He’s homotopy perturbation method for a boundary layer equation in unbounded domain

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Abstract

By means of He’s homotopy perturbation method (HPM) an approximate solution of a boundary layer equation in unbounded domain is obtained. Comparison is made between the obtained results and those in open literature. The results show that the method is very effective and simple.

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1. Introduction

With the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics and fluid mechanics. In particular, a boundary layer equation in unbounded domain is of much interest. Various kinds of analytical solutions methods and numerical solutions methods [1,2] were used to handle the problem. Homotopy perturbation method [3–8] is proven to be a very effective and convenient way for handling nonlinear problems. The method was successfully applied to nonlinear equations with discontinuities [9], to finding bifurcation, limit cycle of nonlinear wave equations [10–13], to solving nonlinear problems arising in textile engineering [14], to boundary value problems [15], to Blasius equation [16], to Laplace transform [17], to nonlinear oscillators [18,19], to integro-differential equation [20], to Helmholtz equation [21], to non-Newtonian flow [22,23], etc.; a complete review on application of the method is given in Refs. [24,25].

In this paper we aim to apply He’s homotopy perturbation method to solve a boundary layer equation in unbounded domain [2]

\[ f''' + (n - 1) f f'' - 2n (f')^2 = 0, \quad n > 0 \] (1)

with boundary conditions

\[ f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \] (2)

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which appears in boundary layers in fluid mechanics [1], where \( f''(0) < 0 \). An analytical treatment is much needed to find an accurate value of \( f''(0) \) for different \( n \).

2. He’s homotopy perturbation method

Homotopy perturbation method [3,4,15] provides an alternative approach to introducing an expanding parameter, and can solve various nonlinear equations. The method has eliminated limitations of the traditional perturbation methods; on the other hand, it can take full advantage of the traditional perturbation techniques [24,25].

Now we will use the method in order to obtain the solution of Eq. (1). Assuming \( u = f \), Eq. (1) can be written in the following form:

\[
u'' + F(u) = 0, \quad n > 0 \tag{3}\]

in which

\[
F(u) = (n - 1)u u'' - 2n(u')^2. \tag{4}
\]

According to the homotopy perturbation method [3,4], we construct a homotopy in the form

\[
u'' + \alpha^3 u + p[F(u) - \alpha^3 u] = 0, \tag{5}\]

with the initial conditions

\[
u(0) = 0, \quad \nu'(0) = 1, \quad \nu'(\infty) = 0. \tag{6}\]

When \( p = 0 \), Eq. (5) becomes a linearized equation, \( u'' + \alpha^3 u = 0 \), where \( \alpha \) is an unknown parameter to be further determined; when \( p = 1 \), it turns out to be the original one. The embedded parameter \( p \) monotonically increases from zero to unit as the trivial problem, \( u'' + \alpha^3 u = 0 \), is continuously deformed to the original problem, Eq. (3). According to the homotopy perturbation method, we assume that the solution to Eq. (5) may be written as a power series in \( p \):

\[
u = u_0 + p u_1 + p^2 u_2 + \cdots. \tag{7}\]

Substituting Eq. (6) into Eq. (4) and equating the terms with the identical powers of \( p \), we have

\[
\begin{align*}
p^0: & \quad u_{0''}'' + \alpha^3 u_0 = 0, \quad u_0(0) = 0, \quad u_0'(0) = 1, \quad u_0'(\infty) = 0 \\
p^1: & \quad u_{1''}'' + \alpha^3 u_1 + (n - 1)u_0 u_{0''} - 2n(u_0')^2 - \alpha^3 u_0 = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0, \quad u_1'(\infty) = 0. \tag{9}\end{align*}\]

The solution of Eq. (8) can be readily obtained, which reads

\[
u_0(t) = \frac{1}{\alpha}(1 - e^{-\alpha t}). \tag{10}\]

Substituting Eq. (10) into Eq. (9) results in

\[
u_1'' + \alpha^3 u_1 = (n - 1 - \alpha^2)e^{-\alpha t} + (n + 1)e^{-2\alpha t} + \alpha^2. \tag{11}\]

In case \( n > 1 \) we can solve \( u_1 \) from Eq. (11) under the initial/boundary conditions \( u_1(1) = 0, u_1'(0) = 0 \) and \( u_1'(\infty) = 0 \) and obtain the following solution with ease:

\[
u_1(t) = \frac{n + 1}{7(\sqrt{n - 1})^3}(2e^{-\sqrt{n-1}t} - e^{-2\sqrt{n-1}t} - 1), \quad (n > 1). \tag{12}\]

Therefore, we obtain the first-order approximate solution for \( n > 1 \), which reads

\[
u(t) = \nu_0(t) + \nu_1(t) = \frac{1}{\sqrt{n - 1}}(1 - e^{-\sqrt{n-1}t}) + \frac{n + 1}{7(\sqrt{n - 1})^3}(2e^{-\sqrt{n-1}t} - e^{-2\sqrt{n-1}t} - 1), \quad (n > 1). \tag{13}\]
Table 1

Numerical values of $f''(0)$ for $n > 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>HPM</th>
<th>Padé approximants [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-2.5568</td>
<td>-2.483954032</td>
</tr>
<tr>
<td>10</td>
<td>-4.0476</td>
<td>-4.026385103</td>
</tr>
<tr>
<td>100</td>
<td>-12.8501</td>
<td>-12.8434315</td>
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<tr>
<td>1000</td>
<td>-40.6556</td>
<td>-40.65538218</td>
</tr>
<tr>
<td>5000</td>
<td>-90.9127</td>
<td>-104.8420672</td>
</tr>
</tbody>
</table>

It is obvious that

$$u''(t) = -\sqrt{n-1}e^{-\sqrt{n-1}t} + \frac{n+1}{7\sqrt{n-1}}(2e^{-\sqrt{n-1}t} - 4e^{-2\sqrt{n-1}t}), \quad (n > 1). \quad (14)$$

We, therefore, can easily obtain the initial slope which reads $f''(0) = u''(0) = -\sqrt{n-1} - \frac{2(n+1)}{7\sqrt{n-1}}$ for $n > 1$. For different $n$, the values of the initial slope are illustrated in Table 1.

The main merit of homotopy perturbation method is that only one iteration leads to a highly accurate solution. In case $n > 1$ comparison of the first-order approximate solution with those obtained by Padé approximants [2] is tabulated in Table 1, showing a remarkable agreement. If a higher-order approximate solution is required, the parameter-expanding method can be applied. For example, we can construct the following homotopy: [25]

$$u''' + 0 \cdot u + p F(u) = 0, \quad (15)$$

and we expand the coefficient of the linear term, 0, into a series of $p$

$$0 = \alpha^3 + p \alpha_1 + p^2 \alpha_2 + \cdots. \quad (16)$$

Proceeding as the above method, we can obtain a higher-order approximate solution.

3. Conclusion

In this paper, we have applied He’s homotopy perturbation method in solving a boundary layer equation in an unbounded domain. It is of utter simplicity and effectiveness; the first-order approximate solution leads to a very highly accurate solution. It is a promising method and might find wide applications.

References


