Norm estimates for generalized translation operators associated with a singular differential operator

by C. Markett

Lehrstuhl A für Mathematik, Rheinisch-Westfälische Technische Hochschule Aachen,
Templergraben 55, D-5100 Aachen, B.R.D.

Communicated by Prof. J. Korevaar at the meeting of March 26, 1984

ABSTRACT

Delsarte's approach via the solution of a Cauchy problem is used to estimate generalized translation operators associated with a singular Sturm-Liouville differential operator of the form

\[ D_\alpha x = \left( \frac{d^2}{dx^2} \right) + \left( \frac{2\alpha + 1}{x} \right) \frac{d}{dx} - q(x), \quad \alpha \geq -1/2, \quad 0 < x < \infty. \]

Such estimates are fundamental for establishing a convolution structure. As a general purpose, the relation between translations belonging to different potential functions \( q \) are studied. This leads to norm estimates for classes of "related" translation operators mainly under hypotheses on the Riemann function associated with an explicitly given member of the class. Specifically, our aim is to investigate perturbations of the Hankel translation as well as of the Laguerre translation, the underlying Sturm-Liouville equations of which are prototypes of singular equations on the positive half-axis with continuous and discrete spectra, respectively.

1. INTRODUCTION

For many problems of harmonic analysis of eigenfunction expansions associated with a Sturm-Liouville equation it is indispensable to have a group structure, i.e., to find an appropriate convolution which plays the part of the usual convolution in case of ordinary Fourier series. The convolution is fundamental for questions of convergence and summability of eigenfunction expansions, maximal function inequalities and multiplier theorems, for instance. The most developed example in this respect is the ultraspherical polynomial expansion, cf., e.g., [5], [15].

* Supported by the Deutsche Forschungsgemeinschaft under grant No. GO 261/5-1.
Let a Sturm-Liouville equation (S.-L. equation) be given in self-adjoint form by

\[ \frac{d}{dx} \left[ p(x) \frac{d}{dx} u(x) \right] + \left[ \lambda w(x) - \tilde{q}(x) \right] u(x) = 0 \]

on some interval \( I \), where \( \lambda \) is a parameter, \( p, w \) are positive in the interior of \( I \), and \( p, w, \tilde{q} \) satisfy certain smoothness conditions. If \( p \) or \( w \) vanishes at one or both endpoints of \( I \) or if \( I \) is unbounded, equation (1.1) is called singular. We are mainly interested in singular equations whereas regular equations appear as limiting cases. According as the spectrum of (1.1), say \( S \), is discrete or continuous, the expansion of a given function in the associated eigenfunction system \( \{ u_\lambda \}_{\lambda \in S} \) is of series or integral form, respectively. Both types will occur here.

The convolution structure for such an expansion consists in the definition of an appropriate convolution product, together with a set of function spaces on which there are "nice" estimates of the product, e.g., in the form of a Young-type inequality. This can be achieved by introducing a generalized translation operator and by deriving suitable norm estimates. The latter task will be the subject of the present paper.

In connection with the self-adjoint DE (1.1), the most natural choice of function spaces will be \( L^p_w, 1 \leq p \leq \infty \), with weight \( w(x) \). But this does not always guarantee the existence of satisfactory norm estimates since the given S.-L. equation may have to be "normalized" by means of a suitable transformation. Mostly, however, the main obstacle is the lack of a closed representation of the generalized translation. Due to the orthogonality of \( \{ u_\lambda \}_{\lambda \in S} \) and in view of the defining property

\[ T_\gamma(u_\lambda; x) = u_\lambda(x)u_\lambda(y) \quad (x, y \in I; \ \lambda \in S) \]

of the translation operator \( T_\gamma \), a formal representation, at least, is given by an integral operator in kernel form, the kernel

\[ K(x, y, z) = \int c_\lambda \varphi_\lambda(x)\varphi_\lambda(y)\varphi_\lambda(z) \]

being a triple product sum or integral. For some of the classical (singular) eigenfunctions, like the Jacobi polynomials on \([-1, 1]\) or the Jacobi, Bessel, and Laguerre functions on \([0, \infty)\), a closed representation of (1.3) is available via the corresponding product formula, by means of which \( T_\gamma \) can be estimated, cf., e.g., [8], [10], [11], [14]. In the Jacobi polynomial case, one has also dealt with (1.3) directly by using asymptotic representations of \( \varphi_\lambda \) and other special function arguments, cf. [1], [9]. In particular cases it could be shown that \( K(x, y, z) \) is positive, the uniform boundedness of \( T_\gamma \) then being immediate. Even though these results are deep and important, they only cover an infinitely small part of the set of general singular S.-L. expansions. Moreover, they depend on arguments which are either too special or too complicated to be readily extendable to some larger class of expansions.

A promising way to treat more general eigenfunction expansions is to draw
directly from the **DE** (1.1) the information needed. By (1.1) the product on the right hand side of (1.2), when interpreted as a function of two variables, is the solution of a Cauchy problem (cf. (1.5) below), so that the rich theory of partial differential equations is at hand. For instance, Weinberger's maximum theorem for hyperbolic **PDE** has been used in verifying the positivity of generalized translations; cf., e.g., Koornwinder [12] for Jacobi series and Chébli [4] or Connett and Schwartz [6] for certain other S.-L. expansions, as well as the literature cited there. This method fails, however, in cases like the Laguerre translation, which is not positive, but nevertheless of favorable norm behavior [10].

Another approach is to solve the hyperbolic initial value problem by Riemann's method. The translation kernel is then obtained from the associated Riemann function. This approach will be used in the present paper. It goes back to an idea of Delsarte [7] (cf. [3] for a summary of the historical background). Though it is a rare event that a Riemann function can be determined explicitly (cf. [13], [14]), the advantage of the approach is that, given two "related" equations, there exists an integral relation between the corresponding Riemann functions. This will be used to carry over the norm estimates from one translation which is explicitly known (and whose Riemann function must be known) to a class of "related" translations, without needing explicit representations for them.

More explicitly we consider singular differential equations \( D^\alpha q,u(x) + \lambda u(x) = 0 \) on the positive half-axis, where the differential operator is of the form (or can be transformed into)

\[
D^\alpha q = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx} - q(x) \quad (\alpha \geq -\frac{1}{2}),
\]

and where \( u \) is supposed to satisfy the boundary conditions \( u(0) = 1, u'(0) = 0 \). The parameter \( \alpha \) describes the order of the regular singularity at the origin, while further important information is contained in the potential function \( q \). Typical differential equations to be admitted are (suitably transformed versions of) the Bessel **DE**, the Laguerre **DE**, as well as the non-compact Jacobi **DE**.

The generalized translation operator \( T^\alpha q,u(f;x) = u(x,y) \), where \( u(x,y) \) is the solution of the Cauchy problem

\[
\begin{align*}
D^\alpha q,x,y u(x,y) &= 0 \quad (0 < y \leq x) \\
u(x,0) &= f(x), u_x(x,0) = \frac{\partial}{\partial y} u(x,y) \bigg|_{y=0} = 0 \quad (x > 0),
\end{align*}
\]

with \( D^\alpha q,x,y = D^\alpha q,x - D^\alpha q,y \), where \( f \) is supposed to be sufficiently smooth and \( u(x,y) \) is extended symmetrically to \( 0 < x < y \). Clearly (1.5) implies the relation (1.2). In the preliminary § 2 we provide the basic representation of a translation kernel in terms of the corresponding Riemann function as it follows by
Riemann's method. Moreover, for two particular cases the Riemann functions as well as the resulting translation kernels will be given explicitly, namely for the potential functions $q_1 = 0$ and $q_2(x) = x^2$. In the first case, the operator (1.4) is the Bessel differential operator which has been considered by Delsarte already, while in the second case (1.4) is a certain form of the Laguerre differential operator. The corresponding translations are called the Hankel and Laguerre translation, respectively. Our purpose will be to investigate translations which arise from perturbations of such potentials. Therefore $q_1$ and $q_2$ will be called reference potentials here.

Actually, the paper aims in two directions: On the one hand we study, from a more general point of view, the ingredients which are crucial for transferring norm boundedness from a reference translation to its perturbations (cf. § 3). This results in the general Theorem 1 in § 4 which is based on hypotheses described in terms of the Riemann function of the known translation and in terms of some measure for the perturbation. Here we strive for practicable conditions upon the Riemann function. Indeed, the latter can be rather complicated: In the present cases it is a simple or double sum, cf. (2.4); it may consist of triple or quadruple sums in other cases.

On the other hand, the applications of the general theorem to perturbed Hankel and Laguerre translations (cf. Corollaries 1 and 2 in § 5) are of independent interest. It will turn out that, due to the deep relationship between the Riemann functions of the Hankel and Laguerre cases (cf. (2.5) below), these can be treated in parallel. Nevertheless, the two cases are independent of each other and they differ, for instance, as to positivity and shape of spectrum. In the Hankel case, the Riemann function approach has been used by Braaksma and de Snoo [2], [3], papers which have much stimulated the present one. See also [17] and, for $\alpha = -\frac{1}{2}$, [16]. As yet the Laguerre case has hardly been investigated. However, compared with the well-developed harmonic analysis of the Jacobi expansion, the Laguerre expansion deserves equal attention since it can be considered as a prototype of discrete, singular eigenfunction expansions on a non-compact interval.

Let us add a few remarks on possible further applications of Theorem 1. A first candidate is the Jacobi function case, for which the translation kernel has recently been derived from the corresponding Riemann function [14]. Moreover, in recent years much progress was made in determining new Riemann functions to which our approach may be applied; cf., e.g., Wallner [18] and the literature cited there. Finally, the present methods may be modified to study also generalized translations associated with S.-L. expansions on a compact interval as well as on the real axis, for which the Jacobi and Hermite polynomial expansions are typical examples. The latter case may be easier to handle, since boundary effects are to be considered in the former.

2. TRANSLATION AND RIEMANN FUNCTION

Let $A_q^{\alpha}(\xi, \eta; x, y)$ denote the Riemann function associated with (1.5), i.e. let $u(\xi, \eta) = A_q^{\alpha}(\xi, \eta; x, y)$ solve the characteristic boundary value problem

302
\[
\begin{aligned}
(D^a_{\alpha, \xi, \eta})^* v(\xi, \eta) & = 0 \text{ if } (\xi, \eta) \in \Delta_{xy} \\
v(\xi, \eta; x, y) &= \left( \frac{\xi \eta}{xy} \right)^{\frac{\alpha + 1}{2}} \text{ if } \xi - \eta = x - y \text{ or } \xi + \eta = x + y,
\end{aligned}
\]

where the asterisk denotes the adjoint operator and \( \Delta_{xy} \) stands for the triangle in the \((\xi, \eta)\)-plane with vertices \((x, y), (x - y, 0), \text{ and } (x + y, 0), 0 < y \leq x\). Supposing that \( A^a_\eta \) and \((2\alpha + 1)/\eta - \partial/\partial \eta) A^a_\eta \) are continuous on \( \Delta_{xy} \), an application of Green’s theorem yields (cf. [3], [13])

\[
u(x, y) = \begin{cases}
\int_{x-y}^{x+y} f(\xi) w^a_\eta(x, y, \xi) d\xi, & \alpha > -\frac{1}{2} \\
\frac{1}{2} \{ f(x - y) + f(x + y) \} + \int_{x-y}^{x+y} f(\xi) w^{-1/2}_\eta(x, y, \xi) d\xi, & \alpha = -\frac{1}{2},
\end{cases}
\]

where

\[w^a_\eta(x, y, \xi) = \lim_{\eta \to 0^+} w^a_\eta(\xi, \eta; x, y),\]

\[
G^a_\eta(\xi, \eta; x, y) = \left( \frac{2\alpha + 1}{\eta} - \frac{\partial}{\partial \eta} \right) A^a_\eta(\xi, \eta; x, y)
\]

\[= -\frac{1}{2} \eta^{2\alpha + 1} \frac{\partial}{\partial \eta} [\eta^{-2\alpha - 1} A^a_\eta(\xi, \eta; x, y)].\]

For \(0 < x < y\) one defines \(u(x, y) = u(y, x)\). Hence, for the generalized translation operator \(T^a_{\alpha, y}\) one obtains the representation

\[
T^a_{\alpha, y}(f; x) = \begin{cases}
\int_0^\infty f(z) K^a_\eta(x, y, z) z^{2\alpha + 1} dz, & \alpha > -\frac{1}{2} \\
\frac{1}{2} \{ f(|x - y|) + f(x + y) \} + \int_0^\infty f(z) K^{-1/2}_\eta(x, y, z) dz, & \alpha = -\frac{1}{2}
\end{cases}
\]

with kernel

\[K^a_\eta(x, y, z) = \begin{cases}
z^{-2\alpha - 1} w^a_\eta(x, y, z), & 0 \leq x - y < z < x + y \\
z^{-2\alpha - 1} w^a_\eta(y, x, z), & 0 \leq y - x < z < x + y \\
0 \text{ elsewhere.}
\end{cases}\]

Under the mild condition \(q \in L(0, 1)\), it can be shown that \(K^a_q\) is symmetric in its three variables [3; Thm. 6].
In particular, for \( q_1 = 0 \) and \( q_2(x) = x^2 \) the above quantities can be specified as follows (cf. [13]). Let

\[
\chi(\xi, \eta; x, y) = \frac{1}{16} \left[ (x + y)^2 - (\xi + \eta)^2 \right] \left[ (\xi - \eta)^2 - (x - y)^2 \right],
\]

\[
\psi(\xi, \eta; x, y) = \frac{1}{16} \left[ (x + y)^2 - (\xi - \eta)^2 \right] \left[ (\xi + \eta)^2 - (x - y)^2 \right],
\]

\[\phi(\xi, \eta; x, y) = \chi/\psi,\]

so that \( 1 - \phi = \xi \eta xy / \psi \geq 0 \). Then for each \( \alpha \geq -\frac{1}{2} \), the corresponding Riemann functions are

\[
A_\alpha^\alpha(\xi, \eta; x, y) = \left( \frac{\xi \eta}{xy} \right)^{\alpha + 1/2} (1 - \phi)^{1/2 - |\alpha|} a_\alpha(\xi, \eta; x, y) \quad (i = 1, 2),
\]

where

\[
a_\alpha(\xi, \eta; x, y) = F(\frac{1}{2} - |\alpha|, \frac{1}{2} - |\alpha|; 1; \phi),
\]

\[
a_\alpha(\xi, \eta; x, y) = \sum_{n=0}^{\infty} S(n, \frac{1}{2} - |\alpha|; \phi, \chi) \frac{(1/2 - |\alpha|)_n (1/2 - |\alpha|)_n}{n!} \phi^n.
\]

Here \( F \) denotes the hypergeometric function and

\[
S(n, \frac{1}{2} - |\alpha|; \phi, \chi) = \sum_{k=0}^{\infty} F(n + \frac{1}{2} - |\alpha|, k; k + n + 1; \phi) \frac{n!(1 - \phi)^k}{k!(k + n)!}.
\]

In [13] it has been shown that \( S \) is uniformly bounded by 1, so that

\[
|A_\alpha^\alpha(\xi, \eta; x, y)| \leq A_\alpha^\alpha(\xi, \eta; x, y) \quad (\alpha \geq -\frac{1}{2}; (\xi, \eta) \in \Delta xy).
\]

An application of (2.2), (2.3) yields the respective translation kernels

\[
K_{\alpha_1}^\alpha(x, y, z) = \begin{cases} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1/2)\Gamma(1/2)} (xyz)^{-2\alpha} y^{2\alpha - 1}, & \alpha > -\frac{1}{2} \\ 0, & \alpha = -\frac{1}{2}, \end{cases}
\]

\[
K_{\alpha_2}^\alpha(x, y, z) = \begin{cases} K_{\alpha_1}^\alpha(x, y, z) y^{1/2}(\alpha), & \alpha > -\frac{1}{2} \\ -\frac{1}{2}xyz f_1(\alpha), & \alpha = -\frac{1}{2} \end{cases}
\]

for \(|x - y| < z < x + y\), where \( f_v \) denotes the Bessel function

\[
f_v(x) = 2^v \Gamma(v + 1)x^{-v} f_v(x) = \sum_{k=0}^{\infty} \frac{\Gamma(v + 1)(-1)^k (x/2)^{2k}}{k! \Gamma(k + v + 1)} \quad (v > -1)
\]

and

\[
\varphi = \varphi(x, y, z) = \frac{1}{4} (2[x^2 y^2 + x^2 z^2 + y^2 z^2] - x^4 - y^4 - z^4)^{1/2}
\]

\[
= \frac{1}{4} [(x+y)^2 - z^2, (x-y)^2)]^{1/2}.
\]
In (2.2) the limits as $\eta \rightarrow 0^+$ hold uniformly with respect to the other parameters. For instance, one has (cf. [13])

$$G_{q_1}^\alpha(\xi, \eta; x, y) = K_{q_1}^\alpha(x, y, \xi)\xi^{2\alpha + 1} + O(h(\eta)) \quad (\eta \rightarrow 0^+),$$

where $h(\eta) = \eta$ if $|\alpha| \geq \frac{1}{2}$, $= \eta^{2\alpha}$ if $0 < \alpha < \frac{1}{2}$, $= \eta \log 1/\eta$ if $\alpha = 0$, and $= \eta^{2\alpha + 1}$ if $-\frac{1}{2} < \alpha < 0$.

Since $|j,v(x)| \leq 1$ if $v \geq -\frac{1}{2}$ [19; 3.31(1)], it follows that

$$|K_{q_1}^\alpha(x, y, z)| \leq K_{q_1}^\alpha(x, y, z) \quad (\alpha \geq 0).$$

Notice that, in contrast to (2.5), one has to confine $\alpha$ to non-negative values here.

3. INTEGRAL RELATIONS FOR RIEMANN FUNCTIONS AND TRANSLATION KERNELS

Suppose that for a certain reference potential $q_r$ the Riemann function $A_{q_r}^\alpha$ of $D^\alpha_{q_r,x,y}$ is explicitly known, and let $A_{q_r,Q}^\alpha(\xi, \eta; x, y)$ denote the Riemann function of

$$D^\alpha_{q_r,x,y} - Q(x, y),$$

where $Q(x, y)$ is a continuous function defined for $0 < y \leq x$. Then one has

$$(3.2) \quad A_{q_r,Q}^\alpha(\xi, \eta; x, y) = A_{q_r}^\alpha(\xi, \eta; x, y)$$

$$- \frac{1}{2} \int_{\Omega} A_{q_r}^\alpha(\xi, \eta; s, t)Q(s, t)A_{q_r,Q}^\alpha(s, t; x, y)dsdt,$$

where $\Omega$ is the rectangle

$$\Omega(x, y; \xi, \eta) = \{(s, t); \xi + \eta < s + t < x + y, x - y < s - t < \xi - \eta\}.$$ 

Obviously the structure of (3.2) does not depend on the particular choice of $q_r$ since the boundary conditions of (2.1) do not. Hence (3.2) is of the same form as in [3; (2.5)].

The usual method for solving (3.2) is to use successive approximations. Thus principally, $A_{q_r,Q}^\alpha$ can be represented by an infinite series as far as its convergence is verified. But since the terms of the series are defined only recursively, in general they cannot be determined explicitly. Nevertheless, one can give the following estimates (cf. [3; Theorem 1, Corollary 1]). For convenience we throughout omit the case $\alpha = 0$, which may be treated by an analytic continuation argument with respect to $\alpha$ (cf. [3]). We further use the notation

$$L_w(0, y) = \{f; \int_0^y |f(x)|x^ydx < \infty\}.$$ 

PROPOSITION 1. Let $0 < y \leq x$ and suppose that $Q(s, t)$ is a measurable function on $A_{xy}$ satisfying $|Q(s, t)| \leq Q(t)$, where $Q(t) \in L_w^1(0, y)$. Moreover, for $\alpha \geq -\frac{1}{2}, \alpha \neq 0$, let a function $R_0(\xi, \eta; x, y) \leq M_0$ be given as well as constants $M_0, M_1$ such that

$$(3.3) \quad |A_{q_r}^\alpha(\xi, \eta; x, y)| \leq M_0 R_0(\xi, \eta; x, y)$$
for each $Q \in L_1^\infty(0,y)$. Then for each $(\xi, \eta) \in A_{x,y}$
\[ |A^a_{q_1, Q}(\xi, \eta; x, y) - A^a_{q_2}(\xi, \eta; x, y)| \leq M_0 R_a(\xi, \eta; x, y) \{ \exp \left( M \int_0^y tQ(t)dt \right) - 1 \} \]
holds, where $M$ can be chosen to be $\frac{1}{4}M_0 M_1$. Consequently,
\[ |A^a_{q_1, Q}(\xi, \eta; x, y)| \leq M_0 R_a(\xi, \eta; x, y) \exp \left( M \int_0^y tQ(t)dt \right). \]

In the two cases $q_r = q_1$ and $q_r = q_2$, the hypotheses (3.3), (3.4) are satisfied if, for $R_a(\xi, \eta; x, y)$,
\[ R_a(\xi, \eta; x, y) = \left( \frac{\xi u}{xy} \right)^{a + 1/2} (1 - \phi)^{1/2 - |\alpha|} \]
is chosen, as well as $M_0 = \Gamma(2|\alpha|)/\Gamma^2(|\alpha| + \frac{1}{2})$, $M_1 = 2$ if $|\alpha| \geq \frac{1}{2}$, and
\[ M_1 = 2^{2 - 4|\alpha|} \Gamma(\frac{1}{2}) \Gamma(|\alpha| + \frac{1}{2})/\Gamma(|\alpha| + 1) \text{ if } 0 < |\alpha| < \frac{1}{2}. \]
Indeed, (3.3) then follows by (2.4), (2.5), using the upper bound of the hypergeometric function. Inequality (3.4) was verified for $\alpha < \frac{1}{2}$ in [2], while, for $\alpha > \frac{1}{2}$, it can be deduced from
\[ 0 \leq \frac{\psi(\xi, \eta; s, t)\psi(s, t; x, y)}{\psi(\xi, \eta; x, y)s^2 t^2} \leq 1 \quad ((s, t) \in \Omega(x, y; \xi, \eta)), \]
which is obtained by a somewhat wearisome determination of the relative maxima with respect to $s$ and $t$ of the above quotient. We omit the details.

In view of (2.2), equation (3.2) can also be used to derive an integral relation between the respective translation kernels. Indeed,
\[ G^a_{q_1, Q}(\xi, \eta; x, y) = G^a_{q_1}(\xi, \eta; x, y) \]
\[ - \frac{1}{2} \int_{\Omega(\xi, \eta; x, y)} G^a_{q_1}(\xi, \eta; s, t)Q(s, t)A^a_{q_1, Q}(s, t; x, y)dsdt \]
\[ - \frac{1}{2} \left( \int_{x-y}^{x+y} + \int_{\xi-\eta}^{\xi+\eta} \right) A^a_{q_1}(\xi, \eta; \sigma, \tau)Q(\sigma, \tau)A^a_{q_1, Q}(\sigma, \tau; x, y)du, \]
where
\[ (\sigma, \tau) = \left( \frac{\xi + \eta + u}{2}, \frac{\xi + \eta - u}{2} \right) \text{ if } x - y < u < \xi - \eta \]
and
\[(\sigma, \tau) = \left(\frac{u + \xi - \eta}{2}, \frac{u - \xi + \eta}{2}\right) \text{ if } \xi + \eta < u < x + y.\]

In view of the boundary values of \(A_q^\alpha\), one has
\[
A_q^\alpha(\xi, \eta; \sigma, \tau) = \begin{cases} 
\frac{(4\xi \eta)}{(\xi + \eta)^2 - u^2} \sigma^{1/2} & \text{if } x - y < u < \xi - \eta \\
\frac{(4\xi \eta)}{u^2 - (\xi - \eta)^2} \sigma^{1/2} & \text{if } \xi + \eta < u < x + y,
\end{cases}
\]
which is equal to 1 if \(\alpha = -\frac{1}{4}\) and tends to 0 as \(\eta \to 0^+\) if \(\alpha > -\frac{1}{4}\). Under the assumption that \(w(x, y, u, 0, t) = \lim_{\eta \to 0^+} G_{q, Q}(\xi, \eta; s, t)\) holds uniformly for \((s, t) \in \Omega(x, y; \xi, \eta)\), it follows that \(G_{q, Q}(\xi, \eta; x, y)\) is continuous for \((\xi, \eta) \in A_{xy}\), \(0 < y \leq x\), and in the limit \(\eta \to 0^+\) one arrives at
\[(3.7) \quad w_{q, Q}^\alpha(x, y, \xi) = w_{q, Q}^\alpha(x, y, \xi) - \frac{1}{\Omega(x, y, 0)} \int_0^\infty w_{q, Q}^\alpha(s, t, \xi)Q(s, t)A_{q, Q}(s, t, x, y)dsdt\]
for \(\alpha > -\frac{1}{4}\). In case \(\alpha = -\frac{1}{4}\) one still has to add
\[(3.8) \quad -\frac{1}{8} \int_{x-y}^{x+y} Q\left(\frac{u + \xi}{2}, \left|\frac{u - \xi}{2}\right|\right)A_{q, Q}\left(\frac{u + \xi}{2}, \left|\frac{u - \xi}{2}\right|, x, y\right)du\]
to the right-hand side of (3.7).

4. NORM ESTIMATES FOR TRANSLATION OPERATORS

The weighted Lebesgue spaces to be used are
\[L_w^p(2\alpha + 1)(0, \infty) = \begin{cases} 
\{f; \{ \int_0^\infty |f(x)|^p x^{2\alpha + 1}dx\}^{1/p} < \infty\}, & 1 \leq p < \infty \\
\{f; \text{ess sup}_{x>0} |f(x)| < \infty\}, & p = \infty,
\end{cases}\]
where the weight function \(w(2\alpha + 1; x) = x^{2\alpha + 1}\) is chosen according to the self-adjoint form of the underlying S.-L. equation.

From now on we assume that the potential function \(q\) in the differential operator (1.4) is decomposed into a reference and a perturbation potential, \(q = q_r + q_p\), and that \(Q(x, y) = q_p(x) - q_p(y)\) in (3.1). Henceforth we write \(A_q^\alpha\) instead of \(A_{q_r, Q}\), etc. If the reference kernel \(w_q^\alpha(x, y, z)\) is explicitly known (as a uniform limit of \(G_{q, Q}(z, \eta; x, y)\) as \(\eta \to 0^+\)), one may derive via (3.7) and (3.5) an upper bound for \(w_q^\alpha(x, y, z)\) to obtain an estimate for
\[(4.1) \quad \|T_{q, y}^\alpha\|_{L_w^p(2\alpha + 1)(0, \infty)} = \sup_{0 \leq z < \infty} \int_0^\infty |K_q^\alpha(x, y, z)| x^{2\alpha + 1}dx.\]
Braaksma and de Snoo [2], [3] went this way for \(q_r = q_1\). In the present paper, however, we postpone using special information about \(w_q^\alpha\) as long as possible.
The gain for application will be a considerable simplification of the hypotheses, since estimating the double integral over $\Omega$ in (3.7) and subsequently integrating with respect to $x$, which may be lengthy and involved, is now replaced by just verifying the conditions (4.2) and (4.3) below.

**THEOREM 1.** Let $\alpha \geq -\frac{1}{2}$, $\alpha \neq 0$, $y > 0$, and let $q$ be decomposed into $q_r$ and $q_p$ such that the following conditions are satisfied.

(i) There is a function $Q \in L^1_w(0, y) \cap L^w_{w(\alpha - |\alpha| + 1)}(0, y)$ such that

$$|q_r(s) - q_p(t)| \leq Q(t) \quad \text{for} \quad 0 < t < y, \quad t < s < \infty;$$

(ii) there is a function $R_q(\xi, \eta; x, y)$ for $x > 0$, $0 < \eta < \min (x, y)$, $|x - y| + \eta < \xi < x + y - \eta$ and there are constants $M_0, M_1, M_2$ such that, simultaneously, the conditions (3.3), (3.4) of Proposition 1 are satisfied if $x \geq y$, as well as

$$R_q(\xi, \eta; x, y) = R_q(\xi, \eta; y, x)$$

and

$$\int_{|x - y| + \eta}^{x + y - \eta} R_q(\xi, \eta; x, y) x^{2\alpha + 1} d\xi \leq M_2 x^{2\alpha + 1} \eta^{|\alpha| + 1} y |\alpha - \alpha|.$$

Then the translation $T^\alpha_q$, is a bounded operator from $L^1_w(0, \infty)$ into itself, provided $T^\alpha_q$, is bounded for $0 \leq t \leq y$, and one has

$$\sup_{0 \leq t \leq y} \|T^\alpha_q, t\|_{L^1_w(0, \infty)} \leq \frac{M_3 \exp \{M y |\alpha - \alpha| Q\}}{M_1 M_2 M_3^2}.$$

where, e.g., $M = \frac{1}{2}M_0 M_1$ and $M_3 = \max (1, M_2/M_1)$ if $\alpha > -\frac{1}{2}$, $\alpha \neq 0$, and $M_3 = \max (1, 2M_2/M_1)$ if $\alpha = -\frac{1}{2}$. Moreover, if $\alpha > 0$ and $Q \in L^1_w(0, \infty)$ and if the norm of $T^\alpha_q$, is uniformly bounded with respect to $y$, the same holds for $T^\alpha_{q, y}$.

**PROOF.** Let $x > 0$ and $|x - y| < z < x + y$. If $x \geq y$, we apply (3.5) of Proposition 1 to (3.7), and, otherwise, we interchange the roles of $x$ and $y$ to get (with the substitution $\sigma = s + \tau$, $\tau = s - t$)

$$|K^\alpha_q(x, y, z) - K^\alpha_{q', r}(x, y, z)| \leq \frac{M_0}{4} \left( \int_{|x - y|}^z \left| K^\alpha_{q, \tau}(\sigma + \tau, \frac{\sigma - \tau}{2}, z) \right| Q\left(\frac{\sigma - \tau}{2}\right) d\sigma \right)$$

$$\cdot \left( \int_{|x - y|}^z u Q(u) du \right) d\tau.$$
monotonely increasing with respect to $y$. Integrating both sides of (4.5) with respect to $x$ over $I = ([y-z], y+z)$ (the kernels vanish outside of $I$), interchanging the integral over $x$ with the two others over $\sigma$ and $\tau$, and returning to the variables $s$ and $t$ (we omit the somewhat lengthy details) one obtains for $\alpha > -\frac{1}{2}$, $\alpha \neq 0$,

$$
\int_I |K^{\alpha}_q(x, y, z)| x^{2\alpha+1} dx \leq \int_I |K^{\alpha}_q(x, y, z)| x^{2\alpha+1} dx
$$

$$
+ \frac{M_0}{2} \int_0^y \left\{ \left[ \int_{\max(z-t, \sigma)}^{s+z-t} R_\sigma(s, t; x, y) x^{2\alpha+1} dx \right] \right\} \cdot |K^{\alpha}_q(s, t, z)| ds \cdot Q(t) \exp \left\{ M \int_I u Q(u) du \right\} dt.
$$

Now apply the assumption (4.3) with $(\xi, \eta)$ replaced by $(s, t)$, extend the $s$-interval of integration to $(|z-t|, z+t)$ and pass over to the suprema with respect to $z$ on both sides to arrive at

$$
\| T^{\alpha}_{q, y} \| \leq \| T^{\alpha}_{q, y} \| + \sup_{0 \leq s \leq y} \| T^{\alpha}_{q, z} \|.
$$

$$
\cdot \frac{1}{2} M_0 M_2 \int_0^y t^{\alpha-|\alpha|+1} Q(t) y^{\alpha} \exp \left\{ M \int_I u Q(u) du \right\} dt.
$$

For $\alpha > 0$, the latter integral can be evaluated at once, whereas for $\alpha < 0$, we first majorize $u$ by $u^{\alpha-|\alpha|+1} y^{\alpha} - \alpha$, so that

$$
(4.6) \quad \int_0^y \ldots dt \leq \frac{1}{M} \left\{ \exp \left\{ M y^{\alpha} \int_0^y u^{\alpha-|\alpha|+1} Q(u) du \right\} - 1. \right\}
$$

This proves (4.4) for $\alpha > -\frac{1}{2}$.

For $\alpha = -\frac{1}{2}$, in view of (3.8), there occurs the additional term

$$
\frac{M_0}{4} \int_0^y \left\{ \int_{\max(-z/-2, -y)}^{s+z+y-|t|} R_{-1/2}(t+z, t, x, y) dt \right\} \cdot Q(|t|) \exp \left\{ M \int_I u Q(u) du \right\} dt.
$$

Using (4.3) once more and replacing max $(-z/2, -y)$ by $-y$, this can be majorized by

$$
\frac{1}{2} M_0 M_2 y \int_0^y Q(t) \exp \left\{ M \int_I u Q(u) du \right\} dt
$$

or, in view of (4.6), by

$$
M_0 M_2 (2M)^{-1} \left( \exp \left\{ M y \int_0^y Q(u) du \right\} - 1. \right\}
$$

309
5. APPLICATIONS

Generally, for \( \alpha > 0 \), Theorem 1 can be used to introduce a bounded convolution structure associated with a differential operator \( D^\alpha_{a,x} \) by

\[
(f * g)(y) = \int_0^\infty T^\alpha_{a,y}(f;x)g(x)x^{2a+1}dx \quad (0 < y < \infty),
\]

provided there is a reference translation with uniformly bounded \( L^p_k \) norm and \( q \) satisfies the conditions of Theorem 1 as well as that for symmetry of the kernel \( K^a_q \). As a straightforward consequence one then obtains the usual properties of a convolution product, i.e., to be commutative, associative, and to satisfy \((f * g)^*(\lambda) = f^* (\lambda)g^*(\lambda)\) for \( \lambda \in S \), as far as the transforms exist, where

\[
h^*(\lambda) = \int_0^\infty h(x)u_\lambda(x)x^{2a+1}dx.
\]

Moreover, if \( f \in L^p_{w(2a+1)}(0, \infty) \) and \( g \in L^q_{w(2a+1)}(0, \infty) \) for some \( 1 \leq p \leq q < \infty \), the \( f * g \) belongs to \( L^r_{w(2a+1)} \) with \((1/r) = (1/p) + (1/q) - 1\) and satisfies the Young-type inequality

\[
\|f * g\|_r \leq C\|f\|_p\|g\|_q.
\]

In order to apply Theorem 1 to our examples \( q_r = q_1 \) and \( q_r = q_2 \), it has to be verified that the function \( \beta_{\alpha, q;x,y} \) defined in (3.6) satisfies the conditions (4.2) and (4.3). Since \( 1 - \phi = \xi \eta x y / \psi \) is symmetric in \( x \) and \( y \), property (4.2) is obvious. Using the symmetry of \( \psi \) with respect to \( \xi \) and \( x \), the integral in (4.3) can be written as

\[
I_\alpha = \int_{|\xi - y| + \eta}^{\xi + y - \eta} \beta_{\alpha}(\xi, \eta; x, y)x^{2a+1}dx = \frac{(\xi\eta)^{\alpha - |a| + 1}}{y^{\alpha + |a|}} \cdot
\]

\[
\int_{|\xi - y| + \eta}^{\xi + y - \eta} \left\{ \frac{1}{16} [(\xi + y)^2 - (\eta - x)^2][((\eta + x)^2 - (\xi - y)^2)]^a \right\} |a| - 1/2 x^{2a - |a| + 1}dx.
\]

Since \( \frac{1}{16} [(\xi + y)^2 - x^2][x^2 - (\xi - y)^2] \leq \psi(x, \eta; \xi, y) \leq (\xi y)^2 \) for all \( x \) admitted, one has

\[
I_\alpha \leq 2^{2a} \eta \int_{|\xi - y|}^{\xi + y} xdx = 2^{2a + 1}\eta
\]

if \( \alpha > \frac{1}{2} \), and

\[
I_\alpha \leq \frac{\xi\eta}{2^{2a}} \int_{|\xi - y|}^{(\xi + y)^2} \left\{ \frac{1}{16} [(\xi + y)^2 - x][x - (\xi - y)^2] \right\} ^{\alpha - 1/2}dx,
\]

\[
= 2^{1 - 2a} \frac{\Gamma(\alpha + 1/2)\Gamma(1/2)}{\Gamma(\alpha + 1)} \xi^{2a + 1}\eta
\]

310
if $0<\alpha \leq \frac{1}{2}$. Finally, if $-\frac{1}{2} \leq \alpha < 0$, it follows that

$$I_\alpha \leq (\xi \eta)^{2\alpha + \frac{1}{2}} \frac{(\xi + y)^2}{(\xi - y)^2} \left\{ \frac{1}{16} [(\xi + y)^2 - \lambda] [x - (\xi - y)^2] \right\}^{-\alpha} \frac{\lambda^{1/2} x^\alpha}{dx}$$

$$= (2\xi \eta)^{2\alpha + \frac{1}{2}} (\xi y)^{-\alpha} \left\{ \frac{1}{16} [(1 - u)u]^{-\alpha} \left( u + \frac{(\xi - y)^2}{4\xi y} \right) \right\} du$$

$$= 2B(\frac{1}{2} - \alpha, \frac{1}{2} - \alpha)(\xi \eta)^{2\alpha + 1} \left( \frac{\xi - y}{\xi y} \right)^{2\alpha} F \left( -\alpha, \frac{1}{2} - \alpha; 1 - 2\alpha; -\frac{4\xi y}{(\xi - y)^2} \right)$$

$$= 2B(\frac{1}{2} - \alpha, \frac{1}{2} - \alpha)(\xi \eta)^{2\alpha + 1} \left\{ \max \left( \frac{\xi, y}{} \right) \right\}^{2\alpha}$$

$$\leq 2^{2\alpha + 1} \frac{\Gamma(1/2) - \alpha \Gamma(1/2)}{\Gamma(1 - \alpha)} (\xi \eta)^{2\alpha + 1} y^{-2\alpha}.$$

This proves (4.3) with $M_2 = 2$ if $\alpha > \frac{1}{2}$ and $M_2 = 2^{1 - 2|\alpha|} \Gamma(|\alpha| + \frac{1}{2})/\Gamma(\frac{3}{2})/\Gamma(|\alpha| + 1)$ if $0 < |\alpha| \leq \frac{1}{2}$.

Denoting the class of admissible perturbation potentials by

$$\mathcal{Q}_\alpha(0, y) = \{ q_p; q_p \text{ satisfies condition (i) of Theorem 1} \},$$

and using the uniform boundedness of the Hankel- and Laguerre translation operators (for the latter see [10]), we have the following two corollaries.

**COROLLARY 1** (Braaksma [2]). For $\alpha \geq -\frac{1}{2}$, $\alpha \neq 0$, and $y \geq 0$ let $T^\alpha_{q_p}$ be the generalized translation operator associated with

$$D^\alpha_{q_p} = \frac{d^2}{dx^2} - \frac{2\alpha + 1}{x} \frac{d}{dx} - q_p(x),$$

where $q_p \in \mathcal{Q}_\alpha(0, y)$. Then

$$\| T^\alpha_{q_p} \|_{L^1([0, \infty])} \leq \exp \{ M y^{\alpha} - \alpha |Q|_{L^1([0, \infty])} \}$$

with $M = 2^{2\alpha - 1} \Gamma(\alpha) [\Gamma(\frac{1}{2}) \Gamma(\alpha + \frac{1}{2})]^{-1}$ if $\alpha > \frac{1}{2}$, and $M = 2^{-2|\alpha|} |\alpha|^{-1}$ if $0 < |\alpha| \leq \frac{1}{2}$.

**COROLLARY 2.** For $\alpha > 0$ and $y \geq 0$ let $T^\alpha_{q_p}$ be the generalized translation operator associated with

$$D^\alpha_{q_p} = \frac{d^2}{dx^2} - \frac{2\alpha + 1}{x} \frac{d}{dx} - x^2 - q_p(x),$$

where $q_p \in \mathcal{Q}_\alpha(0, y)$. Then

$$\| T^\alpha_{q_p} \|_{L^1([0, \infty])} \leq \exp \{ M |Q|_{L^1([0, \infty])} \}$$

with $M$ as in Corollary 1.

More generally than in Corollary 1, Braaksma [2] established norm estimates of translations also in other Lebesgue spaces, where the weight parameter ranges over a certain interval. Theorem 1 may also be extended in this direction.
If $q_r$ is the Laguerre potential $q_2$, the assumptions of Theorem 1 are even satisfied for negative $\alpha$ in view of (2.5), and formally, (4.4) remains valid. It is the lack of boundedness of the Laguerre translation which leads to the restriction of $\alpha$. The situation may be more favorable when the topology is changed.

ACKNOWLEDGEMENT

The author would like to thank Prof. E. Görlich for helpful comments and suggestions.

REFERENCES