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# On the Minimal Solution of the Problem of Primitives

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We characterize the primitives of the minimal extension of the Lebesgue integral which also integrates the derivatives of differentiable functions (called the $C$-integral). Then we prove that each BV function is a multiplier for the $C$-integral and that the product of a derivative and a BV function is a derivative modulo a Lebesgue integrable function having arbitrarily small $L^1$-norm. © 2000 Academic Press

## 1. INTRODUCTION

The question of providing a minimal constructive integration process which includes the Lebesgue integral and also integrates the derivatives of differentiable functions was solved in [1, 3] by the following Riemann-type integral (called the $C$-integral):

DEFINTION 1.1. Given a function $f$: $[a, b] \to \mathbf{R}$ we say that $f$ is $C$-integrable on $[a, b]$ if there exists a constant $A$ such that for each $\varepsilon > 0$ there is a gage $\delta$ such that

$$\left| \sum_{i=1}^{p} f(x_i)|I_i| - A \right| < \varepsilon,$$

for each $\delta$-fine McShane partition $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ satisfying the condition

$$\sum_{i=1}^{p} \mathrm{dist}(x_i, I_i) < 1/\varepsilon. \tag{1}$$

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The number $A$ is called the $C$-integral of $f$ on $[a,b]$, and we write $A = \int_a^b f$.

We recall that a gage $\delta$ on $[a,b]$ is a positive function defined on $[a,b]$, and that a $\delta$-fine McShane partition of $[a,b]$ is a collection $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ of pairwise nonoverlapping intervals $I_i \subset [a,b]$ and points $x_i \in [a,b]$ such that $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ and $\sum_{i=1}^{p} |I_i| = b - a$.

A first descriptive characterization of $C$-primitives was obtained by Bruckner et al. [7]. They proved that a function $F$ is a $C$-primitive if and only if it is the limit in variation of a sequence of absolutely continuous functions. Note that, in general, $F$ is not a function of bounded variation; however, its associate variational measure is absolutely continuous with respect to the Lebesgue measure. This follows from the trivial observation that a $C$-primitive is also a Denjoy–Perron primitive and by [4, Theorem 3].

In effect the last mentioned theorem gives a characterization of Denjoy–Perron primitives. The idea of considering appropriate variational measures to characterize the primitives of some integral has been also used in [2, 8, 9] for many multidimensional integrals and in [5] for the Henstock-dyadic integral and for the Henstock-symmetric integral.

In this paper we define a useful variational measure that allows us to extend [4, Theorem 3] to the $C$-integral (theorem 4.1). The resulting characterization is then used to prove that each BV function is a multiplier for our integral (Theorem 4.2), and consequently the product of a derivative, and a BV function is a derivative modulo a Lebesgue integrable function having an arbitrarily small $L^1$-norm (Theorem 4.3).

## 2. PRELIMINARIES

$\mathbf{R}$ denotes the set of all real numbers. Given a Lebesgue measurable set $E \subset \mathbf{R}$, $|E|$ denotes the Lebesgue measure of $E$. For a function $F$ and for an interval $I = [\alpha, \beta]$, we use the notation $F(I) = F(\beta) - F(\alpha)$. A $\delta$-fine McShane partial partition of $[a,b]$ is a collection $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ of pairwise nonoverlapping intervals $I_i \subset [a,b]$ and points $x_i \in [a,b]$ such that $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$.

LEMMA 2.1. *If $f$ is a $C$-integrable function on $[a,b]$, then given $\varepsilon > 0$ there exists a gage $\delta$ such that*

$$\sum_{i=1}^{p} \left| f(x_i)|I_i| - \int_{I_i} f \right| < \varepsilon,$$

*for each $\delta$-fine McShane partial partition $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ satisfying the condition $\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon$.*

The proof follows by standard techniques (see [11, Lemma 9.11]).

LEMMA 2.2. *If $F$ is a function differentiable at $x$, then given $\varepsilon > 0$ there exist $\delta(x)>0$ such that*

$$|F(I) - F'(x)|I|| < \varepsilon(\text{dist}(x, I) + |I|),$$

*for each interval $I \subset (x - \delta(x), x + \delta(x))$.*

*Proof.* By the existence of $F'(x)$ there is $\delta(x) > 0$ such that

$$|F(y) - F(x) - F'(x)(y - x)| < \frac{\varepsilon}{2}|y - x|,$$

for each $y \in (x - \delta(x), x + \delta(x))$.

Therefore, given $I = (\alpha, \beta) \subset (x - \delta(x), x + \delta(x))$ we have

$$\begin{aligned}
&|F(\beta) - F(\alpha) - F'(x)(\beta - \alpha)| \\
&\quad \leq |F(\beta) - F(x) - F'(x)(\beta - x)| + |F(\alpha) - F(x) - F'(x)(\alpha - x)| \\
&\quad < \frac{\varepsilon}{2}|\beta - x| + \frac{\varepsilon}{2}|\alpha - x| \\
&\quad < \frac{\varepsilon}{2} \text{ dist } (x, I) + \frac{\varepsilon}{2} (\text{dist } (x, I) + |I|) \\
&\quad = \varepsilon (\text{dist } (x, I) + |I|).
\end{aligned}$$

∎

## 3. THE VARIATIONAL MEASURE

For a given function $F$ on $[a, b]$, a gage $\delta$, a set $E \subset [a, b]$, and a positive $\varepsilon$ we denote

$$V_\varepsilon(F, \delta, E) = \sup\Big\{ \textstyle\sum_i |F(I_i)| : \{(I_1, x_1), \ldots, (I_p, x_p)\} \text{ is a } \delta\text{-fine McShane partial partition of } [a, b] \text{ such that } x_i \in E, i = 1, 2, \ldots, p, \text{ and } \sum_{i=1}^p \text{dist } (x_i, I_i) < 1/\varepsilon \Big\}.$$

We also denote

$$V_0(F, \delta, E) = \sup\Big\{ \textstyle\sum_i |F(I_i)| : \{(I_1, x_1), \ldots, (I_p, x_p)\} \text{ is a } \delta\text{-fine McShane partial partition of } [a, b] \text{ such that } x_i \in I_i \cap E, i = 1, 2, \ldots, p \Big\}.$$

It is clear that

$$V_0(F, \delta, E) \leq V_{\varepsilon_1}(F, \delta, E) \leq V_{\varepsilon_2}(F, \delta, E) \qquad \text{whenever } \varepsilon_1 > \varepsilon_2. \tag{2}$$

Then we define

$$V_C F(E) = \sup_\varepsilon \inf_\delta V_\varepsilon(F, \delta, E)$$

and

$$V_0 F(E) = \inf_\delta V_0(F, \delta, E).$$

From (2) it follows that

$$V_0 F(E) \leq V_C F(E). \tag{3}$$

By the same argument used in [18] for proving Theorems 3.7 and 3.15, we can show that the extended real-valued set functions $V_0 F$ and $V_C F$ are *Borel regular measures* in $[a, b]$.

## 4. MAIN RESULTS

THEOREM 4.1. *$F$ is an indefinite $C$-integral if and only if the variational measure $V_C F$ is absolutely continuous with respect to the Lebesgue measure.*

*Proof.* Let $F(x) = F(a) + \int_a^x f(t)dt$ be the indefinite $C$-integral of $f$ on $[a, b]$, and let $E \subset [a, b]$ with $|E| = 0$. Without loss of generality we can assume $f(x) = 0$ for each $x \in E$. Then by Lemma 2.1, given $\varepsilon > 0$ there exists a gage $\delta$ such that

$$\sum_{i=1}^{p} |F(I_i)| < \varepsilon$$

for each $\delta$-fine partial McShane partition $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ with $x_i \in E$, $i = 1, 2, \ldots, p$, and $\sum_{i=1}^p \operatorname{dist}(x_i, I_i) < 1/\varepsilon$. Then $V_\varepsilon(F, \delta, E) \leq \varepsilon$ and by (2), we have $V_C F(E) = 0$.

Conversely, if $V_C F$ is absolutely continuous, then by (3), so is $V_0 F$. Therefore by [4, Theorem 2], the function $F$ is differentiable almost everywhere in $[a, b]$.

Let $N = \{t \in [a, b] : F \text{ is not differentiable at } t\}$ and let

$$f(t) = \begin{cases} F'(t) & \text{if } t \in [a, b]\setminus N, \\ 0 & \text{if } t \in N. \end{cases}$$

We prove that $F$ is the indefinite $C$-integral of $f$. To this end, let $0 < \varepsilon < 1/(b - a)$ and let $x \in [a, b]$. For each $t \in [a, b]\setminus N$ there exists (by Lemma 2.2) $\delta_1(t) > 0$ such that

$$|f(t)|B| - F(B)| < \frac{\varepsilon^2}{4}(\operatorname{dist}(t, B) + |B|),$$

for each interval $B \subset (t - \delta_1(t), t + \delta_1(t))$.

Moreover, since $V_C F(N) = 0$, there exists a gage $\delta_2$ such that

$$\sum_{i=1}^{p} |F(B_i)| < \frac{\varepsilon}{2},$$

for each $\delta_2$-fine McShane partial partition $\{(B_1, x_1), \ldots, (B_p, x_q)\}$ with $x_i \in N$, $i = 1, 2, \ldots, q$, and $\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon$.

Let

$$\delta(t) = \begin{cases} \delta_1(t) & \text{if } t \in [a, b]\backslash N, \\ \delta_2(t) & \text{if } t \in N, \end{cases}$$

and let $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ be a $\delta$-fine McShane partition of $[a, x]$ satisfying condition (1). Then

$$\left| \sum_{i=1}^{p} f(x_i)|I_i| - (F(x) - F(a)) \right| \le \sum_{i=1}^{p} |f(x_i)|I_i| - F(I_i)|$$

$$< \sum_{x_i \in N} |F(I_i)| + \frac{\varepsilon^2}{4} \sum_{x_i \notin N} (\operatorname{dist}(x_i, I_i) + |I_i|)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon^2}{4}(b - a) < \varepsilon.$$

By the arbitrariness of $\varepsilon$, the function $f$ is $C$-integrable on $[a, x]$,

$$\int_a^x f(t)dt = F(x) - F(a),$$

and by the arbitrariness of $x \in [a, b]$, the function $F$ is the indefinite $C$-integral of $f$ on $[a, b]$. ∎

Recall that a function $g\colon [a, b] \to \mathbf{R}$ is said to be a BV function whenever there exists a function of bounded variation $\tilde{g}\colon [a, b] \to \mathbf{R}$ such that $g = \tilde{g}$ a.e. in $[a, b]$.

THEOREM 4.2. *Each BV function is a multiplier for the C-integral.*

*Proof.* Let $f$ be a $C$-integrable function and let $F$ be its primitive. If $g$ is a BV function, then by [17, Chap. 8, Theorem 2.5], $fg$ is Denjoy–Perron integrable, and for each $x \in [a, b]$,

$$(\mathrm{DP}) \int_a^x fg\, dt = [Fg]_a^x - (\mathrm{L}) \int_a^x F\, dg; \tag{4}$$

here (DP) $\int$ and (L)$\int$ denote the Denjoy–Perron and Lebesgue integrals, respectively. Let $\Phi(x) = (\mathrm{DP}) \int_a^x fg\, dt$,

$$E = \{x \in [a, b] : \Phi'(x) = f(x)g(x)\},$$

and observe the set $N = [a, b]\setminus E$ is negligible. Without loss of generality we can assume that $f(x) = 0$ for each $x \in N$, and that $g(x)$ is increasing and positive on $[a, b]$.

Now fix an $0 < \varepsilon < 1/(b-a)$. By Lemma 2.2, for each $x \in E$ there exists $\delta_1(x) > 0$ such that

$$|f(x)g(x)|I| - \Phi(I)| < \frac{\varepsilon^2 (\text{dist}\,(x, I) + |I|)}{6}, \tag{5}$$

for each interval $I \in (x - \delta_1(x), x + \delta_1(x))$. As the variational measure $V_C F$ is absolutely continuous by Theorem 4.1, there exists a gage $\delta_2$ such that

$$\sum_{i=1}^{p} |F(I_i)| < \frac{\varepsilon}{3(||g||_\infty + 1)} \tag{6}$$

for each $\delta^2$-fine McShane partial partition $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ with $x_i \in N$, $i = 1, 2, \ldots, p$, and $\sum_{i=1}^{p} \text{dist}(x_i, I_i) < 1/\varepsilon$. Choose a $\sigma > 0$ so that

$$|F(x) - F(y)| < \frac{\varepsilon}{6(||g||_\infty + 1)} \tag{7}$$

for each $x, y \in I$ with $|x - y| < \sigma$, and define a function $\delta$ by the formula

$$\delta(x) = \begin{cases} \delta_1(x) & \text{if } x \in E, \\ \min(\delta_2(x), \sigma) & \text{if } x \in N. \end{cases} \tag{8}$$

Let $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ be a $\delta$-fine McShane partition of $[a, b]$ satisfying condition (1). Then

$$\begin{aligned} &\left| \sum_i f(x_i)g(x_i)|I_i| - \Phi([a, b]) \right| \\ &\qquad \le \sum_i |f(x_i)g(x_i)|I_i| - \Phi(I_i)| \le \sum_{x_i \in E} + \sum_{x_i \in N}. \end{aligned} \tag{9}$$

An estimate of $\sum_{x_i \in E}$ follows from (5):

$$\begin{aligned} &\sum_{x_i \in E} |f(x_i)g(x_i)|I_i| - \Phi(I_i)| \\ &\qquad < \frac{\varepsilon^2}{6} \sum_{x_i \in E} (\text{dist}(x_i, I_i) + |I_i|) < \frac{\varepsilon^2}{3} \cdot \frac{1}{\varepsilon} = \frac{\varepsilon^2 (b-a)}{6} < \frac{\varepsilon}{3}. \end{aligned} \tag{10}$$

Next we estimate $\sum_{x_i \in N}$. Note that $f(x_i) = 0$ for $x_i \in N$. Using (4) and letting $I_i = [\alpha_i, \beta_i]$, we obtain

$$\begin{aligned}
\sum_{x_i \in N} &= \sum_{x_i \in N} |\Phi(I_i)| \\
&= \sum_{x_i \in N} \left| \left( F(\beta_i) g(\beta_i) - F(\alpha_i) g(\alpha_i) - (L) \int_{\alpha_i}^{\beta_i} F\, dg \right) \right| \\
&= \sum_{x_i \in N} |(F(\beta_i) - F(\alpha_i)) g(\beta_i) \\
&\quad + F(\alpha_i)(g(\beta_i) - g(\alpha_i)) - F(\xi_i)(g(\beta_i) - g(\alpha_i))| \\
&\le \sum_{x_i \in N} |(F(\beta_i) - F(\alpha_i)) g(\beta_i)| \\
&\quad + \sum_{x_i \in N} |F(\alpha_i) - F(\xi_i)| (g(\beta_i) - g(\alpha_i)),
\end{aligned} \tag{11}$$

where $\xi_i \in [\alpha_i, \beta_i]$. By (6),

$$\sum_{x_i \in N} |(F(\beta_i) - F(\alpha_i)) g(\beta_i)| \le \frac{\varepsilon}{3(||g||_\infty + 1)} \cdot ||g||_\infty < \frac{\varepsilon}{3}, \tag{12}$$

and by (7) and (8),

$$\sum_{x_i \in N} |F(\alpha_i) - F(\xi_i)| (g(\beta_i) - g(\alpha_i)) \le \frac{\varepsilon}{6(||g||_\infty + 1)} \cdot 2||g||_\infty < \frac{\varepsilon}{3}. \tag{13}$$

Summing up the inequalities (10), (12), and (13) and taking into account (9) and (10), we obtain

$$\left| \sum_i f(x_i) g(x_i) |I_i| - \Phi([a, b]) \right| < \varepsilon,$$

which completes the proof. ∎

THEOREM 4.3. *The product of a derivative and a BV function is a derivative modulo a Lebesgue integrable function of an arbitrarily small $L^1$-norm.*

*Proof.* Let $f$ be a derivative, and let $g$ be a BV function. By [1, Teorema 1], $f$ is $C$-integrable. According to Theorem 4.2, $fg$ is also $C$-integrable. Thus by [3, Main Theorem], there exists a derivative $f_1$ such that $fg - f_1$ is a Lebesgue integrable function. Choose an $\varepsilon > 0$. The absolute continuity of the Lebesgue integral and Lusin's theorem imply that there is a continuous function $h_1$ with

$$\int_a^b |fg - f_1 - h_1| < \frac{\varepsilon}{4}.$$

Let $h_2$ be a continuous function such that $h_2(a) = f_1(a) + h_1(a)$, $h_2(b) = f_1(b) + h_1(b)$, and $\int_a^b |h_2| < \varepsilon/4$. Moreover, let $h_3$ be a continuous function for which $h_3(a) = h_3(b) = 0$ and

$$\int_a^b h_3 = \int_a^b (f_1 + h_1 - h_2) - \int_a^b fg$$
$$= \int_a^b (f_1 + h_1 - fg) - \int_a^b h_2.$$

Clearly, we may assume $h_3 \geq 0$ or $h_3 \leq 0$. Thus

$$\int_a^b |h_3| = \left| \int_a^b h_3 \right| \leq \int_a^b |fg - f_1 - h_1| + \int_a^b |h_2|$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Observe that the function $h_\varepsilon = f_1 + h_1 - h_2 - h_3$ is a derivative, since a continuous function is a derivative and the sum of derivatives is a derivative. Furthermore,

$$\int_a^b |fg - h_\varepsilon| \leq \int_a^b |fg - f_1 - h_1| + \int_a^b |h_2| + \int_a^b |h_3|$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, the claim follows by the obvious identity $fg = h_\varepsilon + (fg - h_\varepsilon)$. ∎

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