Scattering of surface waves over an uneven sea-bed

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Abstract

This work is concerned with the scattering of a train of progressive waves by a small undulation of the bottom of a laterally unbounded sea, using two-dimensional linear water wave theory. Assuming irrotational motion, a simplified perturbation analysis is employed to obtain the first-order corrections to the velocity potential by using the Green’s integral theorem in a suitable manner, and the reflection and transmission coefficients in terms of integrals involving the shape of the function $c(x)$ representing the bottom undulation. Particular forms of the shape function are considered and the integrals for reflection and transmission coefficients are evaluated for these shape functions. For a varying sinusoidal sea bottom when Bragg resonance occurs, the reflection coefficient becomes a multiple of the number of ripples, and high reflection of the incident wave energy occurs if this number is large. We present the result in graphical form for one, three and five ripples in the patch. The results find absolute agreement with available results.

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1. Introduction

The problems of free surface flow over an obstacle or a geometrical disturbance at the bottom of a sea are important for their possible applications in the areas of coastal and marine engineering, and as a result these have been being studied by researchers for a long time. The problem of reflection of surface waves by patches of bottom undulations has received an increasing amount of attention as its mechanism is important in the development of shore-parallel bars.

Porter and Porter [1] investigated the interactions between surface water waves over the periodic bed forms in three different situations, one of which was the scattering of given incident waves by a finite section of periodic topography. A transfer matrix method incorporating evanescent modes was developed for the scattering problem which reduces the computation to that required for a single period without deviating from linear theory.

The problem of scattering of water waves by a varying bottom topography was solved by Evans and Linton [2] using a mapping method in which the problem was first transferred into a uniform strip resulting in a variable free
surface boundary condition. This was then approximated by a finite number of sections on which the free surface boundary condition was assumed to be constant. Staziker et al. [3] considered the problem of two-dimensional wave scattering by a local bed elevation of any shape on an otherwise horizontal bed using linearized water wave theory and they derived an integral equation of the second kind for the velocity potential over the varying part of the bottom profile and avoided the numerical problem of solving this equation by converting it into a new first-kind integral equation for the tangential fluid velocity on the deformation, which is amenable to numerical solution. However, for a small deformation of the bottom, by using perturbation analysis, Mandal and Basu [4] solved the problem of water wave diffraction by a small cylindrical elevation of the bottom of a laterally unbounded ocean covered by an ice sheet that was modelled as a thin elastic plate. They derived the reflection and transmission coefficients up to the first order in terms of integrals involving the shape function representing the bottom elevation. Porter and Porter [5] investigated the effect on wave propagation of an ice sheet of varying thickness floating on water of varying depth in three dimensions. Miles [6] obtained the reflection and transmission coefficients approximately up to the first order in terms of integrals involving the shape function of the bottom deformation by using small perturbation theory for when the wave train was obliquely incident.

Davies [7] considered the reflection of normally incident surface waves by a patch of sinusoidal undulations on the sea-bed in a finite region. The problem was solved by using a Fourier transform after introducing a linear friction term in the free surface condition for the purpose of ensuring the convergence of various integrals occurring in the analysis. Recently Warke et al. [11] obtained closed form solutions for scattering of surface waves by wavy or exponential bed topography. Numerical computations indicated that when solitary or sinusoidal wave conditions were applied at the boundary, water surface elevation attained a near Gaussian profile. Bhattacharya and Debnath [12] developed a two-layer analysis of the transient development of water waves over a viscoelastic bed and the solution to the scattering problem was obtained by using Laplace and Fourier transform methods. Martha and Bora have dealt with the oblique scattering of surface wave propagation over a small undulation on the bottom of a sea. By employing perturbation analysis, the velocity potential, reflection coefficient and transmission coefficients up to the first order were obtained by using the Green’s function technique [13] and finite cosine transformation [14]. The results were demonstrated with a number of practical examples. Martha et al. [15] considered the scattering problem when the sea-bed was a porous one and employed a Fourier transform technique in obtaining the reflection and transmission coefficients up to the first order.

Amos, Li and others, over the last ten years, have taken up practical problems of the uneven bottom of the sea or an estuary and have provided solutions to various hydrodynamics and geophysics related problems. Some of the notable works can be found in [8–10].

In our work, we employ perturbation analysis involving a small parameter $\varepsilon$ directly to the governing boundary value problem (BVP) and reduce the original problem to a simpler BVP for the first-order correction of the potential, the solution for which is obtained by a suitable application of Green’s integral theorem. The reflection and transmission coefficients are evaluated approximately up to the first order of $\varepsilon$ in terms of integrals involving the shape function when a train of progressive waves propagating from negative infinity is incident on the sea-bed having a small undulation. The integrals are evaluated explicitly for two different special forms of the undulation. For the case of a patch of sinusoidal sea-bed undulations with a wavenumber equal to twice the wavenumber of the incident wave
field, the first-order reflection coefficient is found to increase with the number of undulations. This is consistent with
the result obtained earlier by Davies [7] who had studied the reflection of wave energy by bottom undulations in an
ocean with a free surface.

2. Statement and formulation

A right-handed rectangular Cartesian coordinate system is considered in which $y$ is measured vertically downward
from the undisturbed free surface and the $x$-axis is measured along the undisturbed free surface $y = 0$ of the sea.
The bottom of the sea with small undulation is described by $y = h + \varepsilon c(x)$ where $c(x)$ is a function with compact
support and describes the bottom undulation, $h$ the uniform finite depth of the sea far to either side of the undulation
of the bottom so that $c(x) \to 0$ as $|x| \to \infty$ and the non-dimensional number $\varepsilon \ll 1$ a measure of the smallness
of the undulation. It is also assumed that the fluid is incompressible and inviscid, and the motion is irrotational.
The harmonic time dependence can be removed from the velocity potential $\Phi$ which describes the fluid motion by
setting

$$\Phi(x, y, t) = \text{Re}[\phi(x, y)e^{-i\sigma t}], \quad (1)$$

where $\sigma$ is the angular frequency of the incoming water wave with time dependence $e^{-i\sigma t}$.

Assuming the linear theory of water waves, $\phi$ satisfies the following set of equations:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < h + \varepsilon c(x), \quad (2)$$

$$\frac{\partial \phi}{\partial y} + K \phi = 0, \quad y = 0, \quad (3)$$

$$\frac{\partial \phi}{\partial n} = 0, \quad y = h + \varepsilon c(x), \quad (4)$$

where $K = \sigma^2/g$, and $\partial/\partial n$ denotes the normal derivative at a point $(x, y)$ on the bottom.

It is assumed that a progressive wave train represented by the velocity potential

$$\phi_0(x, y) = \cosh k(y - h)e^{ikx} \quad (5)$$

is incident upon the bottom undulation from negative infinity, where $k$ is the unique positive root of the equation

$$K = k \tanh kh. \quad (6)$$

Then it is partially reflected by and partially transmitted over the undulation so that $\phi$ has an asymptotic behavior
given by

$$\phi(x, y) \sim \begin{cases} (e^{ikx} + Re^{-ikx}) \cosh k(y - h), & x \to -\infty, \\ Te^{ikx} \cosh k(y - h), & x \to +\infty, \end{cases} \quad (7)$$

where $R$ and $T$ are the usual reflection and transmission coefficients in water wave problems defined to be the ratios
of the amplitudes of the reflected and transmitted waves, respectively, to that of the incident wave, and are to be
determined.

Assuming $\varepsilon$ to be very small and neglecting $O(\varepsilon^2)$ terms, the boundary condition $\partial \phi/\partial n = 0$ on $y = h + \varepsilon c(x)$
can be expressed in an appropriate form as

$$\frac{\partial \phi}{\partial y} - \varepsilon \frac{\partial}{\partial x} \left\{ c(x) \frac{\partial \phi}{\partial x} \right\} + O(\varepsilon^2) = 0 \quad \text{on} \quad y = h. \quad (8)$$
3. Method of solution

The boundary condition (8) and the fact that a wave train propagating in an ocean of uniform finite depth experiences no reflection together suggest that $\phi$, $R$ and $T$ can be expressed in terms of the small parameter $\varepsilon$ as

\[
\begin{align*}
\phi &= \phi_0 + \varepsilon \phi_1 + O(\varepsilon^2) \\
R &= \varepsilon R_1 + O(\varepsilon^2) \\
T &= 1 + \varepsilon T_1 + O(\varepsilon^2)
\end{align*}
\]

Using (9) in (2), (3), (7) and (8) we find that $\phi_1(x, y)$ satisfies the BVP described by

\[
\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0, \quad -\infty < x < \infty, \ 0 < y < h,
\]

(10)

\[
\frac{\partial \phi_1}{\partial y} + K \phi_1 = 0, \quad y = 0,
\]

(11)

\[
\frac{\partial \phi_1}{\partial y} = ik \frac{d}{dx} (c(x)e^{ikx}) \equiv q(x) \quad \text{on} \quad y = h,
\]

(12)

and

\[
\phi_1(x, y) \sim \begin{cases} R_1e^{-ikx} \cosh k(y-h), & x \to -\infty, \\ T_1e^{ikx} \cosh k(y-h), & x \to +\infty. \end{cases}
\]

(13)

For solving the above boundary value problem we need to find a Green’s function $G(x, y; \xi, \eta)$ which will satisfy

\[
\nabla^2 G = 0, \quad \text{in} \ -\infty < x < \infty, \ 0 < y < h
\]

(14)

except at $(\xi, \eta)$ where $0 < \eta < h$,

\[
\frac{\partial G}{\partial y} + K G = 0, \quad \text{on} \ y = 0,
\]

(15)

\[
\frac{\partial G}{\partial y} = 0, \quad \text{on} \ y = h,
\]

(16)

\[
G \sim \ln r \quad \text{as} \quad r = \{(x - \xi)^2 + (y - \eta)^2\}^{1/2} \to 0,
\]

(17)

\[
G \sim \text{multiple of} \ \cosh k(y-h)e^{ik|x-\xi|} \quad \text{as} \ |x-\xi| \to \infty.
\]

(18)

The condition (18) means that $G$ represents an outgoing wave as $|x-\xi| \to \infty$. The solution is given by

\[
G(x, y; \xi, \eta) = \frac{-4\pi i \cosh k(y-h) \cosh k(\eta-h)e^{ik|x-\xi|}}{2k(h + \sinh 2kh)} - \sum_{n=1}^{\infty} \frac{4\pi \cos k_n(y-h) \cos k_n(\eta-h)e^{-k_n|x-\xi|}}{2k_nh + \sin 2k_nh},
\]

(19)

where $k_n$ are real and positive roots of

\[
K + k_n \tan k_nh = 0.
\]

(20)

The behavior of $G$ at $|x| \to \infty$ is given by the first term of Eq. (19). Now applying the Green’s integral theorem to $\phi_1(x, y)$ and $G(x, y; \xi, \eta)$, we get

\[
\int \left( \phi_1 \frac{\partial G}{\partial n} - G \frac{\partial \phi_1}{\partial n} \right) ds = 0
\]

(21)

over the region bounded externally by the lines

\[
y = 0, -X \leq x \leq X; \quad x = X, 0 \leq y \leq h;
\]

\[
y = h, -X \leq x \leq X; \quad x = -X, 0 \leq y \leq h;
\]
and internally by a circle $C_0$ with center at $(\xi, \eta)$ and the radius $\delta$, and ultimately letting $X \to \infty$ and $\delta \to 0$. Finally the resultant integral equation (21) will give rise to the determination of $\phi_1$ as

$$\phi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, h; \xi, \eta)q(x)dx. \quad (22)$$

4. Reflection and transmission coefficients

The first-order reflection and transmission coefficients $R_1$ and $T_1$ are now obtained by letting $\xi \to \mp \infty$ and using the radiation condition (13) (with $(x, y)$ replaced by $(\xi, \eta)$) in Eq. (22). For this, from (19) we require the result

$$G(x, y; \xi, \eta) = -\frac{4\pi i \cosh k(y-h)\cosh k(\eta-h)e^{ik|x-\xi|}}{2kh + \sinh 2kh} \quad \text{as } |x-\xi| \to \infty, \quad (23)$$

and we get $R_1$ and $T_1$, respectively, as

$$R_1 = \frac{-2i}{2kh + \sinh 2kh} \int_{-\infty}^{\infty} e^{ikx}q(x)dx = \frac{-ik}{[h + \frac{\sinh^{2}kh}{k}]} \int_{-\infty}^{\infty} e^{-ikx}c(x)dx, \quad (24)$$

and

$$T_1 = \frac{-2i}{2kh + \sinh 2kh} \int_{-\infty}^{\infty} e^{-ikx}q(x)dx = \frac{ik}{[h + \frac{\sinh^{2}kh}{k}]} \int_{-\infty}^{\infty} c(x)dx. \quad (25)$$

Eqs. (24) and (25) can be evaluated once the shape function $c(x)$ is known. Below we consider two special forms for the function $c(x)$.

5. Special forms of the bottom deformation

The following functional forms of the bottom disturbance closely resemble some naturally occurring obstacles formed at the bottom due to sedimentation and ripple growth of sands.

Example I

$$c(x) = be^{-k_1|x|}(k_1 > 0), \quad -\infty < x < \infty. \quad (26)$$

In this case, the top of the elevation lies at the point $(0, b)$, and on either side it decreases exponentially. In order to evaluate $R_1$, using (26) in (24) we have

$$R_1 = \frac{-ik}{[h + \frac{\sinh^{2}kh}{k}]} b \frac{2k_1}{4k^2 + k_1^2}. \quad (27)$$

In order to evaluate $T_1$, using (26) in (25) we have

$$T_1 = \frac{ik}{[h + \frac{\sinh^{2}kh}{k}]} \frac{2}{k_1}. \quad (28)$$

Example II

$$c(x) = \begin{cases} a \sin (lx + \delta), & L_1 \leq x \leq L_2, \\ 0 & \text{otherwise}. \end{cases} \quad (29)$$

For continuity of bed elevation we can take

$$L_1 = \frac{-n\pi - \delta}{l}, \quad L_2 = \frac{m\pi - \delta}{l},$$

where $m$ and $n$ are positive integers. This represents a patch of sinusoidal ripples on an otherwise flat bottom (Fig. 1), the patch consisting of $(n + m)/2$ ripples having the same wavenumber $l$. For this case we obtain the reflection and
transmission coefficients, respectively, as

\[
R_1 = \frac{-ik}{[h + \sinh^2 kl_k]} \frac{l}{l^2 - (2k)^2} \left[ (-1)^n e^{2ikL_1} - (-1)^m e^{2ikL_2} \right],
\]

and

\[
T_1 = \frac{ik}{[h + \sinh^2 kl_k]} \frac{(-a)}{l} \left[ (-1)^m - (-1)^n \right].
\]

In the situation in which there is an integer number of ripple wavelengths in the patch \( L_1 \leq x \leq L_2 \) such that \( m = n \) and \( \delta = 0 \), we find \( T_1 \) and \( R_1 \), respectively, as

\[
T_1 = 0,
\]

and

\[
R_1 = \frac{1}{[h + \sinh^2 kl_k]} \frac{\beta}{\beta^2 - 1} (-1)^m \sin (m\pi \beta),
\]

where

\[
\beta = \frac{2k}{l}.
\]

These results exactly match those obtained in [7]. The main observation in this case is that the first-order reflection coefficient is an oscillatory function of \( \beta \) which is twice the ratio of the wavenumbers \( k \) and \( l \). It is also oscillatory in the quotient of length of ripple patch and the surface wavelength.

Consider the critical case \( \beta = 1 \) where the ratio of the wavenumbers \( k \) and \( l \) is 0.5. This is equivalent to saying that the wavelength of the sinusoidal ripples is half the wavelength of the incident wave field. In this situation, Bragg resonance occurs. So we find from (33) that

\[
R_1 = \frac{a}{2h} \frac{m\pi}{2}.
\]

In this case \( R_1 \) becomes a constant multiple of \( m \), the number of ripples in the patch. Hence, the reflected wave amplitude increases linearly with \( m \) which implies the possibility of a large amount of reflection of the incident wave energy.

For long waves \((kh \ll 1)\) having a wavenumber such that \( \beta = 1 \), Eq. (34) gives rise to

\[
R_1 = \frac{a}{2h} \frac{m\pi}{2}.
\]

If \( \frac{a}{2h} \approx \frac{1}{15} \) and \( m = 10 \), then \( R_1 \approx 1 \) which indicates that in the physically interesting case of a ripple having long wavelength with, say, \( \frac{a}{2h} \approx \frac{1}{15} \), the theory predicts that the total reflection of wave energy will occur for \( m \geq 10 \).
6. Numerical results

Of the two special forms of the bottom deformation discussed above, we take up the case of a patch of sinusoidal ripples due to its physical importance. We focus our attention on numerical results for the first-order reflection coefficient $|R_1|$ given by Eq. (33) for the patch with wavenumber $l$, having $m$ number of ripple wavelengths in the patch. $|R_1|$ is depicted against the wavenumber $K_h$ for one, three and five ripples for the values $a/h = 0.1, lh = 1$. From Fig. 2 it is clear that, for each of the cases, the maximum value of $|R_1|$ is attained when the wavenumber of the uneven bottom ($lh$) becomes approximately twice as large as the surface wavenumber ($kh$). The same general feature of $|R_1|$ is observed, that as the overall value of $R_1$ increases, the oscillating nature of $|R_1|$ against $K_h$ is more pronounced and the number of zeros of $|R_1|$ also increases. The results match exactly with one set of results in [4].

7. Conclusion

A simplified perturbation technique is employed in the study of two-dimensional free surface flow upon a small undulation at the bottom of the sea-bed. If the sea is of uniform finite depth, then an incoming wave train does not experience any reflection, so $R = 0, T = 1$. But the presence of a small undulation at the bottom causes the incoming wave train to experience a certain amount of reflection and transmission. Up to the first order, the reflection and transmission coefficients are evaluated in terms of integrals involving the shape function $c(x)$ describing the bottom undulation provided the undulation is small. The correction of the transmission coefficient $T_0 (= 1)$ up to the first order is given by $T_1$. Since far from the undulation, the depth on both sides is uniform and finite, $c(x)$ is of compact support (i.e., $c(x) \to 0$ as $|x| \to \infty$), so the integrals are convergent and can be evaluated once $c(x)$ is known explicitly. Integrals for reflection and transmission coefficients are evaluated for two different shape functions.

The particular case of a patch of sinusoidal ripples on the sea-bed is of considerable significance due to the ability of an undulating bed to reflect incident wave energy, which is important in respect of both coastal protection, and possible ripple growth if the bed can be eroded. Erosion sand-ripple bottoms exist on Sable Island Bank, located in the Atlantic Ocean, approximately 290 km south-east of Halifax, Nova Scotia, Canada, and our work can be observed as an analysis of how waves interact with these kinds of bottoms. We observe that the first-order transmission coefficient vanishes identically, and the reflection coefficient becomes a constant multiple of the number of ripples in the patch when the wavelength of the sinusoidal ripples is half the wavelength of the incident wave field. Consequently, if the number of ripples is large, then there occurs a large amount of reflection of the incident wave energy. This result may be useful in the construction of an effective reflector of the incident wave energy for protecting coastal areas from the rough sea in the arctic regions. But in a real sense, it will be too costly to do. This wave erosion related problem is evident for parts of the Arctic region on the Southern Beaufort Sea.
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