The Generalized Vector Quasi-Variational-Like Inequalities*

XIE PING DING
Department of Mathematics, Sichuan Normal University
Chengdu, Sichuan 610066, P.R. China
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Abstract—In this paper, we introduce and study a class of Generalized Vector Quasi-Variational-Like Inequality Problem (GVQVLIP) involving set-valued mappings with certain monotonicity. By employing the scalarization technique, several existence results for solutions of the (GVQVLIP) are established under noncompact setting in topological vector spaces. These new existence results improve, unify, and generalize many known results for scalar and vector variational inequalities in recent literature. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Generalized vector quasivariational-like inequality, $C_+\cdot\eta$-monotone, Topological vector space.

1. INTRODUCTION

In 1980, Giannessi [1] first introduced and studied the vector variational inequality problem in finite-dimensional Euclidean space. Chen and Cheng [2] studied the vector variational inequality in infinite-dimensional spaces and applied it to vector optimization problem. Since then, many authors [3–11] have intensively studied the vector variational inequality problem on different assumptions in infinite-dimensional spaces. Lee et al. [12,13], Lin et al. [14], Konnov and Yao [15], and Daniilidis and Hadjisavvas [16] studied the generalized vector variational inequality and obtained some existence results. Chen et al. [17] and Lee et al. [18] introduced and studied the generalized vector quasi-variational inequality and established some existence theorems. Ansari [19,20] studied the generalized vector variational-like inequalities.

In this paper, we shall introduce and study a class of generalized vector quasivariational-like inequality problem (GVQVLIP) involving $C_+\cdot\eta$-monotone and weakly $C_+\cdot\eta$-monotone set-valued mappings. By employing the scalarization method, some existence theorems for solutions of the (GVQVLIP) are established in noncompact setting of topological vector spaces. Our results improve, unify, and generalize many recent results for generalized scalar and vector variational inequalities in the literatures.

2. PRELIMINARIES

Let $X$ be a nonempty set, we shall denote by $2^X$ the family of all nonempty subsets of $X$. Let $E$ and $F$ be Hausdorff real topological vector spaces. A nonempty subset $D$ of $F$ is said to...
be a cone if $D + D \subset D$ and $\lambda D \subset D$, $\forall \lambda \geq 0$. A cone $D$ is said to be pointed if $D \cap (-D) = \{0\}$. A cone $D$ is said to be solid if the topological interior $\text{int}(D)$ of $D$ in $F$ is nonempty. Let $F^*$ be the topological dual space of $F$. The set $D^* = \{f \in F^* : (f, x) \geq 0, \forall x \in D\}$ is called the dual cone of $D$. We denote by $L(E, F)$ the space of all continuous linear mappings from $E$ into $F$ and by $(u, y)$ the evaluation of $u \in L(E, F)$ at $y \in E$. Let $\sigma$ and $\delta$ be the families of all finite subsets and all bounded subsets of $E$, respectively. Let $B$ be a neighborhood base of 0 in $F$. When $S$ runs through $\sigma$, $V$ through $B$, the family

$$M(S, V) = \left\{ l \in L(E, F) : \bigcup_{x \in S} (l, x) \subset V \right\}$$

is a neighborhood base of 0 in $L(E, F)$ for a unique translation-invariant topology, called the topology of pointwise convergence, briefly, the $\sigma$-topology. Similarly, when $S$ runs through $\delta$, $V$ through $B$, the family $\{M(S, V) : S \in \delta, V \in B\}$ is a neighborhood base of 0 in $L(E, F)$ for a unique translation-invariant topology, called the topology of uniform convergence on the set $S \in \delta$, briefly, the $\delta$-topology, see [21, pp. 79–81].

Let $E$ and $F$ be real Hausdorff topological vector spaces, $X$ be a nonempty convex subset of $E$ and $T : X \to 2^{L(E, F)}$ be a set-valued mapping. Let $G : X \to 2^X$ and $C : X \to 2^F$ be set-valued mappings such that for each $x \in X$, $C(x)$ is a closed pointed convex cone with $\text{int} C(x) \neq \emptyset$. Let $\eta : X \times X \to E$ be a single-valued mapping. In this paper, we consider the generalized vector quasivariational-like inequality problem (GVQVLIP), that is, to find $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and

$$\forall y \in G(\hat{x}), \quad \exists \hat{u} \in T(\hat{x}) : (\hat{u}, \eta(y, \hat{x})) \notin -\text{int} C(\hat{x}). \quad (1)$$

The point $\hat{x}$ is said to be a solution of the (GVQVLIP). If there exist $\hat{x} \in X$ and $\hat{u} \in T(\hat{x})$ such that $\hat{x} \in G(\hat{x})$ and

$$(\hat{u}, \eta(y, \hat{x})) \notin -\text{int} C(\hat{x}), \quad \forall y \in G(\hat{x}),$$

then $\hat{x}$ is called a strongly solution of the (GVQVLIP).

**Special Cases.**

(I) If $\eta(x, y) = x - y$, $\forall x, y \in X$, then the (GVQVLIP) reduces to find $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and

$$\forall y \in G(\hat{x}), \quad \exists \hat{u} \in T(\hat{x}) : (\hat{u}, y - \hat{x}) \notin -\text{int} C(\hat{x}). \quad (3)$$

Problem (3) is called the generalized vector quasivariational inequality problem (GVQVIP) which is new. When $C(x) = C$, $\forall x \in X$ is a constant cone, problem (3) was studied by Chen and Li [17] and Lee et al. [18].

(II) If $G(x) = X$, $\forall x \in X$, then the (GVQVLIP) reduces to find $\hat{x} \in X$ such that

$$\forall y \in X, \quad \exists \hat{u} \in T(\hat{x}) : (\hat{u}, \eta(y, \hat{x})) \notin -\text{int} C(\hat{x}). \quad (4)$$

Problem (4) was introduced and studied by Ansari [19,20].

(III) If $G(x) = X$, $\forall x \in X$ and $\eta(x, y) = x - y$, $\forall x, y \in X$, then the (GVQVLIP) reduces to find $\hat{x} \in X$ such that

$$\forall y \in X, \quad \exists \hat{u} \in T(\hat{x}) : (\hat{u}, y - \hat{x}) \notin -\text{int} C(\hat{x}). \quad (5)$$

Problem (5) and its special cases are called the generalized vector variational inequality (GVVII) which was introduced and studied in [12–16].

(IV) If $T$ is a single-valued mapping, then the (GVQVLIP) reduces to find $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and

$$(T(\hat{x}), \eta(y, \hat{x})) \notin -\text{int} C(\hat{x}), \quad \forall y \in G(\hat{x}). \quad (6)$$
When $G(x) = X$, $\forall x \in X$, problem (6) and its special cases were studied by many authors, see [1-11].

(V) If $F = R$ and $C(x) = [0, \infty)$, $\forall x \in X$, then $L(E, F) = E^*$ and the (GVQVLIP) reduces to find $\hat{x} \in X$ and $\hat{u} \in T(\hat{x})$ such that $\hat{x} \in G(\hat{x})$ and

$$\langle \hat{u}, \eta(y, \hat{x}) \rangle \geq 0, \quad \forall y \in G(\hat{x}).$$

Problem (7) includes many classes of scalar type generalized quasivariational inequality and generalized quasivariational-like inequality problems as special cases, see [22-28].

In order to prove the main results, we need the following definitions and lemmas.

**Lemma 2.1.** Let $E$ and $F$ be topological vector spaces. Then for each fixed $x \in E$ and $f \in F^*$, the function $f((\cdot, x)) : (L(E, F), \sigma) \rightarrow \mathbb{R}$ is continuous.

**Proof.** Suppose that $\{w_\alpha\}_{\alpha \in \Gamma} \subset L(E, F)$ converges to $w_0 \in L(E, F)$ in the $\sigma$-topology. Clearly, for any fixed $f \in F^*$ and $\epsilon > 0$, $f^{-1}(-\epsilon, \epsilon)$ is an open neighborhood of 0 in $F$, and hence, there is a $V \in \mathcal{B}$ such that $V \subset f^{-1}(-\epsilon, \epsilon)$. Since, for each fixed $x \in E$, $M(\{x\}, V)$ is an open neighborhood of 0 in $L(E, F)$ under the $\sigma$-topology, there exists $\alpha_0 \in \Gamma$ such that $w_\alpha - w_0 \in M(\{x\}, V)$, $\forall \alpha \geq \alpha_0$. Therefore, we have

$$\langle w_\alpha - w_0, z \rangle \in V \subset f^{-1}(-\epsilon, \epsilon), \quad \forall \alpha \geq \alpha_0.$$ 

It follows that

$$f(\langle w_\alpha - w_0, x \rangle) \in (-\epsilon, \epsilon), \quad \forall \alpha \geq \alpha_0.$$ 

By the arbitrariness of $\epsilon$, we obtain

$$f(\langle w_\alpha, x \rangle) \rightarrow f(\langle w_0, x \rangle).$$

This completes the proof.

**Lemma 2.2.** Let $E$ and $F$ be topological vector spaces. Then the bilinear mapping $\langle \cdot, \cdot \rangle : L(E, F) \times E \rightarrow F$ is continuous on $(L(E, F), \delta) \times E$.

**Proof.** Let $(l_\alpha, x_\alpha)_{\alpha \in \Gamma}$ be a net in $L(E, F) \times X$ and $(l_\alpha, x_\alpha) \rightarrow (l_0, x_0)$, then we have $l_\alpha \rightarrow l_0$ under the $\delta$-topology of $L(E, F)$ and $x_\alpha \rightarrow x_0$ in $X$. Let $V \in \mathcal{B}$ be an arbitrary given neighborhood of 0 in $F$, we can choose a neighborhood $V_1 \in \mathcal{B}$ such that $V_1 + V_1 + V_1 \subset V$. Since $x_\alpha \rightarrow x_0$, $\{x_\alpha - x_0\}_{\alpha \in \Gamma} \cup \{x_0\}$ must be bounded and hence $\{x_\alpha - x_0\}_{\alpha \in \Gamma} \cup \{x_0\} \subset S_0$ for some $S_0 \in \delta$. Let

$$M(S_0, V_1) = \left\{ l \in L(E, F) : \bigcup_{x \in S_0} \langle l, x \rangle \subset V_1 \right\}.$$ 

Then $M(S_0, V_1)$ is an open neighborhood of 0 under the $\delta$-topology of $L(E, F)$. Since $l_\alpha - l_0 \rightarrow 0$ under the $\delta$-topology of $L(E, F)$, there exists $\alpha_1 \in \Gamma$ such that $l_\alpha - l_0 \in M(S_0, V_1)$ for all $\alpha \geq \alpha_0$. It follows that

$$\langle l_\alpha - l_0, x_\alpha - x_0 \rangle \subset V_1 \quad \text{and} \quad \langle l_\alpha - l_0, x_0 \rangle \subset V_1, \quad \forall \alpha \geq \alpha_1.$$ 

Since $l_0 \in L(E, F)$ is a continuous linear operator, there exists $\alpha_2 \in \Gamma$ such that

$$\langle l_0, x_\alpha - x_0 \rangle \in V_1, \quad \forall \alpha \geq \alpha_2.$$ 

Hence there exists $\alpha_3 \in \Gamma$ such that

$$\langle l_\alpha, x_\alpha \rangle - \langle l_0, x_0 \rangle = \langle l_\alpha - l_0, x_\alpha - x_0 \rangle + \langle l_\alpha - l_0, x_0 \rangle + \langle l_0, x_\alpha - x_0 \rangle$$ 

$$\in V_1 + V_1 + V_1 \subset V, \quad \forall \alpha \geq \alpha_3.$$ 

This proves that the bilinear mapping $\langle \cdot, \cdot \rangle : (L(E, F), \delta) \times X \rightarrow F$ is continuous.
DEFINITION 2.1. Let \( X \) and \( Y \) be topological spaces. A set-valued mapping \( T : X \to 2^Y \) is said to be upper semicontinuous (respectively, lower semicontinuous) at \( x_0 \in X \) if for each open set \( V \subset Y \) with \( T(x_0) \subset V \) (respectively, \( T(x_0) \cap V \neq \emptyset \)), there exists an open neighborhood \( U \) of \( x_0 \) in \( X \) such that \( T(x) \subset V \) (respectively, \( T(x) \cap V \neq \emptyset \)) for all \( x \in U \). \( T \) is said to be upper semicontinuous (respectively, lower semicontinuous) on \( X \) if \( T \) is upper semicontinuous (respectively, lower semicontinuous) at each point of \( X \).

DEFINITION 2.2. Let \( X \) be a nonempty subset of a topological vector space \( E \). A set-valued mapping \( G : X \to 2^E \) is said to be upper hemicontinuous on \( X \) if for each \( f \in E^* \), the function \( x \mapsto \sup_{u \in G(x)} \langle f, u \rangle \) is upper semicontinuous on \( X \).

DEFINITION 2.3. Let \( E \) and \( F \) be topological vector spaces, \( X \) be a nonempty subset of \( E \), \( T : X \to 2^{L(E, F)} \), and \( F^* \) be the dual space of \( F \). Then \( T \) is said to be

1. upper hemicontinuous on \( X \) if and only if for each \( x \in E \) and for each \( f \in F^* \setminus \{0\} \), the function \( g(f,x) : X \to \mathbb{R} \cup \{+\infty\} \) defined by
   \[
   g(f,x)(z) = \sup_{v \in T(z)} \langle f, (v,x) \rangle, \quad \forall z \in X,
   \]
   is upper semicontinuous on \( X \) (if and only if for each \( x \in E \) and for each \( f \in F^* \setminus \{0\} \), the function \( h(f,x) : X \to \mathbb{R} \cup \{-\infty\} \) defined by
   \[
   h(f,x)(z) = \inf_{v \in T(z)} \langle f, (v,x) \rangle, \quad \forall z \in X,
   \]
   is lower semicontinuous on \( X \).

2. lower hemicontinuous on \( X \) if and only if for each \( x \in E \) and for each \( f \in F^* \setminus \{0\} \), the function \( g(f,x) : X \to \mathbb{R} \cup \{+\infty\} \) defined by
   \[
   g(f,x)(z) = \sup_{v \in T(z)} \langle f, (v,x) \rangle, \quad \forall z \in X
   \]
   is lower semicontinuous on \( X \) (if and only if for each \( x \in E \) and for each \( f \in F^* \setminus \{0\} \), the function \( h(f,x) : X \to \mathbb{R} \cup \{-\infty\} \) defined by
   \[
   h(f,x)(z) = \inf_{v \in T(z)} \langle f, (v,x) \rangle, \quad \forall z \in X
   \]
   is upper semicontinuous on \( X \).

REMARK 2.1. If \( F = \mathbb{R} \), then \( L(E, F) = E^* \) and the concepts in Definition 2.3 reduce to those in Definition 2.1 of Chowdhury and Tan [29].

LEMMA 2.3. Let \( X \) be a nonempty convex subset of a topological vector space \( E \) and \( F \) be a topological vector space with the dual space \( F^* \). Let \( T : X \to 2^{L(E, F)} \) be upper semicontinuous from the line segments in \( X \) to the \( \sigma \)-topology of \( L(E, F) \). Then \( T \) is upper hemicontinuous along the line segment in \( X \).

PROOF. Let \( L \) be an any given line segment in \( X \). For any fixed \( x \in X \) and \( f \in F^* \setminus \{0\} \), define the function \( g(f,x) : L \to \mathbb{R} \cup \{+\infty\} \) by
   \[
   g(f,x)(z) = \sup_{v \in T(z)} \langle f, (v,x) \rangle, \quad \forall z \in L.
   \]
Let \( r \in \mathbb{R} \) be given and let \( A_r = \{ z \in L : g(f,x)(z) < r \} \). Take any \( z_0 \in A_r \), then there is a \( r' < r \) such that
   \[
   \sup_{v \in T(z_0)} \langle f, (v,x) \rangle < r' < r.
   \]
Hence, we have \( T(z_0) \subset \{ u \in L(E, F) : f((v,x)) < r' \} =: V \) and \( V \) is open in the \( \sigma \)-topology of \( L(E, F) \) by Lemma 2.1. Since \( T \) upper semicontinuous from the line segment in \( X \) to the \( \sigma \)-topology of \( L(E, F) \), there exists a relatively open neighborhood \( U \) of \( z_0 \) in \( L \) such that \( T(z) \subset V, \forall z \in U \). Therefore,
   \[
   \sup_{v \in T(z)} \langle f, (v,x) \rangle \leq r' < r, \quad \forall z \in U.
   \]
Hence \( z_0 \in U \subset A_r \) and \( A_r \) is relatively open in \( L \). This shows that \( g(f,x) \) is upper semicontinuous along the line segment \( L \) in \( X \) and hence \( T \) is upper hemicontinuous along the segment in \( X \).
LEMMA 2.4. Let $X$ be a nonempty convex subset of a topological vector space $E$ and $F$ be a topological vector space with the dual space $F^*$. Let $T : X \to 2^{L(E,F)}$ be lower semicontinuous from the line segment in $X$ to the $\sigma$-topology of $L(E, F)$. Then $T$ is lower hemicontinuous along the line segment in $X$.

PROOF. Let $L$ be any given line segment in $X$. For any fixed $x \in X$ and $f \in F^* \setminus \{0\}$, define the function $g(f,x) : L \to \mathbb{R} \cup \{+\infty\}$ by

$$g(f,x)(z) = \sup_{\nu \in T(z)} f((\nu, x)), \quad \forall z \in L.$$ 

Let $r \in \mathbb{R}$ be given and let $A_r = \{z \in L : g(f,x)(z) > r\}$. Take any $z_0 \in A_r$, then

$$g(f,x)(z_0) = \sup_{\nu \in T(z_0)} f((\nu, x)) > r.$$ 

Let $V = \{v \in L(E, F) : f((v, x)) > r\}$. By Lemma 2.1, $V$ is an open set in the $\sigma$-topology of $L(E, F)$. Clearly, we have $T(z_0) \cap V \neq \emptyset$. Since $T$ is lower semicontinuous from the line segment in $X$ to the $\sigma$-topology of $L(E, F)$, there a relatively open neighborhood $U$ of $z_0$ in $L$ such that $T(z) \cap V \neq \emptyset$ for all $z \in U$. Hence, we have

$$g(f,x)(z) = \sup_{\nu \in T(z)} f((\nu, x)) > r, \quad \forall z \in U$$ 

and $z_0 \in U \subset A_r$. Therefore, $g(f,x)$ is lower semicontinuous on $L$. Hence $T$ is lower hemicontinuous along the line segment in $X$.

REMARK 2.2. Lemmas 2.3 and 2.4 generalize Propositions 2.2 and 2.4 of Chowdhury and Tan [29] to operator-valued set-valued mappings, respectively. The examples 2.3 and 2.5 in [29] show that the converse of Lemma 2.3 and Lemma 2.4 are not true in general.

Let $E$ and $F$ be topological vector spaces, $X$ be a nonempty subset of $E$ and $D$ be a convex cone in $F$. Let $C : X \to 2^F$ be a set-valued mapping such that for each $x \in X$, $C(x)$ is a closed pointed convex cone in $F$ with $\text{Ant} C(x) \neq \emptyset$. The following notations will be used in the sequel:

$$C_+ = \text{co} \{C(x) : x \in X\}.$$ 

It is clear that $C_+$ is a convex cone in $F$ with $\text{int} C_+ \neq \emptyset$.

DEFINITION 2.4. Let $T : X \to 2^{L(E,F)}$ be a set-valued mapping and $\eta : X \times X \to E$ be a single-valued mapping.

1. $T$ is $D$-$\eta$-monotone on $X$ if for each $x, y \in X$ and for all $u \in T(x)$ and $v \in T(y)$,

$$(v - u, \eta(y, x)) \in D.$$ 

2. $T$ is weakly $D$-$\eta$-monotone on $X$ if for each $x, y \in X$ and for each $u \in T(x)$, there exists $v \in T(y)$ such that

$$(v - u, \eta(y, x)) \in D.$$ 

REMARK 2.3. If $\eta(x, y) = x - y$, $\forall x, y \in X$, Definition 2.3 reduces to the parts (i) and (iv) of Definition 2.1 in [15].

DEFINITION 2.5. (See [23].) Let $X$ be a nonempty convex subset of a topological vector space $E$. A function $\psi : X \times X \to \mathbb{R}$ is said to be 0-diagonal concave (in short, 0-DCV) in $y$ if for any finite set $\{y_1, \ldots, y_n\} \subset X$ and $x_0 = \sum_{i=1}^{n} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,

$$\sum_{i=1}^{n} \lambda_i \psi(x_0, y_i) \leq 0.$$ 

The following result is Theorem 2.1 of [30] (also see [28, Theorem 2.1]).
LEMMA 2.5. Let $X$ be a nonempty paracompact convex subset of a topological vector space $E$ with dual space $E^*$ and $E^*$ separate the points of $E$. Suppose that $G : X \to 2^X$ is upper hemicontinuous on $X$ with nonempty compact convex values and $\psi : X \times X \to \mathbb{R}$ such that

1. the set $\{ x \in X : \sup_{y \in G(x)} \psi(x, y) > 0 \}$ is open in $X$,
2. for each $y \in X$, $x \mapsto \psi(x, y)$ is lower semicontinuous in each compact subset of $X$,
3. $\psi(x, y)$ is $0-DCV$ in $y$,
4. there exist a nonempty compact convex subset $X_0$ of $X$ and a nonempty compact subset $K$ of $X$ such that for each $x \in X \setminus K$ there is a $y \in G(x) \cap co(X_0 \cup \{x\})$ satisfying $\psi(x, y) > 0$.

Then there exists a $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and $\sup_{y \in G(\hat{x})} \psi(\hat{x}, y) \leq 0$.

LEMMA 2.6. (See [31].) Let $X$ be a nonempty compact convex subset of a topological vector space $E$ and $Y$ be a nonempty convex subset of a vector space $F$. Suppose that $\phi : X \times Y \to \mathbb{R}$ satisfies:

1. for each $y \in Y$, $x \mapsto \phi(x, y)$ is a lower semicontinuous convex function,
2. (or each $x \in X$, $y \mapsto \phi(x, y)$ is a concave function.

Then

$$\min_{x \in X} \sup_{y \in Y} \phi(x, y) = \sup_{y \in Y} \min_{x \in X} \phi(x, y).$$

3. EXISTENCE OF SOLUTION TO THE (GVQVLIP)

In order to prove our main theorems, we shall need the following useful result.

LEMMA 3.1. Let $X$ be a nonempty convex subset of a topological vector space $E$ and $F$ be a topological vector space. Let $T : X \to 2^{L(E,F)}$ be upper hemicontinuous along the line segment in $X$ with nonempty values and $\eta : X \times X \to E$ be such that $\eta(x, y)$ is affine in first argument and $\eta(x, x) = 0$, $\forall x \in X$. If there exist $f \in F^* \setminus \{0\}$ and $\hat{x} \in X$ such that $\inf_{v \in T(y)} f(-\langle v, \eta(y, \hat{x}) \rangle) < 0$, $\forall y \in X$, then we have

$$\inf_{u \in T(z_t)} f(-\langle u, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in X.$$ 

PROOF. For any fixed $y \in X$, let $z_t = \hat{x} + t(y - \hat{x})$. As $X$ is convex, $z_t \in X$ for all $t \in [0,1]$. Noting that $y \mapsto \eta(y, \hat{x})$ is affine and $\eta(x, x) = 0$, $\forall x \in X$, by the assumption, we have

$$t \inf_{u \in T(z_t)} f(-\langle v, \eta(y, \hat{x}) \rangle)$$

$$= \inf_{u \in T(z_t)} f(-\langle v, (1 - t)\eta(\hat{x}, \hat{x}) + t\eta(y, \hat{x}) \rangle)$$

$$= \inf_{u \in T(z_t)} f(-\langle v, \eta(z_t, \hat{x}) \rangle) \leq 0, \quad \forall t \in [0,1].$$

It follows that

$$\inf_{u \in T(z_t)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall t \in (0,1]. \quad (8)$$

Since $T$ is upper hemicontinuous along the line segment in $X$, by Definition 2.3, the function

$$z \mapsto h_{\langle f, -\eta(y, \hat{x}) \rangle} = \inf_{u \in T(z)} f(-\langle u, -\eta(y, \hat{x}) \rangle)$$

is lower semicontinuous along the line segment in $X$. Since $z_t \to \hat{x}$ as $t \to 0^+$, it follows from (8) that

$$\inf_{u \in T(z_t)} f(-\langle u, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in X.$$ 

REMARK 3.1. If $T : X \to 2^{L(E,F)}$ is upper semicontinuous from the line segment in $X$ to the $\sigma$-topology of $L(E,F)$, then, by Lemma 2.3, $T$ is upper hemicontinuous along the line segment in $X$ and hence the conclusion of Lemma 3.1 still holds.
LEMMA 3.2. Let $X, E,$ and $F$ be same as that in Lemma 3.1. Let $T : X \to 2^{L(E,F)}$ be a lower semicontinuous along the line segment in $X$ with nonempty values. If there exist $f \in F^* \setminus \{0\}$ and $\hat{x} \in X$ such that $\sup_{v \in T(y)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \forall y \in X$, then we have

$$\sup_{u \in T(\hat{x})} f(-\langle u, \eta(\hat{x}, y) \rangle) \leq 0, \quad \forall y \in X.$$ 

PROOF. For any fixed $y \in X$, let $z_t = \hat{x} + t(y - \hat{x})$. Since $X$ is convex, $z_t \in X$ for all $t \in [0, 1]$. By the assumption, for each $t \in [0, 1]$, we have

$$t \sup_{v \in T(z_t)} f(-\langle v, \eta(y, \hat{x}) \rangle) = \sup_{v \in T(z_t)} f(-\langle v, (1-t)\eta(\hat{x}, \hat{x}) + t \eta(y, \hat{x}) \rangle) = \sup_{v \in T(z_t)} f(-\langle v, \eta(z_t, \hat{x}) \rangle) \leq 0.$$ 

Hence, we have

$$\sup_{v \in T(z_t)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall t \in (0, 1]. \quad (9)$$

Since $T$ is lower semicontinuous along the line segment in $X$, by Definition 2.3, the function

$$x \mapsto g(f, -\eta(y, \hat{x}))(x) = \sup_{v \in T(x)} f(-\langle v, \eta(y, \hat{x}) \rangle)$$

is lower semicontinuous along the line segment in $X$. Since $z_t \to \hat{x}$ as $t \to 0^+$, it follows from (9) that

$$\sup_{u \in T(\hat{x})} f(-\langle u, \eta(\hat{x}, y) \rangle) \leq 0, \quad \forall y \in X.$$

REMARK 3.2. If $T : X \to 2^{L(E,F)}$ is lower semicontinuous from the line segment in $X$ to the $\sigma$-topology of $L(E,F)$, then, by Lemma 2.4, $T$ is lower hemicontinuous along the line segment in $X$ and hence the conclusion of Lemma 3.2 still holds.

LEMMA 3.3. Let $E$ and $F$ be topological vector spaces and $X$ be a nonempty subset of $E$. Suppose that $T : X \to 2^{L(E,F)}$ have nonempty compact values in the $\delta$-topology of $L(E,F)$ and $\eta : X \times X \to E$ is continuous in second argument. Then for each $f \in F^* \setminus \{0\}$ and $y \in X$, the function

$$x \mapsto \inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle)$$

is lower semicontinuous on $X$. 

PROOF. For each $r \in \mathbb{R}$, let $A_r =: \{x \in X : \inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle) \leq r \}$ and $\{x_\alpha\}_\alpha \subset A_r$ with $x_\alpha \to x_0 \in X$, then $\inf_{v \in T(y)} f(-\langle v, \eta(y, x_\alpha) \rangle) \leq r, \forall \alpha \in \Gamma$. By Lemma 2.1 and the compactness of $T(y)$, there exists $\{v_\alpha\}_\alpha \subset T(y)$ such that $f(\langle v_\alpha, -\eta(y, x_\alpha) \rangle) \leq r, \forall \alpha \in \Gamma$. Since $T(y)$ is compact in $\delta$-topology of $L(E,F)$, without loss of generality, we assume that $v_\alpha \to v_0 \in T(y)$. By Lemma 2.2 and the continuity of $\eta$, we have $\langle v_\alpha, -\eta(y, x_\alpha) \rangle \to \langle v_0, -\eta(y, x_0) \rangle$. Noting $f \in F^* \setminus \{0\}$, we obtain

$$\inf_{v \in T(y)} f(-\langle v, \eta(y, x_\alpha) \rangle) \leq f(\langle v_0, -\eta(y, x_0) \rangle) = \lim_{\alpha} f(\langle v_\alpha, -\eta(y, x_\alpha) \rangle) \leq r.$$ 

Hence $x_0 \in A_r$ and $A_r$ is closed for each $r \in \mathbb{R}$. This proves that the function $x \mapsto \inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle)$ is lower semicontinuous on $X$.

LEMMA 3.4. Let $E$ and $F$ be topological vector spaces, $X$ be a nonempty convex subset of $E$ and $D$ be a cone in $F$. Suppose that $T : X \to 2^{L(E,F)}$ is weakly $D$-$\eta$-monotone with nonempty values where $\eta : X \times X \to E$ is such that $x \mapsto \eta(x, y)$ is affine and $\eta(x, x) = 0, \forall x \in X$. Then for each $f \in D^*$, the function $\psi(x, y) = \inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle)$ is $0 - DCV$ in $y$.

PROOF. Let $\{y_1, \ldots, y_n\} \subset X$ be any finite set and $x_0 = \sum_{i=1}^n \lambda_i y_i$ ($\lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1$). By the weak $D$-$\eta$-monotonicity of $T$, for each $u \in T(x_0)$, there exists $u_i \in T(y_i)$ such that

$$\langle u_i - u, \eta(y_i, x_0) \rangle \in D, \quad \forall i = 1, \ldots, n.$$
Since $f \in D^*$, we have

$$f((v_i - u, \eta(y_i, x_0))) \geq 0, \quad \forall i = 1, \ldots, n.$$ 

It follows that

$$f(-\langle v_i, \eta(y_i, x_0) \rangle) \leq f(-\langle u, \eta(y_i, x_0) \rangle), \quad \forall i = 1, \ldots, n.$$ 

Hence, we have

$$\sum_{i=1}^{n} \lambda_i \psi(x_0, y_i) = \sum_{i=1}^{n} \lambda_i \inf_{v \in T(y_i)} f(-\langle v, \eta(y_i, x_0) \rangle) \leq \sum_{i=1}^{n} \lambda_i f(-\langle v_i, \eta(y_i, x_0) \rangle) \leq \sum_{i=1}^{n} \lambda_i f(-\langle u, \eta(y_i, x_0) \rangle) = f(-\langle u, \eta(x_0, x_0) \rangle) = 0.$$ 

This proves that $\psi(x, y)$ is $0 - DCV$ in $y$.

**Lemma 3.5.** Let $E$, $F$, $X$, and $D$ be same as in Lemma 3.4. Suppose that $T : X \rightarrow 2^{L(E, F)}$ is $D$-$\eta$-monotone with nonempty values and $\eta : X \times X \rightarrow E$ is such that $x \mapsto \eta(x, y)$ is affine and $\eta(x, x) = 0$, $\forall x \in X$. Then for any $f \in D^*$, the function $\psi(x, y) = \sup_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle)$ is $0 - DCV$ in $y$.

**Proof.** The proof is similar as in the proof of Lemma 3.4, so we omit it.

**Lemma 3.6.** Let $D$ be a cone of a topological vector space $F$ with $\text{int} D \neq \emptyset$ and $D^*$ be the dual cone of $D$. If $f \in D^*$ and $x_0 \in F$ are such that $f(x_0) < 0$, then either $x_0 \notin \text{int} (D)$ or $f = 0$.

**Proof.** If $x_0 \in \text{int} D$, then there is an open neighborhood $V$ of 0 in $F$ such that $x_0 + V \subseteq D$. Since $V$ is absorbed, i.e., for each $x \in F$, there exists a $\lambda > 0$ such that $\lambda x \in V$, and hence, $x_0 + \lambda x \in x_0 + V \subseteq D$. It follows from $f \in D^*$ that $f(x_0 + \lambda x) \geq 0$. Noting $f(x_0) \leq 0$, we have $\lambda f(x) \geq -f(x_0) \geq 0$. Hence we obtain $f(x) \geq 0$, $\forall x \in F$ and $f = 0$.

**Remark 3.3.** Lemma 3.6 generalizes Proposition 19.3(b) in [32] to topological vector spaces.

Now, we shall show several existence theorems for solutions of the (GVQVLIP).

**Theorem 3.1.** Let $E$ and $F$ be topological vector spaces and the dual space $E^*$ of $E$ separate the points of $E$. Let $X$ be a nonempty paracompact convex subset of $E$ and $C : X \rightarrow 2^F$ be such that for each $x \in X$, $C(x)$ is a closed point convex cone in $F$ with $\text{int} C(x) \neq \emptyset$. Suppose that $G : X \rightarrow 2^X$ is upper hemi-continuous with nonempty compact convex values, $\eta : X \times X \rightarrow E$ is such that $y \mapsto \eta(x, y)$ is continuous, $x \mapsto \eta(x, y)$ is affine and $\eta(x, x) = 0$, $\forall x \in X$ and $T : X \rightarrow 2^{L(E, F)}$ is $C_\eta$-$\eta$-monotone and upper hemi-continuous along the line segment in $X$ with nonempty $\sigma$-compact values. Suppose that there exists $f \in C_+ \setminus \{0\}$ such that

1. the set \( \{ x \in X : \sup_{y \in C(x)} \sup_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle) > 0 \} \) is open in $X$,
2. there exist a nonempty compact convex subset $X_0$ of $X$ and a nonempty compact subset $K$ of $X$ such that for each $x \in X \setminus K$, there is $y \in G(x) \cap \text{co } (X_0 \cup \{x\})$ satisfying

$$\sup_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle) > 0.$$ 

Then the (GVQVLIP) has a solution $\hat{x} \in X$.

If $T(\hat{x})$ is also convex, then then (GVQVLIP) has a strong solution $\hat{x} \in X$.

**Proof.** Define a function $\psi : X \times X \rightarrow \mathbb{R}$ by

$$\psi(x, y) = \sup_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle).$$
Since for each $y \in X$, $x \mapsto \eta(y, x)$ is continuous, therefore for any $v \in T(y)$, $x \mapsto f(-\langle v, \eta(y, x) \rangle)$ is also continuous, and hence, for any $y \in X$, the function $x \mapsto \psi(x, y)$ is lower semicontinuous on $X$. By the $C_+\eta$-monotonicity of $T$ and Lemma 3.5, $\psi(x, y) = 0 - DCV$ in $y$. It is easy to see that all conditions of Lemma 2.5 are satisfied. By Lemma 2.5, there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and $\sup_{y \in G(\hat{x})} \psi(\hat{x}, y) \leq 0$, i.e.,

$$\sup_{v \in T(y)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}). \quad (10)$$

It follows that

$$\inf_{v \in T(y)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}). \quad (11)$$

Since $G(\hat{x})$ is convex and $T$ is upper semicontinuous along the line segment in $X$, by (11) and Lemma 3.1, we have

$$\inf_{u \in T(\hat{x})} f(-\langle u, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}). \quad (12)$$

Since for each fixed $y \in X$, $u \mapsto f(-\langle u, \eta(y, \hat{x}) \rangle)$ is continuous in the $\sigma$-topology of $L(E,F)$ by Lemma 2.1 and $T(\hat{x})$ is $\sigma$-compact, it follows from (12) that for each $y \in G(\hat{x})$ there exists $\hat{u} \in T(\hat{x})$ such that

$$f(-\langle \hat{u}, \eta(y, \hat{x}) \rangle) \leq 0.$$

By Lemma 3.6, we have $-\langle \hat{u}, \eta(y, \hat{x}) \rangle \notin \text{int } C_+$. Since $C_+ = \text{co } \{C(x) : x \in X\}$, we must have $\langle \hat{u}, \eta(y, \hat{x}) \rangle \notin \text{int } C(\hat{x})$. This shows that $\hat{x}$ is a solution of the (GVQVIP).

Now assume that $T(\hat{x})$ is also convex. Define the mapping $\phi : T(\hat{x}) \times G(\hat{x}) \to \mathbb{R}$ by

$$\phi(u, y) = f(-\langle u, \eta(y, \hat{x}) \rangle).$$

By assumptions, $T(\hat{x})$ is $\sigma$-compact convex, $G(\hat{x})$ is convex and by Lemma 2.1, the function $u \mapsto f(-\langle u, \eta(y, \hat{x}) \rangle)$ is linear and continuous under the $\sigma$-topology of $L(E,F)$. Since $y \mapsto \eta(y, \hat{x})$ is affine, it is easy to see that $y \mapsto \phi(u, y)$ is concave. By Lemma 2.6, we have

$$\inf_{u \in T(\hat{x})} \sup_{y \in G(\hat{x})} f(-\langle u, \eta(y, \hat{x}) \rangle) = \sup_{y \in G(\hat{x})} \inf_{u \in T(\hat{x})} f(-\langle u, \eta(y, \hat{x}) \rangle) \leq 0.$$

Since $u \mapsto \sup_{y \in G(\hat{x})} f(-\langle u, \eta(y, \hat{x}) \rangle)$ is lower semicontinuous in the $\sigma$-topology of $L(E,F)$ and $T(\hat{x})$ is $\sigma$-compact, there exists $\hat{u} \in T(\hat{x})$ such that

$$f(-\langle \hat{u}, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}).$$

Since $f \in C_+ \setminus \{0\}$ and $\text{int } C_+ \neq \emptyset$, by Lemma 3.6, we obtain

$$-\langle \hat{u}, \eta(y, \hat{x}) \rangle \notin \text{int } C_+, \quad \forall y \in G(\hat{x}).$$

Since $C_+ = \text{co } \{C(x) : x \in X\}$, we must have

$$\langle \hat{u}, \eta(y, \hat{x}) \rangle \notin \text{int } C(\hat{x}), \quad \forall y \in G(\hat{x}),$$

i.e., $\hat{x}$ is a strong solution of the (GVQVLIP).

**Theorem 3.2.** Let $E$, $F$, $E^*$, $X$, $\eta : X \times X \to E$, $C : X \to 2^E$, and $G : X \to 2^X$ be same as that in Theorem 3.1. Suppose that $T : X \to 2^{L(E,F)}$ is weakly $C_+\eta$-monotone and upper hemicontinuous along the line segment in $X$ with nonempty $\delta$-compact values. If there exists $f \in C_+ \setminus \{0\}$ such that

1. the set $\{x \in X : \sup_{y \in G(x)} \inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle) > 0\}$ is open in $X$,
2. there exist a nonempty compact convex subset $X_0$ of $X$ and a nonempty compact subset $K$ of $X$ such that for each $x \in X \setminus K$ there is $y \in G(x) \cap \text{co } (X_0 \cup \{x\})$ satisfying

$$\inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle) > 0.$$

Then the (GVQVLI) has a solution $\hat{x} \in X$. 

If $T(\hat{x})$ is also convex, then the (GVQVLIP) has strong solution $\hat{x} \in X$.

**Proof.** Define a function $\psi : X \times X \rightarrow \mathbb{R}$ by

$$
\psi(x, y) = \inf_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle).
$$

By Lemma 3.3, the function $x \mapsto \psi(x, y)$ is lower semicontinuous. Since $T$ is weakly $C_+\eta$-monotone, by Lemma 3.4, $\psi(x, y)$ is $0 - DCV$ in $y$. It is easy to check that all conditions of Lemma 2.5 are satisfied. By Lemma 2.5, there exists a point $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and $\psi(\hat{x}, y) \leq 0$, $\forall y \in G(\hat{x})$, i.e.,

$$
\inf_{v \in T(y)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}).
$$

The remainder of the proof is same as that in the proof of Theorem 3.1.

**Remark 3.3.** Theorems 3.1 and 3.2 improve Theorem 3.2 of Chen and Li [17] in the following ways:

1. $E$ and $F$ may be infinite-dimensional topological vector spaces;
2. the continuity assumption of $T$ is weaker;
3. $G : X \rightarrow 2^X$ may be upper hemicontinuous;
4. the domain $X$ of $G$ and $T$ may not be compact. If $X$ is compact, then Condition 2 in Theorems 3.1 and 3.2 is satisfied trivially.

**Theorem 3.3.** Let $E, F, E^*, X, \eta : X \times X \rightarrow E$, and $G : X \rightarrow 2^X$ be same as that in Theorem 3.1. Suppose that $T : X \rightarrow 2^{L(E, F)}$ is $C_+\eta$-monotone and lower semicontinuous along the line segment in $X$ with nonempty values. If there exists a $f \in C_+^* \setminus \{0\}$ such that Conditions 1 and 2 of Theorem 3.1 are satisfied, then there exists a $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and

$$
\langle u, \eta(y, \hat{x}) \rangle \notin \text{int } C(\hat{x}), \quad \forall y \in G(\hat{x}) \text{ and } u \in T(\hat{x}).
$$

**Proof.** Define a function $\psi : X \times X \rightarrow \mathbb{R}$ by

$$
\psi(x, y) = \sup_{v \in T(y)} f(-\langle v, \eta(y, x) \rangle).
$$

By using same argument as in the proof of Theorem 3.1, there exists a $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$ and

$$
\sup_{v \in T(y)} f(-\langle v, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}).
$$

Since $G(\hat{x})$ is convex and $T$ is lower hemicontinuous along the line segment in $X$ with nonempty values, by Lemma 3.2, we have

$$
\sup_{u \in T(\hat{x})} f(-\langle u, \eta(y, \hat{x}) \rangle) \leq 0, \quad \forall y \in G(\hat{x}).
$$

By Lemma 3.6, we obtain

$$
\langle u, \eta(y, \hat{x}) \rangle \notin \text{int } C(\hat{x}), \quad \forall y \in C(\hat{x}) \text{ and } u \in T(\hat{x}).
$$

**Remark 3.4.** If $F = \mathbb{R}$ and $P = [0, \infty)$, then Theorem 3.3 generalizes Theorem 1 in [22] in the following ways:

1. $X$ may not be compact;
2. $E$ may not be locally convex space;
3. $T : X \rightarrow 2^{E^*}$ may not be lower semicontinuous form the line segment in $X$ to the weak* topology of $E^*$. 
REFERENCES


