1(6, 3) = S_1(6, 3) - 2 = 8$, and

2. $C_1(6, 3) = T_1(6, 3) + 1 = 11$.

**Proof.** Since it is well known [2, 6] that there does not exist a decomposition of $DK_6$ into 3-circuits, it follows that $P_1(6, 3) < S_1(6, 3) = 10$ and $C_1(6, 3) > T_1(6, 3) = 10$.

On the other hand, a covering of $DK_6$ by 11 3-circuits can be obtained with the following collection $\mathcal{D}$ of 3-circuits:

$$\mathcal{D} = \{(1, 2, 4), (1, 4, 3), (2, 1, 3), (2, 5, 4), (6, 1, 5), (5, 1, 6), (5, 3, 4), (3, 5, 2), (2, 6, 3), (3, 6, 4), (4, 6, 2)\}.$$

Therefore we have $C_1(6, 3) = T_1(6, 3) + 1 = 11$.

Note that the first eight 3-circuits listed in the collection $\mathcal{D}$ above form a packing into $DK_6$. We now show that there is no packing with nine 3-circuits into $DK_6$. If not so, the differences from either $DK_6$ or the packing for any labelling of the vertices will sum to zero. Then the differences from the remaining three arcs will also sum to zero.
We distinguish the following two cases:

1. the three arcs are disjoint,
2. there are two arcs forming a path of length two.

In case (1) suppose the three arcs are \((u_1, u_2), (u_3, u_4),\) and \((u_5, u_6)\) and vertex \(u_i\) is labelled with \(i\). We then have three differences all equal to one, which cannot sum to zero — a contradiction.

In case (2) suppose the three arcs are \((u_1, u_2), (u_2, u_3)\) and \((u_4, u_5)\). Label \(u_1\) and \(u_3\) with 1 and 6, respectively. Label the remaining four vertices with 2, 3, 4 and 5. Since \([f(u_2) - f(u_1)] + [f(u_3) - f(u_2)] + [f(u_5) - f(u_4)] = 0\), we have \(f(u_5) - f(u_1) = -5\).

According to our labelling, we have \(f(u_5) = 1\) and \(f(u_1) = 6\). Thus \((u_i, u_j) = (u_3, u_1)\).

In such a case, the three arcs form a 3-circuit and we obtain a decomposition of \(DK_6\) into 3-circuits, which is impossible. This establishes the result that \(P_1(6, 3) = S_1(6, 3) - 2 = 8\) and the proof is complete.

Summarizing the results of Lemmas 2.1–2.3 and Lemma 1.3, we have proved the following restatement of Theorem 1.5.

Theorem 2.4. Let \(v \geq 3\) and \(\lambda \geq 1\) be integers. Then

1. \(C_{v}(v, 3) = T_{v}(v, 3) + 1\) if \(v \equiv 2 \pmod{3}\) and \(\lambda \equiv 1 \pmod{3}\) or \((v, \lambda) = (6, 1)\); \(C_{v}(v, 3) = T_{v}(v, 3)\) otherwise.
2. \(P_{v}(v, 3) = S_{v}(v, 3) - 2\) if \((v, \lambda) = (6, 1)\); \(P_{v}(v, 3) = S_{v}(v, 3) - 1\) if \(v \equiv 2 \pmod{3}\) and \(\lambda \equiv 2 \pmod{3}\); \(P_{v}(v, 3) = S_{v}(v, 3)\) otherwise.

3. Packings and coverings of \(\lambda DK_6\) by 4-circuits

In this section, we turn our attention to the determination of the packing number \(P_{v}(v, 4)\) and the covering number \(C_{v}(v, 4)\). We commence with some direct constructions.

Lemma 3.1. If \(v \in \{6, 7\}\), then \(P_{1}(v, 4) = S_{1}(v, 4)\).

Proof. From (1.1), we need only construct a packing with \(S_{1}(v, 4)\) 4-circuits for each of the specified values of \(v\). The required packings are presented as follows.

For \(v = 6\), the vertex set of \(DK_v\) is \(\{1, 2, 3, 4, 5, 6\}\) and the 4-circuits are

\[(1, 6, 5, 4), (1, 2, 4, 5), (1, 3, 2, 6), (1, 4, 6, 2),
(1, 5, 2, 3), (3, 5, 6, 4), (2, 5, 3, 4)\].

For \(v = 7\), the vertex set of \(DK_v\) is \(Z_{v-2} \cup \{x, y\}\) and the 4-circuits are obtained by developing the 4-circuits \((x, 0, 1, 3)\) and \((y, 0, 3, 2)\) modulo 5, where \(x\) and \(y\) are fixed.

Lemma 3.2. If \(v \in \{6, 7\}\), then \(C_{1}(v, 4) = T_{1}(v, 4)\).
Proof. The proof is similar to that of Lemma 3.1. A minimum covering of $DK_6$ is given by the following 4-circuits:

- $(5, 6, 2, 4)$, $(5, 6, 1, 3)$, $(5, 4, 3, 1)$, $(3, 4, 2, 5)$,
- $(5, 1, 4, 6)$, $(5, 2, 3, 6)$, $(3, 2, 1, 6)$, $(6, 4, 1, 2)$.

A minimum covering of $DK_7$ is given by the following 4-circuits:

- $(x, y, 2, 3)$, $(x, y, 0, 4)$, $(x, 3, y, 1)$,
- $(x, 4, 1, 2)$, $(y, x, 0, 4)$, $(y, x, 1, 0)$, $(y, 4, 2, 1)$,
- $(0, 1, 3, 2)$, $(0, 3, 1, 4)$, $(0, 2, 4, 3)$.

Lemma 3.3. (1) $C_1(4, 4) = T_1(4, 4) + 1 = 4$, and
(2) $P_1(4, 4) = S_1(4, 4) - 1 = 2$.

Proof. From Lemma 1.2 we have $P_1(4, 4) \leq S_1(4, 4) - 1$ and $C_1(4, 4) \geq T_1(4, 4) + 1$. Now let the vertex set of $DK_4$ be $Z_4$. A minimum covering of $DK_4$ with four 4-circuits is obtained by developing the 4-circuit $(0, 1, 3, 2)$ modulo 4. A maximum packing of $DK_4$ which is given by taking the following two 4-circuits:

- $(0, 1, 2, 3)$, $(3, 2, 1, 0)$.

In order to obtain our main result, we also require the following known result, which is contained in [3, 5].

Lemma 3.4. If $m \geq k$, $n \geq k$ and if $k$ divides $m$ or $n$, then there exists a $2k$-circuit design of $DK_{m,n}$.

The following main result is a restatement of Theorem 1.6.

Theorem 3.5. Let $v \geq 4$ and $\lambda \geq 1$ be integers. Then

(1) $C_\lambda(v, 4) = T_\lambda(v, 4)$ and $P_\lambda(v, 4) = S_\lambda(v, 4)$ except for the pair $(v, \lambda)$ where $v = 4$ and $\lambda$ is odd; and
(2) $C_\lambda(4, 4) = T_\lambda(4, 4) + 1$ and $P_\lambda(4, 4) = S_\lambda(4, 4) - 1$ for each odd $\lambda$.

Proof. From Lemma 1.4 we need to consider only the cases $v = 4$ and $v \equiv 2$ or $3 \mod 4$ for odd $\lambda$.

For the case $v = 4$, we first note that there is no 4-circuit design of $\lambda DK_4$ for odd $\lambda$ by Lemma 1.2. So $P_\lambda(v, 4) \leq S_\lambda(v, 4) - 1$ and $C_\lambda(v, 4) \geq T_\lambda(v, 4) + 1$ for any odd $\lambda$. When $\lambda = 1$, the result was established in Lemma 3.3. When $\lambda > 1$, we write $\lambda = \lambda_1 + 1$, where $\lambda_1$ is even. Then a maximum packing of $\lambda DK_4$ consists of a maximum packing of $DK_4$ and a 4-circuit design of $\lambda_1 DK_4$. Similarly, a minimum covering of $\lambda DK_4$ consists of a minimum covering of $DK_4$ and a 4-circuit design of $\lambda_1 DK_4$.

For the case where $v \equiv 2$ or $3 \mod 4$, we first deal with the case $\lambda = 1$. In view of Lemmas 3.1 and 3.2, we may assume $v \geq 10$. Let the vertex set of $DK_v$ be
\[ V = X_1 \cup X_2 \cup \{\infty\} \text{ with } |X_1| = v - 6 - q \text{ and } |X_2| = 5 + q \text{ where } q = 0 \text{ or } 1 \text{ depending on whether } v \equiv 2 \text{ or } 3 \pmod{4}. \text{ From Lemma 3.4, there exists a 4-circuit design of } DK_{v-6-q,5+q} \text{ based on } X_1 \cup X_2. \text{ Since } v - 6 - q \equiv 0 \pmod{4}, \text{ there also exists a 4-circuit design of } DK_{v-6-q+1} \text{ based on } X_1 \cup \{\infty\}. \text{ Thus the maximum packing and minimum covering of } DK_{v-6-q} \text{ stated in Lemmas 3.1 and 3.2 can be utilized to create a packing and a covering of } DK_{v} \text{ with } S_1(v,4) \text{ and } T_1(v,4) \text{ 4-circuits, respectively. Thus the conclusion follows. When } \lambda > 1, \text{ constructions similar to those preceding can be applied. We can make use of a maximum packing (minimum covering) of } DK_v \text{ and a 4-circuit design of } (\lambda - 1)DK_v \text{ to obtain the desired result. This completes the proof.}

Acknowledgements

The first author would like to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada under Grant A-5320. A portion of this work was carried out while the second author was visiting Mount Saint Vincent University during the Summer of 1992, and he is grateful for the kind hospitality accorded him.

References