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Characterization of Wishart–Laplace distributions via Jordan algebra homomorphisms ☆

Joe Masaro [a], Chi Song Wong [b,*]

[a] 961 Marentette Ave., Windsor, Ontario, Canada N9A 2A2
[b] University of Windsor, Windsor, Ontario, Canada N9B 3P4

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ABSTRACT

For a real, Hermitian, or quaternion normal random matrix Y with mean zero, necessary and sufficient conditions for a quadratic form $Q(Y)$ to have a Wishart–Laplace distribution (the distribution of the difference of two independent central Wishart $W_p(m_i, \Sigma)$ random matrices) are given in terms of a certain Jordan algebra homomorphism ρ. Further, it is shown that $\{Q_k(Y)\}$ is independent Laplace–Wishart if and only if in addition to the aforementioned conditions, the images $\rho_k(\Sigma^+)$ of the Moore–Penrose inverse Σ^+ of Σ are mutually orthogonal: $\rho_k(\Sigma^+)\rho_\ell(\Sigma^+) = 0$ for $k \neq \ell$.

1. Introduction

Let Y be a real, Hermitian, or quaternion normal random matrix with mean zero and covariance Σ_Y. Necessary and sufficient conditions are obtained for a quadratic form $Q(Y)$ to have a Wishart–Laplace distribution (the distribution of the difference of two independent central Wishart $W_p(m_i, \Sigma)$ random matrices). Further, necessary and sufficient conditions are obtained for a family of quadratic forms

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* Corresponding author.
E-mail addresses: jmasaro1cogeco.ca (J. Masaro), whynanhaiyahoo.com (C.S. Wong).

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$\{Q_k(Y)\}$ to be independent Laplace–Wishart. This is a generalization of Cochran's Theorem [3]. For history, see [15] and the references therein.

To obtain the most general solution to our problem requires the theory of Jordan algebras. To see why, consider the case where Y is an $n \times p$ real normal random matrix and $Q(Y) = Y'WY$. Since Σ_Y and Σ (the Laplace–Wishart parameter) are symmetric, we should develop the theory needed within families $\mathcal{S}_N$, of $N \times N$ symmetric matrices where N may be p or np. Let $A, B \in \mathcal{S}_N$. The usual matrix product AB may not be symmetric. So we introduce a Jordan product:

$$A * B = \frac{1}{2}(AB + BA),$$

or more generally,

$$A *_C B = \frac{1}{2}(ACB + BCA),$$

where $C \in \mathcal{S}_N$. Then $(\mathcal{S}_N, *_C)$ is an example of a Jordan algebra; see, e.g., [4]. Now consider the distribution of $Q_Y = Y'WY$ through its moment generating function M_{Q_Y}, where W is symmetric. From Masaro and Wong [9],

$$M_{Q_Y}(t) = Det[I_n \otimes I_p - 2\Sigma_Y^{1/2}(W \otimes t)\Sigma_Y^{1/2}]^{-1/2}, \quad t \in \mathcal{N}_\circ,$$

where $\mathcal{N}_\circ = \{t \in \mathcal{S}_p : I_n \otimes I_p - 2\Sigma_Y^{1/2}(W \otimes t)\Sigma_Y^{1/2}$ is positive definite$\}$, i.e., M_{Q_Y} is determined by the linear map $\rho : \mathcal{S}_p \to \mathcal{S}_{np}$ with

$$\rho(t) = \Sigma_Y^{1/2}(W \otimes t)\Sigma_Y^{1/2}, \quad t \in \mathcal{S}_p.$$

It can be proved (see [9]) that $Y'WY$ is Wishart with nonsingular scale parameter Σ if and only if ρ is homomorphic from $(\mathcal{S}_p, *_\Sigma)$ into $(\mathcal{S}_q, *)$, i.e., it preserves the Jordan products:

$$\rho(A *_\Sigma B) = \rho(A) * \rho(B), \quad A, B \in \mathcal{S}_p.$$

The mapping ρ is referred to as a representation in (Jordan) algebra theory.

Our main result, Theorem 4.3 and its corollaries extend the above result to a result for the Wishart–Laplace distribution. The framework of Jordan algebras provides a unified and natural approach in that the cases where Y is a real, complex or quaternionic normal random matrix can be dealt with simultaneously as part of a general theory. In addition we feel that this point of view provides a deeper insight into the nature of the Wishart–Laplace distribution.

We remark that the application of Jordan algebras to statistics and probability has appeared in numerous papers (see [2,7,8,10,11,12,9]). For brevity, whenever convenient, we shall refer certain notations and results to Faraut and Koryányi [4], Masaro and Wong [9], and the references therein. In particular, we shall assume the Jordan algebra results we need and for readers who are not familiar with such results, give some remarks on the classical case of real symmetric matrices.

2. Preliminaries

Let $\mathcal{A}_d$ denote $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ according to $d = 1, 2$ or 4, where $\mathbf{R}$ denotes the field of real numbers, $\mathbf{C}$ denotes the field of complex numbers and $\mathbf{H}$ denotes the division ring of quaternions over $\mathbf{R}$. Each $x \in \mathbf{H}$ may be represented as $x = x_1 + ix_2 + jx_3 + kx_4$, where $x_i \in \mathbf{R}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The *conjugate* of x is $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$ and the *real part* of x is $Re\, x = x_1$. Analogous definitions apply to $\mathbf{C}$. Note that multiplication in $\mathbf{H}$ is associative but not commutative, and for $x, y \in \mathbf{H}$, $\overline{xy} = \bar{y}\bar{x}$.

Let $\mathcal{M}_{n\times p}^d$ denote the family of $n \times p$ matrices over $\mathcal{A}_d$. A matrix $A \in \mathcal{M}_{n\times p}^4$ may be written as $A = A_1 + iA_2 + jA_3 + kA_4$, where $A_i \in \mathcal{M}_{n\times p}^1$. The *transpose*, *conjugate* and *adjoint* of A are defined by $A' = A_1' + iA_2' + jA_3' + kA_4'$, $\overline{A} = A_1 - iA_2 - jA_3 - kA_4$ and $A^* = \overline{A}'$ respectively. (Here A_i' is the usual matrix transpose for a matrix over $\mathbf{R}$.) Similar notions apply to $\mathcal{M}_{n\times p}^d$ for $d = 1, 2$. For $A \in \mathcal{M}_{n\times p}^1$, it follows that $\overline{A} = A$, $A^* = A'$ and $Re\, A = A$. We remark that the familiar formulas $(AB)' = B'A'$ and

$\overline{AB} = \overline{A}\overline{B}$ hold only for the cases $d = 1$ or 2, but the formulas $(AB)^* = B^*A^*$ and $Re\,Tr(AB) = Re\,Tr(BA)$ hold for $d = 1, 2$ or 4.

A matrix $A \in \mathcal{M}^d_{n\times n}$ is called *Hermitian* if $A^* = A$. The family of $n \times n$ Hermitian matrices over $\mathcal{A}_d$ will be denoted by $\mathcal{H}^d_n$. *We shall view $\mathcal{M}^d_{n\times p}$ as a Euclidean vector space, i.e., a vector space over* **R** with the inner product $\langle A, B\rangle = Re\,Tr(AB^*)$. Thus the dimension of $\mathcal{M}^d_{n\times p}$ is npd. Also we have $\langle A, B\rangle = \langle \overline{A}, \overline{B}\rangle$ and $\langle A, BC\rangle = \langle B^*A, C\rangle = \langle AC^*, B\rangle$ whenever the matrix multiplication is defined.

Let $End(\mathcal{M}^d_{n\times p})$ denote the (real) vector space of endomorphisms of $\mathcal{M}^d_{n\times p}$. The *adjoint* of $T \in End(\mathcal{M}^d_{n\times p})$ is denoted by T^*. So for all $X, Y \in \mathcal{M}^d_{n\times p}$, $\langle T(X), Y\rangle = \langle X, T^*(Y)\rangle$. T is called *self-adjoint* if $T^* = T$. The space of self-adjoint endomorphisms will be denoted by $End_S(M^d_{n\times p})$. For $A \in \mathcal{M}^d_{n\times n}$ and $B \in \mathcal{M}^d_{p\times p}$, the Kronecker product $A \otimes B$ is defined as the element in $End(\mathcal{M}^d_{n\times p})$ such that

$$(A \otimes B)(C) = ACB^*.$$

For $X \in \mathcal{M}^d_{n\times p}$, let $\delta(X)$ (or $[X]$) be the coordinate vector of X in $\mathbf{R}^{npd}$ with respect to an orthogonal basis for $\mathcal{M}^d_{n\times p}$ and for $T \in End(\mathcal{M}^d_{n\times p})$, let $\varphi(T)$ (or $[T]$) be the matrix representation with respect to this basis. Thus $\langle X, Y\rangle = \langle \delta(X), \delta(Y)\rangle$ and $\delta(T(X)) = \varphi(T)\delta(X)$. Also $\varphi(T^*) = \varphi(T)'$ and T is self-adjoint (nonnegative definite, positive definite) according as $\varphi(T)$ is symmetric (nonnegative definite, positive definite).

We shall view $End(\mathcal{M}^d_{n\times p})$ as a real inner product space with inner product

$$\langle S, T\rangle = \langle \varphi(S), \varphi(T)\rangle = Tr(\varphi(S)\varphi(T)').$$

One may also write $\langle S, T\rangle = Tr(ST^*)$ since by definition, $Tr(ST^*) = Tr\,\varphi(ST^*) = Tr(\varphi(S)\varphi(T)')$.

The following well known lemma will be useful. A proof may be found in Masaro and Wong [9].

Lemma 2.1. *Let $A, C \in \mathcal{M}^d_{n\times n}$ and $B, D \in \mathcal{M}^d_{p\times p}$. Then:*

(a) $\langle A \otimes B, C \otimes D\rangle = d\langle A, C\rangle\langle B, D\rangle$;
(b) $Tr(A \otimes B) = dRe\,Tr(A)Re\,Tr(B)$.

Let $N_k(\gamma, \Sigma)$ denote the usual normal distribution over $\mathbf{R}^k$ with mean γ and nonnegative definite covariance matrix Σ. A random variable Y taking values in $\mathcal{M}^d_{n\times p}$, $d = 1, 2$ or 4, is said to have a *real*, *complex* or *quaternion normal distribution* with mean $\mu_Y \in \mathcal{M}^d_{n\times p}$ and covariance matrix $\Sigma_Y \in End_S(\mathcal{M}^d_{n\times p})$ if $\delta(Y) \sim N_{npd}(\delta(\mu_Y), \frac{1}{d}\varphi(\Sigma_Y))$. In this case, we write $Y \sim N^d_{n\times p}(\mu_Y, \Sigma_Y)$. Note that $N_k(\gamma, \Sigma) = N^1_{k\times 1}(\gamma, \Sigma \otimes 1)$. For more information on the complex and quaternion normal models see [1,13,14] and the references therein.

A random variable U taking values in $\mathcal{H}^d_p$, $d = 1, 2$, or 4, is said to have a *real*, *complex* or *quaternion Wishart distribution* with m degrees of freedom and scale matrix $\Sigma \in \mathcal{H}^d_p$ if $U \overset{d}{=} Z^*Z$, where $Z \sim N^d_{m\times p}(0, I_m \otimes \Sigma)$. In this case, we write $U \sim W^d_p(m, \Sigma)$.

Remark 2.1. Usually, in the case $d = 2$, one defines the $W^2_p(n, \Sigma)$ distribution to be the distribution of $\sum_{i=1}^n Y_iY_i^*$, where the Y_i's are *iid* $N^2_{p\times 1}(0, \Sigma \otimes 1)$ (see [5]). Note that one may also write $\sum_{i=1}^n Y_iY_i^*$ as $Y\overline{Y}$, where $Y' = [Y_1, Y_2, \ldots, Y_n]$. Let $Z = \overline{Y}$. Then $Z^*Z = Y'\overline{Y}$ and since the Y_i^*'s are *iid* $N^2_{1\times p}(0, 1 \otimes \Sigma)$, $Z \sim N^2_{n\times p}(0, I_n \otimes \Sigma)$.

A random variable V taking values in $\mathcal{H}^d_p$, $d = 1, 2$, or 4, is said to have a *real*, *complex* or *quaternion Wishart–Laplace distribution* with (m_1, m_2) degrees of freedom and scale matrix $\Sigma \in \mathcal{H}^d_p$ if $V \overset{d}{=} Z^*KZ$, where $K = diag[I_{m_1}, -I_{m_2}]$ and $Z \sim N^d_{m\times p}(0, I_m \otimes \Sigma)$. In this case, we write $V \sim DW^d_p(m_1, m_2, \Sigma)$.

It is clear that $DW_p^d(m_1, m_2, \Sigma)$ is the distribution of the difference $V_1 - V_2$, where V_1 and V_2 are independent $W_p^d(m_1, \Sigma)$ and $W_p^d(m_2, \Sigma)$.

Let $\psi : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ be a linear map. For each $y \in \mathcal{M}_{n\times p}^d$, the linear form on the real vector space $\mathcal{H}_p^d$ defined by $t \to \langle y, \psi(t)y\rangle$ is given by an inner product on $\mathcal{H}_p^d$. (For simplicity, $\psi(t)(y)$ is abbreviated as $\psi(t)y$.) Thus there is an element in $\mathcal{H}_p^d$ depending y and ψ, call it $Q_\psi(y)$, such that

$$\langle y, \psi(t)y\rangle = \langle t, Q_\psi(y)\rangle \tag{2.1}$$

for all $t \in \mathcal{H}_p^d$. We call the map $Q_\psi : \mathcal{M}_{n\times p}^d \to \mathcal{H}_p^d$ the *$\mathcal{H}_p^d$-valued quadratic form* associated with the linear map ψ.

If $Y \sim N_{n\times p}^d(0, \Sigma_Y)$, then $Q_\psi(Y)$ is a random quadratic form taking values in $\mathcal{H}_p^d$. The mean of $Q_\psi(Y)$ can be obtained as follows:

$$\langle t, E(Q_\psi(Y))\rangle = \frac{1}{d}\langle \Sigma_Y, \psi(t)\rangle. \tag{2.2}$$

Indeed, $\langle t, E(Q_\psi(Y))\rangle = E\langle t, Q_\psi(Y)\rangle = E\langle Y, \psi(t)(Y)\rangle = E\langle \delta(Y), \quad \varphi(\psi(t))\delta(Y)\rangle = E\langle \delta(Y)\delta(Y)', \varphi(\psi(t))\rangle = \langle \frac{1}{d}\varphi(\Sigma_Y), \varphi(\psi(t))\rangle = \frac{1}{d}\langle \Sigma_Y, \psi(t)\rangle$.

We shall now give an example of random quadratic form.

Example 2.1. Let $Y \sim N_{n\times p}^d(0, \Sigma_Y)$, $W \in \mathcal{H}_n^d$ and $\psi : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$, $\psi(t) = W \otimes t$. Then $\langle t, Q_\psi(Y)\rangle = \langle Y, \psi(t)(Y)\rangle = \langle Y, WYt^*\rangle = \langle Y^*W^*Y, t^*\rangle = \langle Y^*WY, t\rangle$.

Thus $Q_\psi(Y) = Y^*WY$. Further, in the case $W = diag[I_{n_1}, -I_{n_2}]$ and $\Sigma_Y = I_n \otimes \Sigma$, we have $Q_\psi(Y) \sim DW_p^d(n_1, n_2, \Sigma)$ and by (2.2) and Lemma 2.1, $\langle t, E(Q_\psi(Y))\rangle = \frac{1}{d}\langle I_n \otimes \Sigma, W \otimes t\rangle = \frac{1}{d}(n_1 - n_2)d\langle \Sigma, t\rangle$. Hence

$$E(Q_\psi(Y)) = (n_1 - n_2)\Sigma. \tag{2.3}$$

3. Jordan algebras

In this section we shall summarize the notions and results from the theory of Jordan algebras that we require for the statements and proofs of our results. Where convenient we illustrate a concept with an example from the Jordan algebra of symmetric matrices. Only some results are proven. Other results are known and can be found in Faraut and Koryányi [4] or Jacobson [6]. We shall not repeat the content of Masaro and Wong [9] unless it affects the lucidity of the present paper.

A *Jordan algebra* V over the set $\mathbf{R}$ of real numbers is a real vector space with a product ab such that $ab = ba$, $\lambda(ab) = (\lambda a)b$, $(a_1 + a_2)b = a_1b + a_2b$ and $a(a^2b) = a^2(ab)$ for λ in $\mathbf{R}$ and a, a_1, a_2 and b in V. An element e in V will be called an *identity* if $ex = x$ for all x in V.

An element c in V is an *idempotent* if $c^2 = c$. Two idempotents c and d are called *orthogonal* if $cd = 0$. An idempotent is said to be *primitive*, if it is non-zero and cannot be written as the sum of two non-zero idempotents. A set of idempotents $\{c_1, c_2, \ldots, c_m\}$ in V is called a *Jordan frame* if all c_i are primitive with $c_ic_j = 0$ for $i \neq j$ and $\Sigma_1^m c_i = e$. A mapping ϕ of V to a Jordan algebra $(W, \#)$ is a *homomorphism* if ϕ is linear and $\phi(ab) = \phi(a) \# \phi(b)$ for all a, b in V. Since the *polarization identity* $xy = \frac{(x+y)^2-(x-y)^2}{4}$ holds in V, the linear map ϕ will be a *homomorphism* iff $\phi(a^2) = \phi(a) \# \phi(a)$ for all a in V. If ϕ is one to one and onto W, then ϕ is called an *isomorphism* of V onto W and V and W are said to be *isomorphic*. A subset I of V is an *ideal* in V if I is a linear subspace of V and for any x in I, y in V, xy belongs to I; V is said to be *simple* if its only ideals are $\{0\}$ and V itself.

It can be shown (see [6]) that there exists a unique integer $r > 0$ and unique functions $a_j : V \to \mathbf{R}$ such that the a_j's are homogeneous of degree j and for all x in V,

$$x^r - a_1(x)x^{r-1} + a_2(x)x^{r-2} - \cdots + (-1)^r a_r(x) = 0.$$

The polynomial $m_x(X) = X^r - a_1(x)X^{r-1} + a_2(x)X^{r-2} - \cdots + (-1)^r a_r(x)$ is called the *generic minimum polynomial* for x; the degree 'r' of $m_x(X)$ is called the *rank* of the Jordan algebra V. The *generic trace* and *generic determinant* of x in V are defined by

$$tr(x) = a_1(x) \quad \text{and} \quad det(x) = a_r(x).$$

We shall use upper case notation, *Det*, *Tr* to denote the usual trace and determinant for matrices (endomorphisms) and lower case notation *det*, *tr* to denote the generic trace and determinant of an element in a Jordan algebra. When required, the notation tr_W and det_W will be used to denote the generic trace and determinant with respect to a specific Jordan algebra W.

The spaces $\mathcal{H}_r^d$, $d = 1, 2, 4$ are Jordan algebras when endowed with the Jordan product $A \circ B = \frac{1}{2}[AB + BA]$, with the product on the right side being the usual matrix product. In the case $d = 1$ the generic trace and determinant correspond to the usual trace and determinant for matrices.

A Jordan algebra V is *Euclidean* if there exists an inner product $\langle \cdot, \cdot \rangle$ on V that is associative: $\langle ab, c\rangle = \langle b, ac\rangle$ for all a, b and c in V. In every Euclidean Jordan algebra with identity, the *generic trace form*, $(x, y) \to tr(xy)$ is positive definite and associative. *Unless otherwise stated, we assume that the inner product in a finite dimensional Euclidean Jordan algebra V with identity is given by* $\langle x, y\rangle = tr(xy)$, $x, y \in V$.

The Jordan algebras $\mathcal{H}_r^d$, $d = 1, 2, 4$ are simple and Euclidean. The generic trace and corresponding inner product are given by $tr\, A = Re\, Tr\, A$ and $\langle A, B\rangle = tr(A \circ B)$. Note that since A and B are Hermitian, $Re\, Tr\, A = Tr\, A$ and $\langle A, B\rangle = Re\, Tr(AB) = Re\, Tr(AB^*)$ so that the inner product for $\mathcal{H}_r^d$ is simply the inner product inherited from the space $\mathcal{M}_{r\times r}^d$ as described in Section 2.

For x in V, the linear map $L(x) : V \to V$ is defined by $L(x)(v) = xv$. Further, we define $P(x) = 2L(x)^2 - L(x^2)$ and $P(x, y) = L(x)L(y) + L(y)L(x) - L(xy)$. Since the inner product is associative, $L(x)$, $P(x)$ and $P(x, y)$ are self-adjoint. The map $x \to P(x)$ is called the *quadratic representation* of V. In the Jordan algebra $\mathcal{H}_r^1$ (with product $A \circ B = \frac{1}{2}[AB + BA]$), we have

$$P(A)B = ABA \quad \text{and} \quad P(A, B)C = \frac{1}{2}[ACB + BCA].$$

An element x in V is said to be *positive definite* (*nonnegative definite*) if $L(x)$ is positive definite (nonnegative definite). In the case $V = \mathcal{H}_r^1$ this agrees with the usual matrix definitions. We let $\Omega(V) = \{x \in V : x \text{ is positive definite}\}$ and $\overline{\Omega}(V) = \{x \in V : x \text{ is nonnegative definite}\}$.

For an idempotent c in V, the *Pierce spaces* $V(c, i)$ are defined by

$$V(c, i) = \{x \in V : cx = ix\}, \quad i = 0, 1/2, 1.$$

It is well known that

$$V = V(c, 1) \oplus V(c, 1/2) \oplus V(c, 0) \quad \text{(a vector space direct sum)}.$$

This decomposition (called the *Pierce decomposition*) is orthogonal with respect to any associative inner product on V. Also, $V(c, 1)$ and $V(c, 0)$ are Jordan subalgebras of V and c is an identity for $V(c, 1)$. The projections of V onto $V(c, 1)$, $V(c, 1/2)$ and $V(c, 0)$ are $P(c)$, $I - P(c) - P(e - c)$ and $P(e - c)$ respectively. For example, if we take $V = \mathcal{H}_r^1$, $r = p + q$ and $c = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$, then

$$V(c, 1) = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A \in \mathcal{H}_p^1 \right\},$$
$$V(c, 0) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} : B \in \mathcal{H}_q^1 \right\},$$
$$V(c, 1/2) = \left\{ \begin{bmatrix} 0 & D \\ D' & 0 \end{bmatrix} : D \in \mathcal{M}_{p\times q}^1 \right\}.$$

If V is simple, the value $d = dim\left[V\left(a, \frac{1}{2}\right) \cap V\left(b, \frac{1}{2}\right)\right]$ is invariant for any pair of orthogonal primitive idempotents a, b. The value d is called the *Pierce invariant* and it is related to the dimension and rank of V by $n = r + r(r - 1)\frac{d}{2}$. Moreover, when V is simple, so is $V(c, 1)$ (this follows from Proposition IV.1.2 of [4]); if c is not primitive, the Pierce invariant for $V(c, 1)$ is also equal to d.

Suppose that the rank of V is r. Then for each x in V, there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and $\lambda_i \in \mathbf{R}$ such that

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r. \tag{3.1}$$

The numbers λ_i (with their multiplicities) are uniquely determined by x and are called the *eigenvalues* of x. Further, $tr(x) = \sum_{i=1}^r \lambda_i$ and $det(x) = \prod_{i=1}^r \lambda_i$. The decomposition (3.1) is called the *spectral decomposition* of x. The *rank* of x, $rk(x)$, is the number of non-zero eigenvalues (with multiplicities counted) in its spectral decomposition. For $V = \mathcal{H}_r^1$ $rk(x)$ agrees with the usual definition of matrix rank.

Let

$$x^+ = \sum_{\lambda_i \neq 0} \lambda_i^{-1} c_i, \quad x^0 = \sum_{\lambda_i \neq 0} c_i \quad \text{and} \quad x^\alpha = \sum_{\lambda_i \neq 0} \lambda_i^\alpha c_i, \tag{3.2}$$

where α may be any real number if all λ_i are positive and α may be an integer if some λ_i are negative. Then $x^+,\ x^0,\ x^\alpha$ are well-defined (see [9]). In the special case $V = \mathcal{H}_r^1$ of Hermitian (symmetric) $r \times r$ matrices over $\mathbf{R}$ these definitions correspond to the usual notions of Moore–Penrose inverse, orthogonal projection and matrix powers in linear algebra arising from the consideration of Wishart distributions with singular scale matrix.

Let V be a Jordan algebra over $\mathbf{R}$ and M a vector space over $\mathbf{R}$. A *representation* of V on M is a linear map $\tau : V \to End(M)$ such that

$$\tau(xy) = \frac{1}{2}(\tau(x)\tau(y) + \tau(y)\tau(x)),$$

i.e., the map τ is a Jordan algebra homomorphism of V into $End(M)$ equipped with the Jordan product $A \circ B = \frac{1}{2}(AB + BA)$. The representation τ is said to be *self-adjoint* if for any $x \in V$, $\tau(x)$ is a self-adjoint endomorphism on M.

For example, the *standard representation* for $\mathcal{H}_r^d$, is given by: $\tau : \mathcal{H}_r^d \to End(\mathbf{R}^{dr})$ with

1. For $A \in \mathcal{H}_r^1,\ \tau(A) = A$.
2. For $A = A_1 + iA_2 \in \mathcal{H}_r^2,\ \tau(A) = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$.
3. For $A = A_1 + iA_2 + jA_3 + kA_4 \in \mathcal{H}_r^4,\ \tau(A) = \begin{bmatrix} A_1 & -A_2 & -A_3 & -A_4 \\ A_2 & A_1 & -A_4 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & -A_3 & -A_2 & A_1 \end{bmatrix}$.

The generic trace and determinant for $\mathcal{H}_r^d$ may be obtained from

$$d\, tr\, A = dRe\, Tr\, A = Tr\, \tau(A) \quad \text{and} \quad (det\, A)^d = Det\, \tau(A).$$

For brevity we will only state Theorem 3.1 leaving its proof as an exercise for those readers familiar with Jordan algebras.

Theorem 3.1. I. *Let*:

(i) *L be a Jordan algebra with identity e_L and $\mathcal{A}$ be an associative algebra*;
(ii) *$\rho : L \to \mathcal{A}$ be a linear map*;
(iii) *$E = \rho(e_L)$*;
(iv) *$(\mathcal{A}, *)$ and $(\mathcal{A}, *_E)$ be the Jordan algebras obtained from $\mathcal{A}$ using the products $A * B = \frac{1}{2}(AB + BA)$ and $A *_E B = \frac{1}{2}(AEB + BEA)$.*

Then (a)–(c) *below are equivalent*:

(a) *$\rho : L \to (\mathcal{A}, *_E)$ is a Jordan algebra homomorphism such that $E\rho(x) = \rho(x)E$ for all x in L*;
(b) *$\rho(x) = \rho_1(x) - \rho_2(x)$, where ρ_1 and ρ_2 are Jordan algebra homomorphisms of L into $(\mathcal{A}, *)$ such that $\rho_1(x)\rho_2(y) = \rho_2(y)\rho_1(x) = 0$ for all x, y in L*;

(c) $\rho(x) = \rho_1(x) - \rho_2(x)$, *where* ρ_1 *and* ρ_2 *are Jordan algebra homomorphisms of* L *into* $(\mathcal{A}, *)$ *such that* $\rho_1(e_L)\rho_2(e_L) = \rho_2(e_L)\rho_1(e_L) = 0$.

II. *In the case that one of* (a)–(c) *holds, the homomorphisms* ρ_1 *and* ρ_2 *are uniquely determined and are given by*

$$\rho_1(x) = \frac{\rho(x) + E\rho(x)}{2} \quad \textit{and} \quad \rho_2(x) = \frac{-\rho(x) + E\rho(x)}{2}.$$

Let V be a Euclidean Jordan algebra with identity e. Given an element u in V, one may define a new composition $x * y = P(x, y)u$. Then V equipped with the product $*$ is also a Jordan algebra and is called the mutation of V with respect to u and is denoted by $MV(u)$. Also $MV(u) = M_1V(u) \oplus M_2V(u)$ where $M_1V(u) = P(u^\circ)V$ and $M_2V(u) = (I - P(u^\circ))V$. Note that $M_1V(u)$ is just the mutation of the Pierce space $V(u^\circ, 1)$ with respect to u.

The following example will provide a more concrete understanding of mutations.

Example 3.1. Let $V = \mathcal{H}_r^1$ be the Jordan algebra of $r \times r$ Hermitian matrices over $\mathbf{R}$ with composition $A \circ B = \frac{1}{2}(AB + BA)$ and let $\Sigma \in \mathcal{H}_r^1$ be nonnegative definite. Then:

Σ^+ is the Moore–Penrose inverse of Σ;
$\Sigma^\circ = \Sigma\Sigma^+$, the orthogonal projection of $\mathbf{R}^r$ onto $Im\ \Sigma$;
$M_1V(\Sigma) = \{A \in V : \Sigma^\circ A \Sigma^\circ = A\}$;
$M_2V(\Sigma) = \{A \in V : \Sigma^\circ A \Sigma^\circ = 0\}$.

Moreover in $MV(\Sigma)$, $A * B = \frac{1}{2}[A\Sigma B + B\Sigma A]$ and in $M_1V(\Sigma)$,

$$tr_1(A) = Tr(\Sigma^{1/2}A\Sigma^{1/2})$$

and

$$det_1(A) = Det(I - \Sigma^\circ + \Sigma^{1/2}A\Sigma^{1/2}),$$

where tr_1, det_1 are the generic trace and determinant for the Jordan algebras $MV_1(\Sigma)$ (see Lemma 3.5.1 of [9]).

Theorem 3.2 is an extension of Theorem 3.5.2 in Masaro and Wong [9] so we state it without proof. This theorem is the key to our main result (Theorem 4.3) in Section 4. It is important to note that Theorem 3.2 was, in part, motivated by Example 3.1. Indeed, the conditions (1)–(3) of Theorem 3.2 are satisfied by Example 3.1 with $V = \mathcal{H}_r^1, J = MV(\Sigma)$, $L = M_1V(\Sigma)$, $K = M_2V(\Sigma)$, $P_L = P(\Sigma^o)$, and $P_K = I_r - P(\Sigma^o)$.

Theorem 3.2. *Suppose that* (1)–(5) *hold*:

(1) J *is a Jordan algebra*.
(2) $J = L \oplus K$, *a vector space direct sum, where*

  (i) L *is a Jordan subalgebra of* J *of rank* r *with identity* e_L *and* L *is simple and Euclidean*;
  (ii) K *is an ideal in* J.

(3) P_L *and* P_K *are the projections of* J *onto* L *and* K *respectively*.
(4) M *is a vector space over* $\mathbf{R}$ *of dimension* m, $\mathcal{A} = End(M)$ *and* $\mathcal{A}_S = End_S(M)$ *with the usual composition product in* $\mathcal{A}$ *and* $\mathcal{A}_S$ *denoted by* AB; $(\mathcal{A}, *)$, $(\mathcal{A}, *_C)$, $(\mathcal{A}_S, *)$ *and* $(\mathcal{A}_S, *_C)$ *will denote the Jordan algebras obtained from* $\mathcal{A}$ *and* $\mathcal{A}_S$ *with Jordan products* $A * B = \frac{1}{2}(AB + BA)$ *and* $A *_C B =$

$\frac{1}{2}(ACB + BCA)$ *and* I_M *will denote the identity mapping in* $End(M)$. *The usual trace and determinant for members of* $End(M)$ *will be denoted by Tr and Det.*

(5) $\rho : J \to W_S$ *is a linear map,* $E = \rho(e_L)$ *and* $\rho_1, \rho_2 : J \to W$ *are linear maps defined by*

$$\rho_1(x) = \frac{\rho(x) + E\rho(x)}{2} \text{ and } \rho_2(x) = \frac{-\rho(x) + E\rho(x)}{2}, \quad x \in J.$$

Then:

I. (a)–(e) *below are equivalent*:

(a) ρ *is a Jordan algebra homomorphism of* J *into* $(\mathcal{A}_S, *_E)$ *with* $K = ker\rho$ *and* $E\rho(x) = \rho(x)E$ *for all* x *in* J;

(b) ρ_1, ρ_2 *are Jordan algebra homomorphisms of* J *into* $(\mathcal{A}_S, *)$ *such that* $\rho_1(e_L)\rho_2(e_L) = 0$ *and* $K = ker\, \rho_1 = ker\, \rho_2$;

(c) *there exist integers* $s_1,\ s_2 > 0$ *such that for all* $x \in J$,

$$Det(I_M - \rho(x)) = det_L(e_L - P_L x)^{s_1} det_L(e_L + P_L x)^{s_2}; \tag{3.3}$$

(d) *there exist integers* $s_1,\ s_2 > 0$ *such that for all* $x \in J$ *and* $k = 1, 2, \ldots,$

$$Tr\, \rho(x)^k = [s_1 + (-1)^k s_2] tr_L(P_L x)^k; \tag{3.4}$$

(e) *there exist integers* $s_1,\ s_2 > 0$ *such that for all* $x \in J$, $Tr\, \rho_j(x)^k = s_j tr_L(P_L(x))^k,\ j = 1, 2;\ j = 1, 2,$ *and* $\rho_1(e_L)\rho_2(e_L) = \rho_2(e_L)\rho_1(e_L) = 0$.

II. *In the case one of* (a)–(e) *holds,* $s_j = \frac{Tr\, \rho_j(e_L)}{r} = Tr(\rho_j(c))$, *where* c *is any primitive idempotent in* L. *Further, if* $r \geqslant 2$, *then* $s_j = m_j d,\ j = 1, 2$, *where* d *is the Pierce invariant of* L *and* m_j *is a positive integer. Also* $\rho_1(x)\rho_2(y) = 0$ *for all* $x,\ y$ *in* L.

4. Characterization of the Wishart–Laplace distributions

We shall, in Theorem 4.3, characterize the Wishart–Laplace distributions in terms of Jordan algebra representations (Theorem 4.4). This is accomplished by linking the moment generating function of the Wishart–Laplace distributions with these homomorphisms via Theorem 3.2. For the convenience of the reader, we shall reintroduce some of our earlier notation: $\mathcal{H}_p^d,\ d = 1, 2, 4$, will denote the simple Euclidean Jordan algebras as described in Section 3 and $M\mathcal{H}_p^d(A)$ its mutation with respect to an element $A \in \mathcal{H}_p^d$. Lower case notation '*tr*', '*det*' refers to the generic trace and determinant and upper case notation '*Tr*' '*Det*' is the usual trace and determinant for matrices (operators), in this case, for endomorphisms in $End(\mathcal{M}_{n\times p}^d)$ or $End(\mathbf{R}^{npd})$. We will also make use of the functions δ and φ as described in Section 2. Note that $End(\mathcal{M}_{n\times p}^d)$ is a Euclidean Jordan algebra with identity $I_n \otimes I_p$. Thus for $T \in End(\mathcal{M}_{n\times p}^d),\ P(T)$ is the linear operator given by $P(T)S = TST,\ S \in End(\mathcal{M}_{n\times p}^d)$. Finally note that for any U in $\overline{\Omega}(\mathcal{H}_p^d)$ and $\alpha,\ \beta$ in $\mathbf{R}$ we have $P(U^\alpha)P(U^\beta) = P(U^{\alpha+\beta})$ where U^α is defined as in (3.2) (see Lemma 3.3.1 of [9]).

We begin with some results on moment generating functions of quadratic forms.

Theorem 4.1

(a) *Let* $Y \sim N_{n\times p}^d(0, \Sigma_Y),\ \psi : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ *be a linear map and* $Q_\psi : \mathcal{M}_{n\times p}^d \to \mathcal{H}_p^d$ *the associated quadratic form. Then the moment generating function of* $Q_\psi(Y)$ *is*

$$M_{Q_\psi}(t) = Det\left[I_n \otimes I_p - \frac{2}{d} P(\Sigma_Y^{1/2})\psi(t)\right]^{-1/2}$$

for $t \in \mathcal{H}_p^d$ *such that* $I_n \otimes I_p - \frac{2}{d}P(\Sigma_Y^{1/2})\psi(t)$ *is positive definite.*

(b) *Let* $U \sim DW_p^d(m_1, m_2, \Sigma),\ \Sigma \in \overline{\Omega}(\mathcal{H}_p^d)$. *Then the moment generating function of U is*

$$M_U(t) = det\left[I_p - \frac{2}{d}P(\Sigma^{1/2})t\right]^{-m_1 d/2} det\left[I_p + \frac{2}{d}P(\Sigma^{1/2})t\right]^{-m_2 d/2}$$

for $t \in \mathcal{H}_p^d$ *such that* $I_p \pm \frac{2}{d}P(\Sigma^{1/2})t \in \Omega(\mathcal{H}_p^d)$.

Proof. The proof of (a) may be found in Masaro and Wong [9].

(b) Since $U \stackrel{d}{=} Z^*KZ$ where $Z \sim N_{m\times p}^d(0, I_m \otimes \Sigma)$ and $K = diag[I_{m_1}, -I_{m_2}]$, we may apply (a) with $n = m = m_1 + m_2,\ \Sigma_Y = I_m \otimes \Sigma$ and $\psi(t) = K \otimes t$ to obtain

$$\begin{aligned} M_U(t) &= Det\left[I_m \otimes I_p - \frac{2}{d}P(I_m \otimes \Sigma^{1/2})(K \otimes t)\right]^{-1/2} \\ &= Det\left[I_m \otimes I_p - K \otimes \frac{2}{d}P(\Sigma^{1/2})t\right]^{-1/2} \\ &= Det\left[diag[I_{m_1} \otimes \left(I_p - \frac{2}{d}P(\Sigma^{1/2})t\right), I_{m_2} \otimes \left(I_p + \frac{2}{d}P(\Sigma^{1/2})t\right),]\right]^{-1/2} \\ &= Det\left[I_{m_1} \otimes \left(I_p - \frac{2}{d}P(\Sigma^{1/2})t\right)\right]^{-1/2} Det\left[I_{m_2} \otimes \left(I_p + \frac{2}{d}P(\Sigma^{1/2})t\right)\right]^{-1/2}. \end{aligned}$$

Since the map $x \to I_{m_i} \otimes x$ is a self-adjoint representation of $\mathcal{H}_p^d$ on $\mathcal{M}_{m_i \times p}^d$ such that $I_p \to I_{m_i} \otimes I_p$, we can apply Proposition IV.4.2 of [4] (with $N = m_i pd$ and $r = p$) to obtain

$$M_U(t) = det\left[I_p - \frac{2}{d}P(\Sigma^{1/2})t\right]^{-m_1 d/2} det\left[I_p + \frac{2}{d}P(\Sigma^{1/2})t\right]^{-m_2 d/2}$$

for $t \in \mathcal{H}_p^d$ such that $I_p \pm \frac{2}{d}P(\Sigma^{1/2})t \in \Omega(\mathcal{H}_p^d)$. □

Corollary 4.2. *Let* $Y,\ \psi,\ Q_\psi$ *be as in Theorem* 4.1, $m_1, m_2 \in \{1, 2, 3, \ldots\}$ *and* $\Sigma \in \overline{\Omega}(\mathcal{H}_p^d)$. *Then* (a)–(c) *below are equivalent*:

(a) $Q_\psi(Y) \sim DW_p^d(m_1, m_2, \Sigma)$.

(b) *For all* $t \in \mathcal{H}_p^d$,

$$Det[I_n \otimes I_p - P(\Sigma_Y^{1/2})\psi(t)] = det[I_p - P(\Sigma^{1/2})t]^{m_1 d} det[I_p + P(\Sigma^{1/2})t]^{m_2 d}. \tag{4.1}$$

(c) *For all* $t \in \mathcal{H}_p^d$ *and* $k = 1, 2, \ldots,$

$$Tr[P(\Sigma_Y^{1/2})\psi(t)]^k = [m_1 + (-1)^k m_2] dtr(P(\Sigma^{1/2})t)^k. \tag{4.2}$$

Proof. First assume $Q_\psi(Y) \sim DW_p^d(m_1, m_2, \Sigma)$. Then by Theorem 4.1(a) and (b),

$$\begin{aligned} &Det\left[I_n \otimes I_p - \frac{2}{d}P(\Sigma_Y^{1/2})\psi(t)\right]^{-1/2} \\ &\quad = det\left[I_p - \frac{2}{d}P(\Sigma^{1/2})t\right]^{-m_1 d/2} det\left[I_p + \frac{2}{d}P(\Sigma^{1/2})t\right]^{-m_2 d/2} \end{aligned} \tag{4.3}$$

for all $\frac{2}{d}t \in N_0$, where N_0 is a neighbourhood of 0 in $\mathcal{H}_p^d$. Now (4.3) amounts to

$$Det[I_n \otimes I_p - P(\Sigma_Y^{1/2})\psi(t)] = det[I_p - P(\Sigma^{1/2})t]^{m_1 d} det[I_p + P(\Sigma^{1/2})t]^{m_2 d} \tag{4.4}$$

for all $t \in N_0$. Then by analytic continuation, (4.4) holds for all $t \in \mathcal{H}_p^d$, proving (b).

Conversely, it is clear that (4.1) implies (4.3), which in turn (by Theorem 4.1) implies $Q_\psi(Y) \sim DW_p^d(m_1, m_2, \Sigma)$.

To prove the equivalence of parts (b) and (c), let $t \in H_p^d$, let $\alpha_1, \alpha_2, \ldots, \alpha_{np}$ be the eigenvalues of $P(\Sigma_Y^{1/2})\psi(t)$ and choose a Jordan frame $c_1, c_2, \ldots, c_p$ such that $P(\Sigma^{1/2})t = \sum_{i=1}^p \lambda_i c_i$. Then for $z \in \mathbf{R}$, we have

$$D(z) \equiv Det[I_n \otimes I_p - P(\Sigma_Y^{1/2})\psi(zt)] = \prod_{i=1}^{np}(1 - z\alpha_i)$$

and

$$d(z) \equiv det[I_p - P(\Sigma^{1/2})zt]^{m_1 d} det[I_p + P(\Sigma^{1/2})zt]^{m_2 d} = \prod_{i=1}^{p}(1 - z\lambda_i)^{m_1 d}\prod_{i=1}^{p}(1 + z\lambda_i)^{m_2 d}.$$

Now a comparison of the coefficients of z^k in the power series expansions of $\ln D(z)$ and $\ln d(z)$ shows that $\ln D(z) = \ln d(z)$ if and only if (4.2) holds, which proves the equivalence of (b) and (c). $\square$

We now prove our main result. For better understanding one should keep Example 3.1 in mind.

Theorem 4.3. I. *Suppose that*:

(1) $Y \sim N_{n\times p}^d(0, \Sigma_Y)$;
(2) $\psi : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ *is a linear map*;
(3) $Q_\psi : \mathcal{M}_{n\times p}^d \to \mathcal{H}_p^d$ *is the quadratic form associated with the linear map* ψ;
(4) $\rho : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ *is the linear map defined by* $\rho(x) = P(\Sigma_Y^{1/2})\psi(x)$;
(5) $(\mathcal{A}, *)$ *is the Jordan algebra* $End_S(\mathcal{M}_{n\times p}^d)$ *with the product* $A * B = \frac{1}{2}(AB + BA)$ *and* $(\mathcal{A}, *_E)$ *is the Jordan algebra* $End_S(\mathcal{M}_{n\times p}^d)$ *with the product* $A *_E B = \frac{1}{2}(AEB + BEA)$.

Then $Q_\psi(Y)$ *has a Wishart–Laplace distribution if and only if* (a)–(c) *hold*:

(a) *There exists an element* $\Sigma \in \overline{\Omega}(\mathcal{H}_p^d)$ *such that* $\rho : M\mathcal{H}_p^d(\Sigma) \to (\mathcal{A}, *_E)$ *is a Jordan algebra homomorphism with* $E\rho(x) = \rho(x)E$ *for all* x *in* $M\mathcal{H}_p^d(\Sigma)$, *where* $E = \rho(\Sigma^+)$.
(b) $\ker \rho = M_2\mathcal{H}_p^d(\Sigma)$.
(c) *Either* $rk(\Sigma) \geqslant 2$ *or* $rk(\Sigma) = 1$ *and* $Tr\ \rho_j(\Sigma^+)$ *is divisible by* d, $j = 1, 2$, *where*

$$\rho_1(x) = \frac{\rho(x) + E\rho(x)}{2} \text{ and } \rho_2(x) = \frac{-\rho(x) + E\rho(x)}{2}, \quad x \in \mathcal{H}_p^d.$$

II. *In the case that* (a)–(c) *hold*, $Q_\psi(Y)$ *is* $DW(m_1, m_2, \Sigma)$ *with* $m_1 - m_2 = \frac{Tr\,E}{2d\,rk(\Sigma)}$ *and* $m_1 + m_2 = \frac{Tr\,E^2}{2drk(\Sigma)}$. *Also* $m_j d = tr(\rho_j(c))$, $j = 1, 2$, *where* c *is any primitive idempotent in* $M_1\mathcal{H}_p^d(\Sigma)$ *and for all* x *in* $M\mathcal{H}_p^d(\Sigma)$ *and* $k = 1, 2, \ldots$, $Tr\ \rho_j(x)^k = m_j d tr(\Sigma^{1/2}x\Sigma^{1/2})^k$ *and* $Tr\ \rho(x)^k = d(m_1 + (-1)^k m_2) \times tr(\Sigma^{1/2}x\Sigma^{1/2})^k$.

III. *Condition* (a) *above may be replaced by the condition* (a)$'$: $\rho_1, \rho_2 : M\mathcal{H}_p^d(\Sigma) \to (\mathcal{A}, *)$ *are Jordan algebra homomorphisms such that* $\rho_1(\Sigma^+)\rho_2(\Sigma^+) = 0$.

Proof. I. Assume that $Q_\psi(Y) \sim DW_p^d(m_1, m_2, \Sigma)$, $\Sigma \in \overline{\Omega}(\mathcal{H}_p^d)$. Then for $x \in M\mathcal{H}_p^d(\Sigma)$, $\Sigma^+ - P(\Sigma^\circ)x \in M_1\mathcal{H}_p^d(\Sigma)$ and $P(\Sigma^{1/2})x = P(\Sigma^{1/2})P(\Sigma^\circ)x$. So by Lemma 3.5.1(d) of [9] and Corollary 4.2, we have

$$\begin{aligned} Det[I_n \otimes I_p - \rho(x)] &= det[I_p - P(\Sigma^{1/2})x]^{m_1 d} det[I_p + P(\Sigma^{1/2})x]^{m_2 d} \\ &= det[I_p - P(\Sigma^{1/2})(\Sigma^+ - (\Sigma^+ - P(\Sigma^\circ)x))]^{m_1 d} det[I_p + P(\Sigma^{1/2})(\Sigma^+ - (\Sigma^+ - P(\Sigma^\circ)x))]^{m_2 d} \\ &= det_1[\Sigma^+ - P(\Sigma^\circ)x]^{m_1} det_1[\Sigma^+ + P(\Sigma^\circ)x]^{m_2}, \end{aligned}$$

where det and det_1 are the *generic* determinants in $\mathcal{H}_p^d$ and $M_1\mathcal{H}_p^d(\Sigma)$ respectively.

Now by Theorem 3.2 (with $J = M\mathcal{H}_p^d(\Sigma)$, $L = M_1\mathcal{H}_p^d(\Sigma)$, $K = M_2\mathcal{H}_p^d(\Sigma)$, $W = End_S(\mathcal{M}_{n\times p}^d)$ and $s_j = m_j d$), conditions (a)–(c) hold.

Conversely, assume that (a)–(c) hold. Since this includes condition I(a) of Theorem 3.2 (with J, L, K and W as indicated above), we may apply the equivalent condition I(c) of Theorem 3.2 together with Lemma 3.5.1(d) of [9] to conclude that there exist positive integers $s_j = \frac{Tr\,\rho_j(\Sigma^+)}{rk(\Sigma)}$, $j = 1, 2$, such that

$$\begin{aligned} &Det[I_n \otimes I_p - \rho(x)] \\ &= det_1[\Sigma^+ - P(\Sigma^\circ)x]^{s_1}\, det_1[\Sigma^+ + P(\Sigma^\circ)x]^{s_2} \\ &= det[I_p - P(\Sigma^{1/2})P(\Sigma^\circ)x]^{s_1} det[I_p - P(\Sigma^{1/2})P(\Sigma^\circ)x]^{s_2} \\ &= det[I_p - P(\Sigma^{1/2})x]^{s_1} det[I_p - P(\Sigma^{1/2})x]^{s_2} \end{aligned}$$

for all $x \in M\mathcal{H}_p^d(\Sigma)$. If $rk(\Sigma) \geqslant 2$, then by Theorem 3.2(II), there exist integers $m_j > 0$ such that $s_j = m_j d$, $j = 1, 2$. So by Corollary 4.2, $Q_\psi(Y) \sim DW_p^d(m_1, m_2, \Sigma)$.

Finally, II and III follow from Theorem 3.2 and Lemma 3.5.1(c) and (d) of [9]. $\square$

Corollary 4.4. I. *Suppose that*:

(1) $Y \sim N_{n\times p}^d(0, \Sigma_Y)$;
(2) $Q(Y) = Y^*WY$, *where* $W \in \mathcal{H}_n^d$;
(3) $\rho : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ *is the linear map defined by* $\rho(x) = \Sigma_Y^{1/2}(W \otimes x)\Sigma_Y^{1/2}$.

Then $Q(Y)$ *follows a Wishart–Laplace distribution if and only if there exists an element* $\Sigma \in \overline{\Omega}(\mathcal{H}_p^d)$ *such that* (a)–(c) *below hold*:

(a) *for all* $x \in \mathcal{H}_p^d$, $\rho(x\Sigma x) = \rho(x)\rho(\Sigma^+)\rho(x)$ *and* $\rho(\Sigma^+)\rho(x) = \rho(x)\rho(\Sigma^+)$, *that is*:

$$\Sigma_Y^{1/2}(W \otimes x\Sigma x)\Sigma_Y^{1/2} = \Sigma_Y^{1/2}(W \otimes x)\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes x)\Sigma_Y^{1/2} \tag{4.5}$$

and

$$\Sigma_Y^{1/2}(W \otimes \Sigma^+)\Sigma_Y(W \otimes x)\Sigma_Y^{1/2} = \Sigma_Y^{1/2}(W \otimes x)\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y^{1/2}. \tag{4.6}$$

(b) $ker\,\rho = \{x \in \mathcal{H}_p^d : \Sigma x \Sigma = 0\}$.
(c) *either* $rk(\Sigma) \geqslant 2$ *or* $rk(\Sigma) = 1$ *and* $Tr\,\rho_j(\Sigma^+)$ *is divisible by* $d, j = 1, 2$, *where*

$$\rho_1(x) = \frac{\rho(x) + E\rho(x)}{2} \quad and \quad \rho_2(x) = \frac{-\rho(x) + E\rho(x)}{2} \text{ with } E = \rho(\Sigma^+).$$

II. *In the case that* (a)–(c) *hold,* $Q(Y)$ *is* $DW(m_1, m_2, \Sigma)$ *with* $m_1 - m_2 = \frac{Tr\, E}{2drk(\Sigma)}$ *and* $m_1 + m_2 = \frac{Tr\, E^2}{2drk(\Sigma)}$. *Also* $m_j d = tr(\rho_j(c))$, *where c is any primitive idempotent in* $M_1\mathcal{H}_p^d(\Sigma)$ *and for all x in* $M\mathcal{H}_p^d(\Sigma)$ *and* $k = 1, 2, \ldots, Tr\, \rho_j(x)^k = m_j dtr(\Sigma^{1/2} x \Sigma^{1/2})^k$, $j = 1, 2$, *and* $Tr\, \rho(x)^k = d(m_1 + (-1)^k m_2) tr(\Sigma^{1/2} x \Sigma^{1/2})^k$.

III. *Condition* (a) *above may be replaced by the condition* (a)$'$: *for all* $x \in \mathcal{H}_p^d$, $\rho_j(x\Sigma x) = \rho_j(x)^2$, $j = 1, 2$, *and* $\rho_1(\Sigma^+)\rho_2(\Sigma^+) = 0$.

Proof. This follows from Theorem 4.3 on noting that $Q(Y) = Q_\psi(Y)$, where $\psi(x) = W \otimes x$ and that from Lemma 3.5.1(a) of [9], $M_2\mathcal{H}_p^d(\Sigma)$=ker$P(\Sigma) = \{x \in \mathcal{H}_p^d : \Sigma x \Sigma = 0\}$. □

Remark 4.1

(a) In Corollary 4.4, we may replace $\rho(x) = \Sigma_Y^{1/2}(W \otimes x)\Sigma_Y^{1/2}$ by $\tilde{\rho}(x) = M(W \otimes x)M^*$, where $M^*M = \Sigma_Y$ is any factorization of Σ_Y. This follows from the *cancellation law* for matrices: if $A^*AB = A^*AC$, then $AB = AC$. For example, one may take M^*M as the Cholesky decomposition or an M^*M with $M \in \mathcal{M}_{q\times np}$, $q = \text{rank}(M) = \text{rank}(\Sigma_Y)$. Note also that $Tr\, \rho(x) = Tr\, \tilde{\rho}(x)$.

(b) Let g be a Hilbert space isomorphism of $\mathcal{M}_{n\times p}^d$ onto a Hilbert space F. Then the map

$$T \to T_g = gTg^{-1}$$

is an algebra isomorphism of $End(\mathcal{M}_{n\times p}^d)$ onto $End(F)$. Let ψ and ρ be as in Theorem 4.3 and define ψ_g and ρ_g by

$$\psi_g(x) = (\psi(x))_g \quad \text{and} \quad \rho_g(x) = (\rho(x))_g.$$

Then $\psi_g, \rho_g : \mathcal{H}_p^d \to End_S(F)$ and

$$\rho_g(x) = P((\Sigma_Y^{1/2})_g)\psi_g(x).$$

Further, ρ is a self-adjoint representation of the Jordan algebra $M\mathcal{H}_p^d(\Sigma)$ on $\mathcal{M}_{n\times p}^d$ if and only if ρ_g is a self-adjoint representation of $M\mathcal{H}_p^d(\Sigma)$ on F. Also $ker\ \rho = ker\ \rho_g$. Thus in Theorem 4.3 (and Corollary 4.4), it may be easier to verify condition (5) (conditions (i) and (ii)) by making a judicious choice for g and using ψ_g and ρ_g in place of ψ and ρ. In particular, if one takes $F = \mathcal{M}_{p\times n}^d$ and $g(X) = X^*$, $X \in \mathcal{M}_{n\times p}^d$, then for all $A \in \mathcal{M}_{n\times n}^d$ and $B \in \mathcal{M}_{p\times p}^d$,

$$(A \otimes B)_g = B \otimes A.$$

Also,

$$(\Sigma_Y)_g = \Sigma_{Y^*}.$$

Thus in Corollary 4.4, one may replace $\rho(x)$ and $\psi(x)$ by

$$\rho_g(x) = \Sigma_{Y^*}^{1/2}(x \otimes W)\Sigma_{Y^*}^{1/2} \quad \text{and} \quad \psi_g(x) = x \otimes W$$

or by

$$\rho_g(x) = L(x \otimes W)L^* \text{ and } \psi_g(x) = x \otimes W, \quad \text{where } \Sigma_{Y^*} = L^*L.$$

Example 4.1. Let $Y \sim N_{4\times 2}^1(0, \Sigma_Y)$, where $\Sigma_Y = A \otimes \Sigma$ with

$$A = \begin{pmatrix} 5/8 & -3/8 & 1/8 & 1/8 \\ -3/8 & 5/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 5/8 & -3/8 \\ 1/8 & 1/8 & -3/8 & 5/8 \end{pmatrix} \quad \text{and} \quad \Sigma = I_2.$$

Let $W \in \mathcal{H}_n^1$ and $L = I_2 \otimes R$ with

$$W = \begin{pmatrix} 0 & 1 & 1/2 & 1/2 \\ 1 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1/2\sqrt{2} & 1/2\sqrt{2} & 1/2\sqrt{2} & 1/2\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\Sigma_{Y^*} = L^*L$. By Corollary 4.4 and the last four lines of Remark 4.1(b), $Y^*WY \sim DW_2^1(2, 1, \Sigma)$.

The following result is a consequence of Corollary 4.4 and we leave its proof to the reader (the case $d = 1$ was shown in [15]).

Corollary 4.5. *Let $Y \sim N_{n\times p}^d(0, A \otimes \Sigma)$, $rk(\Sigma) \geqslant 2$, $W \in \mathcal{H}_n^d$. Then Y^*WY has a Wishart–Laplace distribution with scale matrix Σ if and only if*

(a) *$AWA \neq 0$.*
(b) *$AWA = AWAWAWA$.*

In this case, $Y^*WY \sim DW_p(m_1, m_2, \Sigma)$ with $m_1 - m_2 = Tr(WA)$ and $m_1 + m_2 = Tr(WA)^2$.

Remark 4.2. Corollary 4.5 may be extended to the case where $rk(\Sigma) = 1$ by including condition (c) from Corollary 4.4.

The following theorem shows how a random quadratic form that has a Wishart–Laplace distribution may be written in a natural way as the difference of two independent random quadratic forms each with a Wishart distribution.

Theorem 4.6. *Let* (1)–(5) *be as in Theorem* 4.3 *and suppose that $Q_\psi(Y)$ has a Wishart–Laplace distribution $DW_p^d(m_1, m_2, \Sigma)$. Then* (a) *and* (b) *below hold*:

(a) *$Q_\psi(Y) = Q_{\psi_1}(Y) - Q_{\psi_2}(Y)$, where $Q_{\psi_1}(Y)$ and $Q_{\psi_2}(Y)$ are independent Wishart $W_p^d(m_1, \Sigma)$ and $W_p^d(m_2, \Sigma)$ respectively and ψ_1 and ψ_2 are linear maps of $\mathcal{H}_p^d$ into $End_S(\mathcal{M}_{n\times p}^d)$ given by*

$$\psi_1(x) = \frac{1}{2}[\psi(x) + \psi(x) *_{\Sigma_Y} \psi(\Sigma^+)]$$

and

$$\psi_2(x) = \frac{1}{2}[-\psi(x) + \psi(x) *_{\Sigma_Y} \psi(\Sigma^+)].$$

(b) *In the case that $\Sigma_Y = A \otimes \Sigma$ and $\psi(x) = W \otimes x$, $Q_\psi(Y) = Y^*WY$ and ψ_1 and ψ_2 in (a) can be so chosen that*

$$Q_{\psi_1}(Y) = Y^*\left(\frac{W + WAW}{2}\right)Y \quad \text{and} \quad Q_{\psi_2}(Y) = Y^*\left(\frac{-W + WAW}{2}\right)Y.$$

Proof

(a) By Theorem 4.3 (using the notation therein), we have $\rho(x) = \rho_1(x) - \rho_2(x)$ with

$$\rho(x) = P(\Sigma_Y^{1/2})\psi(x), \quad \rho_1(x) = P(\Sigma_Y^{1/2})\psi_1(x), \quad \text{and} \quad \rho_2(x) = P(\Sigma_Y^{1/2})\psi_2(x),$$

where $\rho_1(x)$ and $\rho_2(x)$ are Jordan algebra homomorphisms of $M\mathcal{H}_p^d(\Sigma)$ into $(\mathcal{A}, *)$ with $ker\ \rho_1 = ker\ \rho_2 = M_2\mathcal{H}_p^d(\Sigma)$ and $\rho_1(\Sigma^+)\rho_2(\Sigma^+) = 0$. Then by Theorem 4.10 of [9], $Q_{\psi_1}(Y)$ and $Q_{\psi_2}(Y)$

are independent Wishart $W_p^d(m_1, \Sigma)$ and $W_p^d(m_2, \Sigma)$ respectively with $m_j = \frac{Tr\,\rho_j(\Sigma^+)}{drk(\Sigma)}$. It is also straightforward to check that $\psi = \psi_1 - \psi_2$ and $Q_\psi = Q_{\psi_1} - Q_{\psi_2}$.

(b) In this case,

$$\psi_1(x) = \frac{1}{2}\left(W \otimes x + WAW \otimes \left(\frac{\Sigma^\circ x + x\Sigma^\circ}{2}\right)\right)$$

and

$$\psi_2(x) = \frac{1}{2}\left(-W \otimes x + WAW \otimes \left(\frac{\Sigma^\circ x + x\Sigma^\circ}{2}\right)\right).$$

Since $\Sigma^{1/2}\left(\frac{\Sigma^\circ x + x\Sigma^\circ}{2}\right)\Sigma^{1/2} = \Sigma^{1/2}x\Sigma^{1/2}$, we may replace the term $\frac{\Sigma^\circ x + x\Sigma^\circ}{2}$ in the definition of the ψ_j by x without affecting the values of $\rho_1(x) = P(\Sigma_Y^{1/2})\psi_1(x)$ and $\rho_2(x) = P(\Sigma_Y^{1/2})\psi_2(x)$. The desired result then follows as in part (a). $\square$

The following theorem shows that the Jordan algebra homomorphism associated with a quadratic form that has a Wishart–Laplace distribution may, through a change of basis, be put in a diagonal form.

Theorem 4.7. *Let* (1)–(5) *be as in Theorem* 4.3 *and suppose* $r = \operatorname{rank}(\Sigma) \geqslant 3$ *and* α *is an isomorphism of* $\mathcal{H}_r^d$ *onto* $M_1\mathcal{H}_p^d(\Sigma)$ *so that* $\alpha(I_r) = \Sigma^+$. *Then*:

$Y^*WY \sim DW_p(m_1, m_2, \Sigma)$ *iff there exists an orthonormal basis* $\mathcal{B}$ *of* $\mathcal{M}_{n\times p}^d$ *(viewed as a Euclidean vector space over* $\mathbf{R}$*) such that*:

(a) *for all* $z \in \mathcal{H}_r^d$ *and for all* $x \in \mathcal{M}_2\mathcal{H}_p^d(\Sigma)$,

$$[\rho(\alpha(z))]_{\mathcal{B}} = \begin{bmatrix} I_{m_1} \otimes \tau(z) & 0 & 0 \\ 0 & -I_{m_2} \otimes \tau(z) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{npd\times npd} \quad \text{and} \quad [\rho(x)]_{\mathcal{B}} = 0_{npd}; \qquad (4.7)$$

equivalently,

(b) *for all* $x \in \mathcal{M}H_p^d(\Sigma)$,

$$[\rho(x)]_{\mathcal{B}} = \begin{bmatrix} I_{m_1} \otimes \tau(\alpha^{-1}(\Sigma^\circ x\Sigma^\circ)) & 0 & 0 \\ 0 & -I_{m_2} \otimes \tau(\alpha^{-1}(\Sigma^\circ x\Sigma^\circ)) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{npd\times npd}, \qquad (4.8)$$

where τ *is the standard representation of* $\mathcal{H}_r^d$ *given in Section* 3.

Proof. Note that $M_1\mathcal{H}_p^d(\Sigma)$ is a simple Euclidean Jordan algebra of rank r. So by Theorem V.3.7 of [4], there exists an isomorphism α of $\mathcal{H}_r^d$ onto $M_1\mathcal{H}_p^d(\Sigma)$ with $\alpha(I_r) = \Sigma^+$. Recall that the Jordan product in $\mathcal{H}_r^d$ is given by $a \circ b = \frac{1}{2}(ab + ba)$.

In what follows below, $[T]$ will denote the matrix representation of an operator T with respect to the standard basis of the associated vector space while $[T]_{\mathcal{B}}$ will denote the matrix representation with respect to a basis $\mathcal{B}$.

First assume that $Y^*WY \sim DW_p(m_1, m_2, \Sigma)$. Let $\rho = \rho_1 - \rho_2$ as in Theorem 4.3 and let $\hat{\rho}_j = \rho_j \circ \alpha, j = 1, 2$. Then the $\hat{\rho}_j$'s are 1–1 Jordan algebra homomorphisms of $\mathcal{H}_r^d$ into $End_S(\mathcal{M}_{n\times p}^d)$. Thus for each x in $\mathcal{H}_r^d$, the matrix representations $[\hat{\rho}_j(x)]$ of the $\hat{\rho}_j(x)$'s belong to the family of $npd \times npd$ symmetric matrices over $\mathbf{R}$. Now since $\hat{\rho}_j(I_r) = \hat{\rho}_j(I_r \circ I_r) = \hat{\rho}_j(I_r)^2$ and $\hat{\rho}_1(I_r)\hat{\rho}_2(I_r) = \rho_1(\Sigma^+)\rho_2(\Sigma^+) = 0$, there exists an orthogonal matrix P such that

$$P^*[\hat{\rho}_1(\alpha(I_r))]P = diag[I_{k_1}, 0_{k_2}, 0_{k_3}]$$

and

$$P^*[\hat{\rho}_2(\alpha(I_r))]P = diag[0_{k_2}, I_{k_2}, 0_{k_3}],$$

where $k_j = rk(\hat{\rho}_j(I_r)) = rk(\rho_j(\Sigma^+)) = Tr(\rho_j(\Sigma^+)),\ j = 1, 2$ and $k_3 = npd - k_1 - k_2$.

Also since $\hat{\rho}_j(z) = \hat{\rho}_j(I_r \circ z) = \frac{1}{2}[\hat{\rho}_j(z)\hat{\rho}_j(I_r) + \hat{\rho}_j(I_r)\hat{\rho}_j(z)]$, we must have

$$P^*[\hat{\rho}_1(z)]P = diag[\beta_1(z), 0_{k_2}, 0_{k_3}]$$

and

$$P^*[\hat{\rho}_2(z)]P = diag[0_{k_2}, \beta_2(z), 0_{k_3}],$$

where the $[\beta_j]$'s are 1-1 Jordan algebra homomorphisms of $\mathcal{H}_r^d$ into the family of $k_j \times k_j$ symmetric matrices over $\mathbf{R}$, with $\beta_j(I_r) = I_{k_j}, j = 1, 2$. Then by Theorem 3 of [7], there exist orthogonal matrices $P_j, j = 1, 2$ such that for all z in $\mathcal{H}_r^d$, $P_j^*[\beta_j(z)]P_j = I_{n_j} \otimes \tau(z)$, where $n_j = \frac{k_j}{rd} = \frac{Tr(\rho_j(\Sigma^+)}{rd} = m_j$ and τ is the standard representation of $\mathcal{H}_r^d$ as described in Section 3.4. Setting $U = P\, diag[P_1, P_2, I_{k_3}]$ yields the first representation in (4.7); the second representation follows since $ker\,\rho = M_2\mathcal{H}_p^d(\Sigma)$. To obtain (4.8), note that for x in $M\mathcal{H}_p^d(\Sigma)$, $x = \Sigma^\circ x \Sigma^\circ + (I_p - \Sigma^\circ x \Sigma^\circ)$, where $\Sigma^\circ x \Sigma^\circ \in M_1\mathcal{H}_p^d(\Sigma)$ and $(I_p - \Sigma^\circ x \Sigma^\circ) \in M_2\mathcal{H}_p^d(\Sigma) = ker\ \rho$. Then replace z in (4.7) by $\alpha^{-1}(\Sigma^\circ x \Sigma^\circ)$ and use the fact that $\rho(x) = \rho(\Sigma^\circ x \Sigma^\circ)$.

Assume that (4.7) (equivalently (4.8)) holds. Since for all $x \in M_2\mathcal{H}_p^d(\Sigma)$, $[\rho(x)]_{\mathcal{B}} = 0_{np}$, it is clear that $ker\,\rho = M_2\mathcal{H}_p^d(\Sigma)$. By (4.7),

$$[E]_{\mathcal{B}} = [\rho(\Sigma^+)]_{\mathcal{B}} = [\rho(\alpha(I_r))]_{\mathcal{B}} = diag[I_{m_1} \otimes I_{dr}, -I_{m_2} \otimes I_{dr}, 0].$$

By (4.7) together with the fact that α and τ are Jordan algebra homomorphisms, we have, for $\alpha(z)$ in $M_1\mathcal{H}_p^d(\Sigma)$,

$$\begin{aligned}
[\rho(\alpha(z) *_\Sigma \alpha(z))]_{\mathcal{B}} &= [\rho(\alpha(z \circ z))]_{\mathcal{B}} \\
&= diag[I_{m_1} \otimes \tau(z \circ z), -I_{m_2} \otimes \tau(z \circ z), 0] \\
&= diag[I_{m_1} \otimes \tau(z)^2, -I_{m_2} \otimes \tau(z)^2, 0] \\
&= [\rho(\alpha(z))]_{\mathcal{B}} *_{[E]_{\mathcal{B}}} [\rho(\alpha(z))]_{\mathcal{B}}.
\end{aligned}$$

Finally, by (4.8), it is easily seen that for all x in $M\mathcal{H}_p^d(\Sigma)$, $[E]_{\mathcal{B}}[\rho(x)]_{\mathcal{B}} = [\rho(x)]_{\mathcal{B}}[E]_{\mathcal{B}}$. Thus by Theorem 4.3, $Y^*WY \sim DW_p(m_1, m_2, \Sigma)$. $\square$

Remark 4.3. In the case that $d = 1$ or 2, we may define α in Theorem 4.7 as follows: let R be a unitary matrix such that $R^*\Sigma R = diag[D_r, 0]$, where $D_r = diag[\sigma_1, \sigma_2, \ldots, \sigma_r], \sigma_i > 0$. Define $\alpha : \mathcal{H}_r^d \to \mathcal{M}_1 H_p^d(\Sigma)$ by

$$\alpha(z) = R\, diag[D_r^{-1/2} z D_r^{-1/2}, 0]\, R^*, \quad \alpha^{-1}(x) = [I_r, 0]R^*\Sigma^{1/2} x \Sigma^{1/2} R[I_r, 0]^*.$$

We now prove a very general version of Cochran Theorem.

Theorem 4.8 (Cochran Theorem)**.** *Let I be a finite set. Suppose that:*

(1) $Y \sim N_{n\times p}^d(0, \Sigma_Y)$;
(2) $\psi_i : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ *is a linear map,* $i \in I$;
(3) $Q_{\psi_i} : \mathcal{M}_{n\times p}^d \to \mathcal{H}_p^d$ *is the quadratic form associated with the linear map* $\psi_i, i \in I$;
(4) $\rho_i : \mathcal{H}_p^d \to End_S(\mathcal{M}_{n\times p}^d)$ *is the linear map defined by* $\rho_i(x) = P(\Sigma_Y^{1/2})\psi_i(x), i \in I$;

(5) $(\mathcal{A}, *)$ *is the Jordan algebra* $End_S(\mathcal{M}^d_{n\times p})$ *with the product* $A * B = \frac{1}{2}(AB + BA)$ *and* $(\mathcal{A}, *_E)$ *is the Jordan algebra* $End_S(\mathcal{M}^d_{n\times p})$ *with the product* $A *_E B = \frac{1}{2}(AEB + BEA)$.

Then $\{Q_{\psi_i}(Y)\}_{i\in I}$ *is an independent family of* $DW(m_{1i}, m_{2i}, \Sigma)$ *random matrices if and only if for all i in I,* (a)–(d) *hold*:

(a) $\rho_i : M\mathcal{H}^d_p(\Sigma) \to (\mathcal{A}, *_{E_i})$ *is a Jordan algebra homomorphism and for all x in* $M\mathcal{H}^d_p(\Sigma)$, $E_i\rho_i(x) = \rho_i(x)E_i$, *where* $E_i = \rho_i(\Sigma^+)$;
(b) $ker\ \rho_i = M_2\mathcal{H}^d_p(\Sigma)$;
(c) *either* $rk(\Sigma) \geqslant 2$ *or* $rk(\Sigma) = 1$ *and* $Tr\ \rho_{ji}(\Sigma^+)$ *is divisible by* $d, j = 1, 2$, *where*

$$\rho_{1i}(x) = \frac{\rho_i(x) + E_i\rho_i(x)}{2} \text{ and } \rho_{2i}(x) = \frac{-\rho_i(x) + E_i\rho_i(x)}{2}, \quad x \in \mathcal{H}^d_p;$$

(d) *for all* $j \neq i$ *in I*, $\rho_i(\Sigma^+)\rho_j(\Sigma^+) = 0$.

In case that (a)–(c) *hold, each* Q_{ψ_i} *has a* $DW(m_{1i}, m_{2i}, \Sigma)$ *distribution with* $m_{ji}d\ rk(\Sigma^+) = Tr\ \rho_{ji}(\Sigma^+)$, $j = 1, 2$.

Proof. Let $\{Q_{\psi_i}(Y)\}$ be an independent family of $DW(m_{1i}, m_{2i}, \Sigma)$ random matrices. Then by Theorem 4.3 and Lemma 4.8 of Masaro and Wong [9], (a)–(d) hold.

Conversely, suppose that (a)–(d) hold. By Theorem 4.3, $Q_{\psi_i}(Y) \sim DW(m_{1i}, m_{2i}, \Sigma)$. Also by Theorem 4.4, $\rho_{1i}, \rho_{2i} : M\mathcal{H}^d_p(\Sigma) \to (\mathcal{A}, *)$ are Jordan algebra homomorphisms with $ker\ \rho_{ji} = M_2\mathcal{H}^d_p(\Sigma)$. Now, by (d) and the definition of ρ_{1i} and ρ_{2i}, we have for $i \neq j$ and $k, l = 1, 2$, $\rho_{ki}(\Sigma^+)\rho_{lj}(\Sigma^+) = 0$. Then by Lemma 3.4.2, $\rho_{ki}(x)\rho_{lj}(y) = 0$ for all x, y in $M_1\mathcal{H}^d_p(\Sigma)$ and therefore for all x, y in $M\mathcal{H}^d_p(\Sigma)$. Since $\rho_i = \rho_{1i} - \rho_{2i}$, it follows that $\rho_i(x)\rho_j(y) = 0$. So by Lemma 4.8 of Masaro and Wong [9], the quadratic forms $Q_{\psi_i}(Y)$ are independent. $\square$

Remark 4.4. In the usual way (using Theorem 4.8), Corollary 4.5 may be extended to a Cochran theorem (see [3]).

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