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Optimality conditions for the calculus of variations with higher-order delta derivatives

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ABSTRACT

We prove the Euler–Lagrange delta-differential equations for problems of the calculus of variations on arbitrary time scales with delta-integral functionals depending on higher-order delta derivatives.

1. Introduction

In recent years numerous works have been dedicated to the calculus of variations on time scales and their generalizations — see [1–9] and the references therein. Most of them deal with delta or nabla derivatives of first-order [10–19], only a few with higher-order derivatives [20,21]. Depending on the type of the functional being considered, different time scale Euler-Lagrange type equations are obtained. For variational problems of first-order the Euler-Lagrange equations are valid for an arbitrary time scale $\mathbb{T}$, while for the problems with higher-order delta (or nabla) derivatives they are only valid in a certain class of time scales, more precisely, the ones for which the forward (or backward) jump operator is a polynomial of degree one [20,21]. Here we consider variational problems involving Hilger derivatives of higher order, and prove a necessary optimality condition of the Euler-Lagrange type on an arbitrary time scale, i.e., without imposing any restriction to the jump operators.

2. Preliminaries

Here we recall some basic results and notation needed in the sequel. For the theory of time scales we refer the reader to [22–25].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The functions $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$ are, respectively, the forward and backward jump operators: $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M); $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). The symbol $\emptyset$ denotes the empty set. The graininess function on $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For $\mathbb{T} = \mathbb{R}$ one has $\sigma(t) = t = \rho(t)$ and $\mu(t) \equiv 0$ for any $t \in \mathbb{R}$. For $\mathbb{T} = \mathbb{Z}$ one has $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) \equiv 1$ for every $t \in \mathbb{Z}$. A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense, or left-scattered, if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, or $\rho(t) < t$, respectively.

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Let $\mathbb{T} = [a, b] \cap \mathbb{T}_0$ with $a < b$ and $\mathbb{T}_0$ a time scale. We define $\mathbb{T}^\kappa := \mathbb{T} \setminus (\rho(b), b]$, and $\mathbb{T}^{\kappa^0} := \mathbb{T}$, $\mathbb{T}^{\kappa^n} := \left(\mathbb{T}^{\kappa^{n-1}}\right)^\kappa$ for $n \in \mathbb{N}$. The following standard notation is used for σ (and ρ): $\sigma^0(t) = t$, $\sigma^n(t) = (\sigma \circ \sigma^{n-1})(t)$, $n \in \mathbb{N}$.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta-differentiable* at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \le \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the *delta-derivative* of f at t. We note that if the number $f^\Delta(t)$ exists then it is unique in $\mathbb{T}^\kappa$ (see [24,25]). In the special cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, f^Δ reduces to the standard derivative $f'(t)$ and the forward difference $\Delta f(t) = f(t+1) - f(t)$, respectively. Whenever f^Δ exists, the following formula holds: $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$, where we abbreviate $f \circ \sigma$ by f^σ. Let $f^{\Delta^0} = f$. We define the rth-delta derivative of $f : \mathbb{T}^{\kappa^r} \to \mathbb{R}$, $r \in \mathbb{N}$, to be the function $\left(f^{\Delta^{r-1}}\right)^\Delta$, provided $f^{\Delta^{r-1}}$ is delta differentiable on $\mathbb{T}^{\kappa^r}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous if all its components are rd-continuous. The set of all rd-continuous functions is denoted by C_{rd}. Similarly, C^r_{rd} will denote the set of functions with delta derivatives up to order r belonging to C_{rd}. A function f is a piecewise rd-continuous function, denoted by $f \in C^r_{prd}$, if f^{Δ^i} is continuous for $i = 0, \ldots, r - 1$, and f^{Δ^r} exists and is rd-continuous for all, except possibly at finitely many $t \in \mathbb{T}^{\kappa^r}$.

A piecewise rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ possess an antiderivative $F^\Delta = f$, and in this case the delta integral is defined by $\int_c^d f(t)\Delta t = F(d) - F(c)$ for all $c, d \in \mathbb{T}$. It satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t$, where the integral on the right-hand side is the usual Riemann integral; if $\mathbb{T} = \mathbb{Z}$ and $a < b$, then $\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$.

3. Main results

Consider the following higher-order problem of the calculus of variations up to order r, $r \ge 1$:

$$\mathcal{L}(y(\cdot)) = \int_a^{\rho^{r-1}(b)} L(t, y(t), y^\Delta(t), \ldots, y^{\Delta^r}(t))\Delta t \longrightarrow \min, \tag{1}$$

subject to boundary conditions

$$y(a) = y_a^0, \qquad y\left(\rho^{r-1}(b)\right) = y_b^0, \ldots, y^{\Delta^{r-1}}(a) = y_a^{r-1}, \qquad y^{\Delta^{r-1}}\left(\rho^{r-1}(b)\right) = y_b^{r-1}, \tag{2}$$

where $\mathbb{T}$ is a bounded time scale with $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$, $L : [a, \rho^r(b)]_\mathbb{T} \times \mathbb{R}^{r+1} \to \mathbb{R}$ is a given function, where we use the notation $[c, d]_\mathbb{T} := [c, d] \cap \mathbb{T}$, and $y_a^i, y_b^i \in \mathbb{R}$, $i = 0, \ldots, r - 1$. The results of the paper are trivially generalized for functions $y : [a, b]_\mathbb{T} \to \mathbb{R}^n$, but for simplicity of presentation we restrict ourselves to the scalar case $n = 1$.

A function $y(\cdot) \in C^r_{prd}$ is said to be admissible if it is satisfies condition (2). An admissible $y(\cdot)$ is a *weak local minimizer* for (1)–(2) if there exists $\delta > 0$ such that $\mathcal{L}(y(\cdot)) \le \mathcal{L}(\bar{y}(\cdot))$ for any admissible $\bar{y} \in \mathrm{C}^r_{prd}$ with $\|y - \bar{y}\|_{r,\infty} < \delta$, where

$$\|y\|_{r,\infty} := \sum_{i=0}^r \left\|y^{\Delta^i}\right\|_\infty,$$

$y^{\Delta^0} = y$ and $\|y\|_\infty := \sup_{t\in[a,\rho^r(b)]_\mathbb{T}} |y(t)|$. For simplicity of notation we introduce the operator $[\cdot]$ defined by $[y](t) = \left(t, y(t), y^\Delta(t), \ldots, y^{\Delta^r}(t)\right)$. Then, functional (1) can be written as

$$\mathcal{L}(y(\cdot)) = \int_a^{\rho^{r-1}(b)} L[y](t)\Delta t.$$

We assume that $(u_1, \ldots, u_{r+1}) \to L(t, u_1, \ldots, u_{r+1})$ has continuous partial derivatives $\frac{\partial L}{\partial u_i}$ for all $t \in [a, \rho^r(b)]_\mathbb{T}$, $i = 1, \ldots, r + 1$, and $t \to L[y](t)$ and $t \to \frac{\partial L}{\partial u_i}[y](t)$, $i = 1, \ldots, r + 1$, are piecewise rd-continuous for all admissible functions $y(\cdot)$.

3.1. The higher-order Euler-Lagrange equation

We now prove the Euler-Lagrange equation for problem (1)–(2).

Remark 1. In order for the problem to be nontrivial we require the time scale $\mathbb{T}$ to have at least $2r+1$ points. Indeed, if the time scale has only $2r$ points, then it can be written as $\mathbb{T} = \{a, \sigma(a), \ldots, \sigma^{2r-1}(a)\}$ and

$$\begin{aligned}
&\int_a^{\rho^{r-1}(b)} L(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^r}(t))\Delta t \\
&= \int_a^{\sigma^r(a)} L(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^r}(t))\Delta t = \sum_{i=0}^{r-1} \int_{\sigma^i(a)}^{\sigma^{i+1}(a)} L(t, y(t), y^{\Delta}(t), \ldots, y^{\Delta^r}(t))\Delta t \\
&= \sum_{i=0}^{r-1} (\sigma^{i+1}(a) - \sigma^i(a)) L(\sigma^i(a), y(\sigma^i(a)), y^{\Delta}(\sigma^i(a)), \ldots, y^{\Delta^r}(\sigma^i(a))). \qquad (3)
\end{aligned}$$

Having in mind the boundary conditions and the formula $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$, we can conclude that the sum in (3) is constant for every admissible function $y(\cdot)$.

Theorem 1. *If $y(\cdot)$ is a weak local minimizer for the problem (1)–(2), then $y(\cdot)$ satisfies the Euler-Lagrange equation*

$$\begin{aligned}
&\frac{\partial L}{\partial y^{\Delta^r}}[y](t) - \int_a^{\sigma(t)} \frac{\partial L}{\partial y^{\Delta^{r-1}}}[y](\tau_r)\Delta\tau_r \\
&+ \sum_{i=0}^{r-3} (-1)^i \int_a^{\sigma(t)} \int_a^{\sigma(\tau_r)} \cdots \int_a^{\sigma(\tau_{r-i})} \frac{\partial L}{\partial y^{\Delta^{r-2-i}}}[y](\tau_{r-1-i})\Delta\tau_{r-1-i} \cdots \Delta\tau_{r-1}\Delta\tau_r \\
&+ (-1)^r \int_a^{\sigma(t)} \left\{ \int_a^{\sigma(\tau_r)} \left[\cdots \int_a^{\sigma(\tau_2)} \frac{\partial L}{\partial y}[y](\tau_1)\Delta\tau_1 + c_1 \cdots \right] \Delta\tau_{r-1} - (-1)^{r-1} c_{r-1} \right\} \Delta\tau_r - c_r = 0 \qquad (4)
\end{aligned}$$

for some constants $c_1, \ldots, c_r$ and all $t \in [a, \rho^r(b)]_{\mathbb{T}}$.

Proof. We first introduce some notation: $y_0(t) = y(t), y_1(t) = y^{\Delta}(t), \ldots, y_{r-1}(t) = y^{\Delta^{r-1}}(t), u(t) = y^{\Delta^r}(t)$. Then problem (1)–(2) takes the following form:

$$\begin{aligned}
&\mathcal{L}[y(\cdot)] = \int_a^{\rho^{r-1}(b)} L(t, y_0(t), y_1(t), \ldots, y_{r-1}(t), u(t))\Delta t \longrightarrow \min, \\
&\begin{cases} y_i^{\Delta}(t) = y^{i+1}(t), & i = 0, \ldots, r-2, \\ y_{r-1}^{\Delta}(t) = u(t), \end{cases} \\
&y^j(a) = y_a^j, \qquad y^j\left(\rho^{r-1}(b)\right) = y_b^j, \quad j = 0, \ldots, r-1.
\end{aligned}$$

With the notation $x = (y_0, y_1, \ldots, y_{r-1})$, our problem (1)–(2) can be written as the optimal control problem

$$\begin{aligned}
&\mathcal{L}[x(\cdot)] = \int_a^{\rho^{r-1}(b)} L(t, x(t), u(t))\Delta t \longrightarrow \min, \\
&x^{\Delta}(t) = Ax(t) + Bu(t), \qquad (5)\\
&\varphi(x(a), x(\rho^{r-1}(b))) = \begin{bmatrix} x(a) - x_a \\ x(\rho^{r-1}(b)) - x_b \end{bmatrix} = 0,
\end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Note that assumption A1 of [26, Theorem 9.4] holds: matrix $I + \mu(t)A$ is invertible, and the matrix $\nabla\varphi(x(a), x(\rho^{r-1}(b)))$ has full rank. Therefore, if $(x(\cdot), u(\cdot))$ is a weak local minimum for (5), then there exists a constant λ and a function $p : [a, \rho^{r-1}(b)]_{\mathbb{T}} \to \mathbb{R}^r$, $p \in C^1_{prd}$, such that $(\lambda, p(\cdot)) \neq 0$ and the following conditions hold:

$$\begin{aligned}
&-p^{\Delta}(t) = A^T p^{\sigma}(t) + \lambda \left[\frac{\partial L}{\partial x}(t, x(t), u(t)) \right]^T, \\
&B^T p^{\sigma}(t) + \lambda \frac{\partial L}{\partial u}(t, x(t), u(t)) = 0 \qquad (6)
\end{aligned}$$

for all $t \in [a, \rho^r(b)]_{\mathbb{T}}$. Consequently, if $y(\cdot)$ is a weak local minimizer for (1)–(2), then

$$p^{\sigma}_{r-1}(t) = -\lambda \frac{\partial L}{\partial u}[y](t) \tag{7}$$

holds for all $t \in [a, \rho^r(b)]_{\mathbb{T}}$, where $p^{\sigma}_{r-1}(t)$ is defined recursively by

$$p^{\sigma}_0(t) = -\int_a^{\sigma(t)} \lambda \frac{\partial L}{\partial y_0}[y](\tau_1)\Delta\tau_1 - c_1, \tag{8}$$

$$p^{\sigma}_i(t) = -\int_a^{\sigma(t)} \left[\lambda \frac{\partial L}{\partial y_i}[y](\tau_{i+1}) + p^{\sigma}_{i-1}(\tau_{i+1})\right]\Delta\tau_{i+1} - c_{i-1}, \quad i = 1, \ldots, r-1, \tag{9}$$

with c_i, $i = 0, \ldots, r-1$, constants. From (7)–(9) we obtain that equation

$$\begin{aligned}\lambda \frac{\partial L}{\partial u}[y](t) &- \int_a^{\sigma(t)} \lambda \frac{\partial L}{\partial y_{r-1}}[y](\tau_r)\Delta\tau_r \\ &+ \sum_{i=0}^{r-3}(-1)^i \int_a^{\sigma(t)}\int_a^{\sigma(\tau_r)}\cdots\int_a^{\sigma(\tau_{r-i})} \lambda \frac{\partial L}{\partial y_{r-2-i}}[y](\tau_{r-1-i})\Delta\tau_{r-1-i}\cdots\Delta\tau_{r-1}\Delta\tau_r \\ &+ (-1)^r \int_a^{\sigma(t)}\left\{\int_a^{\sigma(\tau_r)}\left[\cdots\int_a^{\sigma(\tau_2)} \lambda \frac{\partial L}{\partial y_0}[y](\tau_1)\Delta\tau_1 + c_1\cdots\right]\Delta\tau_{r-1} - (-1)^{r-1}c_{r-1}\right\}\Delta\tau_r - c_r = 0\end{aligned} \tag{10}$$

holds for all $t \in [a, \rho^r(b)]_{\mathbb{T}}$. We show next that $\lambda \neq 0$. First observe that if $f \in C^1_{prd}$ and $f^{\sigma}(t) = 0$ for all $t \in [a, b]^{\kappa}_{\mathbb{T}}$, then $f(t) = 0$ for all $t \in [\sigma(a), b]_{\mathbb{T}}$. Suppose, contrary to our claim, that $\lambda = 0$ in Eqs. (6) and (7). Then, we can write the system of equations

$$\begin{cases} p^{\Delta}_0(t) = 0, \\ p^{\Delta}_i(t) = -p^{\sigma}_{i-1}(t), \quad i = 1, \ldots, r-1, \\ p^{\sigma}_{r-1}(t) = 0, \end{cases} \tag{11}$$

for all $t \in [a, \rho^r(b)]_{\mathbb{T}}$. From the last equation we have $p_{r-1}(t) = 0$, $\forall t \in [\sigma(a), \rho^{r-1}(b)]_{\mathbb{T}}$. This implies that $p^{\Delta}_{r-1}(t) = 0$, $\forall t \in [\sigma(a), \rho^r(b)]_{\mathbb{T}}$, and consequently $p^{\sigma}_{r-2}(t) = 0$, $\forall t \in [\sigma(a), \rho^r(b)]_{\mathbb{T}}$. Therefore, $p_{r-2}(t) = 0$, $\forall t \in [\sigma^2(a), \rho^{r-1}(b)]_{\mathbb{T}}$. Repeating this procedure we have $p_1(t) = 0$ for all $t \in [\sigma^{r-1}(a), \rho^{r-1}(b)]_{\mathbb{T}}$. Hence, $0 = p^{\Delta}_1(t) = -p^{\sigma}_0(t) = -p^{\Delta}_0(t)\mu(t) - p_0(t) = -p_0(t)$ for all $t \in [\sigma^{r-1}(a), \rho^r(b)]_{\mathbb{T}}$. Note that the first equation of (11) implies $p_0(t) = c$ for some constant c and all $t \in [a, \rho^{r-1}(b)]_{\mathbb{T}}$. Since the time scale has at least $2r + 1$ points (see Remark 1), the set $t \in [\sigma^{r-1}(a), \rho^{r-1}(b)]_{\mathbb{T}}$ is nonempty and we conclude that $p_0(t) = 0$ for all $t \in [a, \rho^{r-1}(b)]_{\mathbb{T}}$. Substituting this into the second equation we get $p^{\Delta}_1(t) = d$ for some constant d and all $t \in [a, \rho^{r-1}(b)]_{\mathbb{T}}$. Having in mind that $p_1(t_0) = 0$ for some $t_0 \in [a, \rho^{r-1}(b)]_{\mathbb{T}}$ we obtain $p_1(t) = 0$ for all $t \in [a, \rho^{r-1}(b)]_{\mathbb{T}}$. Repeating this procedure we conclude that $p_i(t) = 0$, $i = 1, \ldots, r-1$, for all $t \in [a, \rho^{r-1}(b)]_{\mathbb{T}}$. This contradicts the fact that $(\lambda, p(\cdot)) \neq 0$. Hence, Eq. (10) can be divided by λ and (4) is proved. □

3.2. Corollaries

For illustrating purposes we consider now the two simplest situations, i.e., $r = 1$ and $r = 2$.

Corollary 1 (*Cf.* [14,17]). *If $y(\cdot)$ is a weak local minimizer for the problem*

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y(t), y^{\Delta}(t))\Delta t \longrightarrow \min$$

subject to boundary conditions $y(a) = y_a$ and $y(b) = y_b$, then $y(\cdot)$ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial y^{\Delta}}\left(t, y(t), y^{\Delta}(t)\right) = \int_a^{\sigma(t)} \frac{\partial L}{\partial y}\left(\tau, y(\tau), y^{\Delta}(\tau)\right)\Delta\tau + c_1$$

for some constant c_1 and all $t \in [a, b]^{\kappa}_{\mathbb{T}}$.

Corollary 2 (*Cf.* [20,21]). *If $y(\cdot)$ is a weak local minimizer for the problem*

$$\mathcal{L}(y(\cdot)) = \int_a^{\rho(b)} L(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}(t))\Delta t \longrightarrow \min$$

subject to boundary conditions $y(a) = y_a^0$, $y(\rho(b)) = y_b$, $y^\Delta(a) = y_a^1$, *and* $y^\Delta(\rho(b)) = y_b^1$, *then* $y(\cdot)$ *satisfies the Euler-Lagrange equation*

$$\frac{\partial L}{\partial y^{\Delta\Delta}}\left(t, y(t), y^\Delta(t), y^{\Delta\Delta}(t)\right) - \int_a^{\sigma(t)} \frac{\partial L}{\partial y^\Delta}\left(\tau_2, y(\tau_2), y^\Delta(\tau_2), y^{\Delta\Delta}(\tau_2)\right)\Delta\tau_2 + \int_a^{\sigma(t)}\left[\int_a^{\sigma(\tau_2)} \frac{\partial L}{\partial y}\left(\tau_1, y(\tau_1), y^\Delta(\tau_1), y^{\Delta\Delta}(\tau_1)\right)\Delta\tau_1 + c_1\right]\Delta\tau_2 - c_2 = 0$$

for some constants c_1 *and* c_2 *and all* $t \in [a, \rho(b)]_{\mathbb{T}}^{\kappa}$.

3.3. An example

Let $\mathbb{T} = [a, b] \cap h\mathbb{Z}$, where $h\mathbb{Z} := \{hz | z \in \mathbb{Z}\}$, $h > 0$. Then for any $f \in C_{prd}^r$ we have

$$\underbrace{\left[\int_a^{\sigma(t)}\left(\int_a^{\sigma}\cdots\int_a^{\sigma} f\right)\Delta\tau\right]^{\Delta^j}}_{j-i \text{ integrals}} = f^{\Delta^i\sigma^{j-i}}(t), \quad i \in \{0, \ldots, j-1\}, \tag{12}$$

where $f^{\Delta^i\sigma^{j-i}}(t)$ stands for $f^{\Delta^i}(\sigma^{j-i}(t))$. We will show this by induction. For $j = 1$

$$\int_a^{\sigma(t)} f(\xi)\Delta\xi = \int_a^t f(\xi)\Delta\xi + \int_t^{t+h} f(\xi)\Delta\xi = \int_a^t f(\xi)\Delta\xi + hf(t),$$

and then $\left[\int_a^{\sigma(t)} f(\xi)\Delta\xi\right]^\Delta = f(t) + hf^\Delta(t) = f^\sigma(t)$. Now assume that (12) is true for all $j = 1, \ldots, k$. Then for $j = k + 1$

$$\begin{aligned}
\underbrace{\left[\int_a^{\sigma(t)}\left(\int_a^{\sigma}\cdots\int_a^{\sigma} f\right)\Delta\tau\right]^{\Delta^{k+1}}}_{k+1-i \text{ integrals}} &= \left(\underbrace{\int_a^t\int_a^{\sigma}\cdots\int_a^{\sigma}}_{k+1-i} f\Delta\tau + h\underbrace{\int_a^{\sigma(t)}\cdots\int_a^{\sigma}}_{k-i} f\Delta\tau\right)^{\Delta^{k+1}} \\
&= \left(\underbrace{\int_a^{\sigma(t)}\cdots\int_a^{\sigma}}_{k-i} f\Delta\tau\right)^{\Delta^k} + \left[h\left(\underbrace{\int_a^{\sigma(t)}\cdots\int_a^{\sigma}}_{k-i} f\Delta\tau\right)^{\Delta^k}\right]^{\Delta} \\
&= f^{\Delta^i\sigma^{k-i}}(t) + \left(hf^{\Delta^i\sigma^{k-i}}(t)\right)^\Delta = f^{\Delta^i\sigma^{k+1-i}}(t).
\end{aligned}$$

Delta differentiating r times both sides of Eq. (4) and in view of (12), we obtain the h–Euler-Lagrange equation in delta differentiated form:

$$L_{y^{\Delta^r}}^{\Delta^r}(t, y, y^\Delta, \ldots, y^{\Delta^r}) + \sum_{i=0}^{r-1}(-1)^{r-i} L_{y^{\Delta^i}}^{\Delta^i\sigma^{r-i}}(t, y, y^\Delta, \ldots, y^{\Delta^r}) = 0.$$

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