Twofold Triple Systems and Graph Imbeddings

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A natural bijection between \((v, 3, 2)\)-BIBD's and certain topological and combinatorial structures related to graph imbeddings is used to obtain new classes of designs of this type. The topological viewpoint often makes it quite simple to differentiate between nonisomorphic designs.

1. The Relationship

A \((v, k, \lambda)\)-Balanced Incomplete Block Design (BIBD) consists of a finite set \(X\) of \(v\) objects and a collection of \(b\) subsets of \(X\) called blocks such that every block contains exactly \(k\) objects, every object occurs in exactly \(r\) blocks, and every pair of distinct objects occurs in exactly \(\lambda\) blocks. Standard counting arguments (see Hall [6, p. 101]) may be used to show that if such a system exists, then \(bk = vr\) and \(\lambda(v - 1) = r(k - 1)\). These equations are sometimes referred to in what follows as "the necessary conditions for existence of a \((v, k, \lambda)\) — BIBD." R. Wilson [21] has shown that for fixed \(k\) and \(\lambda\) and all sufficiently large \(v\), these necessary conditions are in fact sufficient for the existence of a \((v, k, \lambda)\)-BIBD.

Interest here centers on twofold triple systems, that is \((v, 3, 2)\)-BIBD's, in which case the necessary conditions for existence reduce to the single requirement that \(v\) must be congruent to 0 or 1 modulo 3. The first proof of the sufficiency of this requirement for the existence of a twofold triple system on \(v\) objects is due to Bhattacharya [1]. Using very different methods, Hanani [7] was able to show that the necessary conditions for existence of a \((v, 3, \lambda)\)-BIBD are in fact sufficient for any choice of \(\lambda\).

In the present paper the relationship of \((v, 3, 2)\)-BIBD's to imbeddings of graphs into surfaces (connected closed 2-manifolds) is elucidated. The

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study of such imbeddings is concerned primarily with the problem of imbedding a given graph $G$ in the simplest possible surface. Because of the classification theorem for closed 2-manifolds (see, for example, [11]), this amounts to asking in the orientable case for the minimum number $g(G)$ of handles which must be attached to the 2-sphere in order to accommodate all of the graph $G$. The integer $g(G)$ is called the genus of $G$. Similarly, the nonorientable genus of $G$ is defined as the minimum number $q(G)$ of crosscaps which must be attached to the 2-sphere in order to accommodate all of $G$. By convention, if $g(G) = 0$, one sets $q(G) = 0$.

If a graph $G$ is imbedded in a surface $M$ its image is also called $G$. The components of $M - G$ are called the faces of the imbedding. A straightforward Euler characteristic argument may be used to show that seeking a simplest imbedding of a graph amounts to trying to maximize the number of faces in an imbedding. In particular, for graphs without loops or multiple edges, if all faces are triangular, then the imbedding is in a simplest possible surface. Thus triangular imbeddings are of particular interest.

It turns out that there is a natural way of associating to any twofold triple system $X$ a family of triangular imbeddings. To see this, associate to each block of $X$ a triangle with vertices labeled by the objects of that block and then perform the standard identification procedure of combinatorial topology on the resulting family of labeled triangles. In this procedure, those edges which are labeled by the same pair of objects are identified in the appropriate way and no other identifications are made. Because $X$ is a $(v, 3, 2)$-BIBD, every edge will have exactly one mate and so the resulting topological space is a 2-manifold whose connected components are triangulated surfaces $M_1, \ldots, M_n$.

It is perfectly possible that the vertices of $M_1, \ldots, M_n$ will not all receive distinct names (an example will follow). However, if the underlying graphs or 1-skeleta are identified along vertices receiving the same name, then the resulting graph is $K_v$ because every possible pair of objects occurs twice among the blocks of $X$.

The properties of $M_1, \ldots, M_n$ are summarized in the following definition. A triangulation system on $v$ names consists of a family $M_1, \ldots, M_n$ of triangulated surfaces such that the vertices of $M_1, \ldots, M_n$ are labeled by $v$ distinct names in such a way so that if the corresponding 1-skeleta are identified along vertices receiving the same name then the resulting graph is $K_v$. The simplest example of a triangulation system on $v$ names would be a triangular imbedding of $K_v$ in some surface.

The preceding arguments show that any two-fold triple system on $v$ objects leads to a triangulation system on $v$ names. Not surprisingly, the converse is also true, as is made evident by the following theorem.
THEOREM 1. The twofold triple systems on \( v \) objects are in bijective correspondence with the triangulation systems on \( v \) names.

Proof. It has already been shown that a triangulation system may be obtained from a twofold triple system by associating to each block a triangle labeled by the objects of that block and performing a standard identification procedure of combinatorial topology on the edges of the resulting family of labeled triangles. In light of this, it is apparent that the proof will be complete if it can be shown that by taking the faces of a triangulation system \( T \) as blocks and the names as objects one obtains a system satisfying the various properties of a \((v, 3, 2)\)-BIBD. Obviously, if this is done there will be \( v \) objects and all blocks will contain 3 objects. Moreover, every edge will appear in exactly 2 faces and because \( T \) is a triangulation system there will be exactly one edge associated to each pair of distinct names. Hence every pair of distinct objects will occur in exactly 2 blocks. Finally, it may be observed that the degree of a vertex of a cellular decomposition of a 2-manifold is the number of faces in its star, so every object will occur in exactly \( v - 1 \) blocks because every vertex of \( K_v \) has degree \( v - 1 \).

We conclude this section with some historical remarks and an example. The first realization that there is a relationship between \((v, 3, 2)\)-BIBD's and triangular graph imbeddings seems to be due to L. Heffter [8] who in 1891 proved the first several cases of the Heawood map-coloring conjecture by computing the genus of \( K_n \) for \( n \leq 12 \). In particular when \( n = 3, 4, 7 \) and 12, the imbeddings are triangular and hence yield twofold triple systems. Heffter realized that triangular imbeddings of complete graphs were combinatorial relatives of Steiner triple systems \((v, 3, 1)\)-BIBD's and made use of this fact in a subsequent paper [9].

Heffter's ideas seem to have received little attention until 1929 when A. Emch [3] constructed triangular graph imbeddings associated with \((v, 3, 2)\)-BIBD's for \( v = 6, 7, \) and 9. His interest in the imbeddings was principally as an aid in calculating the automorphism groups of the designs. Fig. 1 below is taken from Emch's paper. It is reproduced here as

![Figure 1](https://example.com/figure1.png)
an illustration of a triangulation system with one component but more vertices than vertex names. The figure represents an imbedding in the torus, obtained by identifying the opposite edges of the rectangle in the appropriate way. Unfortunately, the work of both Heffter and Emch on triple systems seems to have received less notice than it deserves.

2. NEW CLASSES OF TRIPLE SYSTEMS

The aim of this section is to apply Theorem 1 to some of the known classes of graph imbeddings in order to produce several new classes of \((v, 3, 2)\)-BIBD's. Distinguishing between these classes of designs is quite simple because their associated triangulation systems are obviously different. Indeed, the proof that they are new is completed in the next section by constructing the triangulation systems associated to previously known twofold triple systems.

The most important genus computations are for the complete graphs. This is because in dual form they yield a proof of the Heawood map-coloring theorem. Imbeddings of complete graphs \(K_n\) into nonorientable surfaces of minimal genus were found by Ringel [12] in 1959. For all \(n\) congruent to 0 or 1 modulo 3, except \(n = 7\), the imbeddings are triangular and so give triangulation systems. Consequently, except in the case \(v = 7\), this gives an independent proof that the necessary conditions for the existence of a \((v, 3, 2)\)-BIBD are in fact sufficient.

The computation of the orientable genera of the complete graphs was completed in 1968 by Ringel and Youngs [15] with the aid of Gustin, Mayer, Terry, and Welch. For \(n\) congruent to 0, 3, 4, or 7 modulo 12, the minimal imbeddings are triangular [13, 18, 19, 22, 23], and so yield new twofold triple systems.

For \(n\) congruent to 1, 6, 9, or 10 modulo 12 the computation of \(g(K_n)\) proceeds by finding a triangular imbedding of \(K_n \sim K_3\) in a surface of genus \(g(K_n) - 1\) and accommodating the missing edges on one handle [13, 22, 23]. Since \(K_3\) has a triangular imbedding in the sphere, this procedure gives a triangulation system on \(n\) names with 2 components, one with 1-skeleton \(K_n \sim K_3\), the other with 1-skeleton \(K_3\). It follows that the computation of \(g(K_n)\) yields a \((v, 3, 2)\)-BIBD for every \(v\) congruent to 0 or 1 modulo 3.

As a final example of how known graph imbeddings can be used to construct twofold triple systems, consider the complete tripartite graph \(K_{p,q,r}\). This graph is obtained by removing disjoint copies of \(K_p\), \(K_q\), and \(K_r\) from \(K_{p+q+r}\). Ringel and Youngs [16], and independently White [20], proved that \(K_{n,n,n}\) has a triangular imbedding for all \(n \geq 1\). Since
$K_{n,n,n} = K_{3n} \sim K_n, K_n, K_n$, this yields a triangulation system on $3n$ names with 4 components for all $n$ congruent to 0 or 1 modulo 3.

It is apparent that all of the triangulation systems discussed in this section are distinct. Hence, the associated triple systems are all distinct.

3. Triangulation Systems from Known Triple Systems

The aim of this section is to prove the following proposition.

**Proposition 1.** For $v \geq 7$, the $(v, 3, 2)$-BIBD's constructed in Section 3 are new.

**Proof.** The procedure will be to construct the triangulation systems associated to previously known twofold triple systems.

Bhattacharya's [1] $(v, 3, 2)$-BIBD's for $v$ congruent to 1 or 3 modulo 6 are obtained by taking the blocks of a $(v, 3, 1)$-BIBD, or Steiner triple system, twice. The existence of $(v, 3, 1)$-BIBD's for all $v$ congruent to 1 or 3 modulo 6 is well known and was apparently first demonstrated in 1847 by Kirkman [10]. It is not hard to see that such a system amounts to a partition of $K_v$ into edge-disjoint copies of $K_3$. Taking the blocks of a Steiner triple system twice then corresponds to forming a triangulation system on $v$ names by imbedding each copy of $K_3$ in the sphere.

Bhattacharya's twofold triple systems on $6s + 6$ and $6s + 4$ objects are constructed using Bose's [2] method of symmetrically repeated differences. A simple but lengthy computation, omitted here, may be used to show that the associated triangulation systems are connected but generally have more vertices than names. Hence they do not come from triangular imbeddings of complete graphs.

In the course of his proof that the necessary conditions for the existence of a $(v, k, \lambda)$-BIBD are sufficient for $k = 3$ and any $\lambda$, Hanani [7] constructs twofold triple systems which differ from Bhattacharya's. The procedure is to first show (by recursive methods) that whenever $v$ is congruent to 0 or 1 modulo 3, there is an arrangement of $v$ objects into $b$ blocks such that each block contains 3, 4, or 6 objects, every object occurs in the same number of blocks, and every pair of distinct objects occurs in exactly one block. Obviously, such a system is equivalent to a partition of $K_v$ into edge-disjoint copies of $K_3, K_4$, and $K_6$ whose union is all of $K_v$.

Each block in the above system is then replaced by a twofold triple system on the objects of that block. In terms of triangulation systems, this amounts to forming a triangular imbedding of the appropriate copy of $K_3, K_4$, or $K_6$. In particular, the corresponding triangulation system
has many components, the most complicated of which is an imbedding of $K_6$ in the projective plane.

A straightforward check now shows that the triple systems of Bhattacharya and Hanani are all different from those found in the previous section because the associated triangulation systems are obviously different. This completes the proof of Proposition 1.

4. GENERALIZATIONS

Part of the interest of the ideas developed in this paper is due to the fact that they can be extended to deal with more general types of block designs and graph imbeddings. Some of these generalizations are therefore discussed in this section.

A $t - (v, k, \lambda)$ design is an arrangement of $v$ objects into blocks of size $k$ so that every set of $t$ distinct objects occurs in exactly $\lambda$ blocks. J. Gross [4] has pointed out that, just as $(v, 3, 2)$-BIBD's correspond to certain simplicial 2-complexes, so $t - (v, t + 1, 2)$ designs correspond to certain simplicial $t$-complexes. More generally, a $(v, k, \lambda)$-BIBD may be obtained by taking as blocks the $(k - 1)$-dimensional simplices of a simplicial $n$-complex which has the property that every 1-simplex is incident upon exactly $\lambda$ $(k - 1)$-simplices and every 0-simplex is incident upon exactly $r$ $(k - 1)$-simplices. In particular, this offers one approach for obtaining BIBD's with $k > 3$ and $\lambda > 2$.

Another generalization of triangulation systems which leads to a topological theory of BIBD's with $k > 3$ and $\lambda > 2$ is obtained by studying graph imbeddings in which all faces are bounded by $k$ edges. This requires some care because diagonals of faces do not appear as graph edges and yet are counted as pairs of objects in the associated blocks. Increasing $\lambda$ in this context corresponds to adding edges (possibly multiple) to the underlying graph.

It is also interesting to consider applying the ideas used in the construction of block designs to the study of graph imbeddings. For example, the direct construction method of Bose [2] has an intimate relationship to the theory of voltage graphs described by Gross [5]. Voltage graph theory is dual to that of current graphs, the technique used in the proof of the Heawood map-coloring theorem [15].

It is unfortunate that the recursive methods which have been so useful in the study of block designs have so few counterparts in the theory of graph imbeddings. An interesting example of a recursive construction of graph imbeddings is found in Schanuel's [17] unpublished computation of the genus of complete bipartite graphs. This computation was first made using direct construction methods by Ringel [14].
TRIPLE SYSTEMS AND GRAPH IMBEDDINGS

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