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On cop-win graphs ☆

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In memory of Martin Farber, 21 August 1951–9 August 1989

Abstract

Following a question of Anstee and Farber we investigate the possibility that all bridged graphs are cop-win. We show that infinite chordal graphs, even of diameter two, need not be cop-win and point to some interesting questions, some of which we answer.

1. Prologue

In 1986, Martin Farber asked the first author whether the result that finite bridged graphs are cop-win extended to infinite bridged graphs of finite diameter. It took a few years to get a counterexample, the difficulty being the *finite diameter*. From the example, we then extracted some information about the structure of bridged graphs and produced a general construction of a class of counterexamples. Along the way, other interesting questions were raised, some of which have only recently been answered in [4].

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2. Introduction and preliminaries

Notation and terminology not explicitly given can be found in [3,5]. Our graphs are simple, finite or infinite, and our subgraphs are not necessarily induced. Recall that the *neighbourhood* of a vertex u in a graph G is set $N(u)$ of vertices adjacent to u, and the *closed neighbourhood* of a vertex u is $N[u]=N(u)\cup\{u\}$. The following game can be played on a given graph G. There are two players, the *cop* and the *robber*. They move alternately, the cop beginning. On the first move each player chooses a starting vertex, on each subsequent move the players move to some vertex in the closed neighbourhood of their current position. The object of the game is for the cop to occupy the same vertex as the robber and for the robber to prevent this from happening. Since on any given graph one of the players must have a winning strategy, it is interesting to characterize those graphs on which (say) the cop can always win. Such graphs are called *cop-win* by Nowakowski and Winkler who characterize them in [12]. A characterization of finite cop-win graphs was also obtained by Quilliot in [14].

Theorem 1 (Nowakowski and Winkler [12] and Quilliot [14]). *A finite graph is cop-win if and only if there is a linear ordering* $v_0,\dots,v_n$ *of its vertices so that for each* $i<n$ *there is a* $i<j\leqslant n$ *such that* $N[v_i]\cap\{v_i,\dots,v_n\}\subseteq N[v_j]\cap\{v_i,\dots,v_n\}$.

The above theorem implies that, for example, finite trees are cop-win. It is more interesting to notice that finite connected chordal graphs are also cop-win, as are finite connected bridged graphs.

Definition 2. Let G be a graph and let C be a cycle of length at least four in G.

(1) A *bridge* of C is a shortest path in G between two vertices in C whose distance in G is strictly smaller than their distance on C. If a bridge is an edge, it is called a *chord*.
(2) The graph G is *chordal* if each cycle of length at least four has a chord.
(3) The graph G is *bridged* if each cycle of length at least four has a bridge.
(4) A vertex of G is *simplicial* if its neighbourhood induces a complete graph.
(5) A vertex u of G is *isometric* if the distances between the vertices of $G\backslash\{u\}$ are the same as those between corresponding vertices in G.

In the above definition, and throughout the paper, all cycles are simple, that is, without repetition of vertices. The key to the claim that finite chordal graphs are cop-win is the following (see [6,10]).

Theorem 3. *Every finite chordal graph contains a simplicial vertex whose neighbourhood induces a complete graph. Further, the deletion of such a vertex leaves a chordal graph.*

It is not difficult to see that by successively deleting simplicial vertices from a finite chordal graph and by numbering them $v_0,v_1,\dots,v_n$ as they are deleted we satisfy the conditions of Theorem 1. A similar result holds for finite bridged graphs [1], introduced by Farber and Jamison in their studies of geodesic convexity, see [7–9].

Theorem 4 (Anstee and Farber [1]). *Every finite bridged graph contains an isometric vertex whose neighbourhood is contained in the neighbourhood of some other vertex. Further, the deletion of an isometric vertex leaves a bridged graph.*

In fact, more is true.

Theorem 5 (Anstee and Farber [1]). *A finite connected graph is bridged if and only if it is cop-win and contains no induced cycles of length four or five.*

It is often reasonable to ask if a property of finite elements of a class of graphs is also true of the infinite ones. In this case the property of being cop-win is clearly *not* shared by all infinite chordal (hence bridged) graphs: it suffices to consider any infinite tree containing a ray (an infinite path). But one could hope that infinite bridged graphs of *finite diameter* are cop-win. This is what Farber and Anstee [2] asked.

One would think that some sort of compactness would help. This does not seem to work directly. Breaking the problem down to three reasonable subquestions, however, could provide an answer.

(1) Is there is a function $f:\mathbb{N}\to\mathbb{N}$ such that on any finite bridged graph of diameter k the cop needs at most $f(k)$ moves to win?
(2) Does every finite subset of vertices of an infinite bridged graph G of diameter k lie in a finite induced bridged subgraph of G of diameter at most k?
(3) Assuming the existence of $f(k)$ for finite graphs, does the cop need at most $f(k)$ moves to win on an infinite graph of diameter k?

Since, as we shall prove, the answer to the first question is *no*, the third one disappears. The second question had been open until recently (see [4]). It is now known that the answer is *yes* under some (reasonable) conditions. In the last section we consider a natural generalization of this problem which is, at this writing, open.

We devote the next section to showing that there is no bound on the length of a game in terms of the diameter of the graph. In the section after that we show that there are infinite chordal graphs of diameter two which are not cop win. In Section 5 we prove that the second statement holds if the diameter $k=2$. We will conclude with open problems.

3. Finite chordal graphs for long games

In this section we give a recursive construction of a sequence $\{G_i\}_{i<\omega}$ of finite chordal graphs of diameter two such that on G_i the cop needs at least i moves to win. Throughout this section we assume G to be a *finite* chordal graph; unless otherwise indicated, all vertices, edges, and sets of vertices are in G.

We begin by defining a rank function ρ on $V(G)$. Let R_0 be the set of simplicial vertices of G and put $\rho(u)=0$ for each u in R_0. For $i>0$ let R_i be the set of vertices simplicial in $G-\{u\colon\ \rho(u)<i\}$, the graph obtained from G by deleting all the vertices of rank less than i. Let $\rho(u)=i$ for each $u\in R_i$. For $S\subset V(G)$ let $\rho(S)=\min\{\rho(u)\colon\ u\in S\}$ and $R(S)=\max\{\rho(u)\colon\ u\in S\}$. When $S=V(G)$ we will write just $\rho(G)$ and $R(G)$. When necessary, we will write $\rho_G(u)$ in order to stress that the rank is computed in the graph G.

Lemma 6. *For every u with $\rho(u)>0$ there is a v such that $\rho(u)>\rho(v)$ and uv is an edge.*

Proof. Clearly u is not simplicial in G. Hence there are two vertices $x,\ y$ adjacent to u but not to one another. If the rank of both x and y were at least that of u, the latter would not be simplicial in $G-\{z\colon\ \rho(z)<\rho(u)\}$. Thus at least one of $x,\ y$ can serve as v. □

Corollary 7. *For each vertex u there is a vertex v and a (possibly trivial) path $u=x_0,x_1,\ldots,x_n=v$ with $\rho(v)=0$ and $\rho(x_i)>\rho(x_{i+1})$.*

Call a set S of vertices *closed* (in G) if, whenever, $u\in S$ and $v\in N(u)$ with $\rho(v)>\rho(u)$ then $v\in S$. The *closure* of a set S of vertices of G, denoted by $[S]_G$, is the smallest closed set containing S. When $S=\{u\}$ we write simply $[u]_G$ for $[\{u\}]_G$. Of course, we omit the subscript G if no confusion is likely.

Lemma 8. *Let $u=x_0,x_1,\ldots,x_n=v$ be a shortest path from u to v. Then there is a k, $0\leqslant k\leqslant n$, such that $\rho(x_i)\leqslant\rho(x_{i+1})$ for $0\leqslant i<k$ and $\rho(x_i)\geqslant\rho(x_{i+1})$ for $k\leqslant i<n$. Furthermore, there is at most one i such that $\rho(x_i)=\rho(x_{i+1})$ and in such a case $i=k$ or $i=k+1$.*

Proof. We induct on n. The statements are clearly true for $n=0$ and $n=1$. Assume $n\geqslant 2$ and consider the path $x_1,\ldots,x_n$. Let μ be such that $\rho(x_\mu)=R(\{x_i\colon\ 1\leqslant i\leqslant n\})$. If $\rho(x_0)<\rho(x_1)$ we are done by induction. Suppose $\rho(x_0)\geqslant\rho(x_1)$. Since, for $2\leqslant i\leqslant n$, x_0 and x_i are not adjacent, $\rho(x_1)>\rho(x_i)$ (x_1 cannot be simplicial while both x_0 and x_i are in the graph). Hence $\mu=1$ and we can put $k=0$. The uniqueness and value of an i such that $\rho(x_i)=\rho(x_{i+1})$, if it exists, follows from the induction hypothesis. □

Corollary 9. *If $v\notin[u]$ and $\rho(v)>\rho(u)$ then every path from v to u contains a vertex other than v of rank greater than $\rho(v)$.*

Proof. Consider any path from u to v, say $u=x_0,x_1,\ldots,x_n=v$. The rank of the x_i's cannot be increasing with i (lest v be in $[u]$). Hence there is a k, $0\leqslant k<n$, as in the lemma. But then $\rho(x_k)\geqslant\rho(v)$. □

Corollary 10. *If G is connected then $[u]\cap[v]\neq\emptyset$ for all u,v.*

Definition 11. Let G_1 and G_2 be disjoint finite chordal graphs and let R be a set disjoint from both $V(G_1)$ and $V(G_2)$. A *composite* of G_1 and G_2 with base R is a graph L such that, for $i=1,2$,

(1) $V(L)=V(G_1)\cup V(G_2)\cup R$;
(2) the graph induced by $V(G_i)$ is G_i;
(3) the graph induced by R is complete;
(4) for each $u\in R$, the neighbourhood $N_i(u)=N(u)\cap V(G_i)$ is a set closed in G_i;
(5) for each pair of vertices $u_i\in V(G_i)$ simplicial in G_i there is a $z\in R$ adjacent to both u_1, u_2.

The composite L is *strict* if for each $z\in R$ there are simplicial vertices $u_i\in V(G_i)$, $i=1,2$ such that z is adjacent *precisely* to $[u_i]_{G_i}$ in G_i.

It follows easily that if L is the composite of (disjoint finite chordal graphs) G_1 and G_2 with base R, $z\in R$ and $u_i,v_i\in V(G_i)$ for $i=1,2$ such that $zv_i\in E(L)$ and $u_i\in[v_i]$ in G_i, then $zu_i\in E(L)$.

Lemma 12. *If both G_1 and G_2 have diameter two then so does each of their composites.*

Proof. Let L be a composite of the G_i's with a base R and let $u_i\in V(G_i)$, $i=1,2$. By Lemma 8, there are vertices v_i in G_i of rank zero and shortest paths from u_i to v_i whose vertices have strictly decreasing ranks. Further, there is a vertex $z\in R$ adjacent to both v_1, v_2. Hence, by the definition of a closed set and part 4 of the definition of a composite, z is adjacent to both u_1 and u_2.

If $u_i\in V(G_i)$ and $v\in R$, note that since u_i is in the closure of some simplicial vertex of G_i (by 8), there is a vertex $z\in R$ adjacent to u_i. Since the graph induced by R is complete, the distance between v and u_i is at most two. □

Lemma 13. *If both G_1 and G_2 are chordal then so is any of their composites.*

Proof. Let L be a composite of the G_i's with a base R. Any cycle containing only vertices of one of G_1, G_2 or R has a chord, by hypothesis and construction. Each cycle C containing vertices from both G_1 and G_2 must contain at least two vertices in R that are not consecutive on C, and hence have a chord since the graph induced by R is complete. It remains to check the cycles with vertices from exactly R and, without loss of generality, G_1. Let C be such a cycle. If three or more vertices of C are in R, C has a chord. Suppose, therefore, that C has at most two vertices in R and let $P=a_1,\ldots,a_{n-1}$ be the path induced in G_1 by $V(C)\cap V(G_1)$. Let a_0,a_n be the vertices of C in R (it is possible that $a_0=a_n$) and suppose the edge a_0a_1 is on C. Let a_i be the vertex of minimum rank on P. If a_i is not an endpoint of P (i.e., if $i\neq 1,n-1$) then, by definition of rank, a_{i-1} is adjacent to a_{i+1}. If a_i is an endpoint of P, then the vertex in $\{a_0,a_n\}$ adjacent to it is also adjacent to its neighbour on P, this time by definition of closed set. □

Lemma 14. *Let L be a composite of two finite chordal graphs G_1 and G_2. Then $\rho_L(u)=\rho_{G_i}(u)$ for each $u\in V(G_i)$, $i=1,2$.*

Proof. We first prove that a simplicial vertex of G_1 remains simplicial in L. Let u be a simplicial vertex of G_1. Since the base of L induces a complete graph, we only need to check that two neighbours $x\in V(G)$ and $y\in R$ are adjacent. But this follows from the construction of a composite: $x\in[u]_G$ and so is adjacent to y. If we now write $G\backslash R_0$ for the graph obtained form G_1 by deleting all the simplicial vertices and, similarly, $L\backslash R_0$ for the graph obtained from L by deleting the simplicial vertices of G_1, we note that, with the same base R, the composite of $G\backslash R_0$ and G_2 is just $L\backslash R_0$. This implies an easy inductive proof of the claim. □

Lemma 15. *Let L be a composite of G_1 and G_2 and let $R(G_i)=k$, $i=1,2$. Then $R(L)=k+1$.*

Proof. Let $u_i\in V(G_i)$, $i=1,2$ be vertices of rank k in the respective G_i. By Corollary 7, there are simplicial (in both the respective G_i and in L) vertices v_i such that $u_i\in[v_i]_{G_i}$. Let z be a vertex in the base of L adjacent to both v_1, v_2. Then z is also adjacent to the u_i, which, however, are not adjacent to one another. Hence, by Lemma 13, $\rho_L(u_i)=k$ and $\rho_L(z)\geqslant k+1$. It is trivial that $\rho_L(z)\leqslant k+1$. □

Lemma 16. *Let L be the strict composite of finite chordal graphs G_1 and G_2 such that $R(G_1)=R(G_2)=k$ and such that $R([u])=k$ for each simplicial vertex $u\in G_i$, $i=1,2$. Then every vertex z in the base of L has rank $k+1$.*

Proof. Analogous to that of Lemma 14. □

Lemma 17. *Let $x=x_0x_1\ldots x_n=y$ be a path such that $\rho(x)>\rho(y)$. Then there is an i, $0\leqslant i<n$ such that x_i is adjacent to y and $\rho(x_i)>\rho(y)$.*

Proof. We induct on n. For $n=1$ the claim is trivial. Suppose $n>1$. If $\rho(x_{n-1})>\rho(y)$ there is nothing more to prove. Assume, therefore, that $\rho(x_{n-1})\leqslant\rho(y)$. Applying the induction hypothesis to the shorter path $x_0\ldots x_{n-1}$ we obtain a $j<n-1$ with x_j adjacent to x_{n-1} and $\rho(x_j)>\rho(x_{n-1})$. Since $\rho(x_{n-1})\leqslant\rho(x_j),\rho(x_n)$, x_j and $x_n=y$ are adjacent. Another application of the hypothesis, this time to the path $x_0\ldots x_jx_n$ yields the desired i. □

We will say that a chordal graph G is *uniformly deep* if $R([u])=R(G)$ for each vertex u. We say that G is *splitting* if for every simplicial vertex u and every $v\in[u]$ with $0<\rho(v)$ there exists a simplicial vertex z such that $v\in[z]$ and $\rho([u]\cap[z])=\rho(v)$. The graph G has the *escape* property if for each pair of adjacent vertices u,v with $0<\rho(u)<\rho(v)$ there is an (escape) vertex z adjacent to u but not to v and such that $\rho(z)=\rho(u)-1$. We call G *rank connected* if for all u,v with $v\in[u]$ there is a path $u=x_0x_1\ldots x_n=v$ such that $\rho(x_i)=\rho(u)+i$; such a path is said to be *ranked*.

Lemma 18. *Every finite connected chordal graph is uniformly deep.*

Proof. Let G be a finite connected chordal graph with $R(G)=k$ and let u be a vertex of G. Let $v\in V(G)$ have $\rho(v)=k$. Let $u=x_0\dots x_n=v$ be a shortest path from u to v. By Lemma 8, the vertices of this path are monotonically increasing in rank. Hence $v\in[u]$. □

Lemma 19. *A strict composite L of finite chordal connected graphs G_1 and G_2 with $R(G_i)=k$, $i=1,2$, which are rank connected, splitting and have the escape property is itself rank connected, splitting, and has the escape property.*

Proof. (1) *Rank connected.* Let u,v be vertices of L such that $v\in[u]_L$ and such that $\rho(v)-\rho(u)>0$. Without loss of generality suppose that v is in the base of L and $u\in V(G_1)$. Let w be a simplicial vertex of G_1 adjacent to v and let $z\in[w]_{G_1}\cap[u]_{G_1}$; this can be done by Corollary 10. Since G_1 is uniformly deep, there is an $x\in[z]_{G_1}$ with $\rho(x)=k$. As $x\in[w]_{G_1}$, v is adjacent to x. By putting together the ranked paths from u to z, from z to x and from x to v we obtain a ranked path from u to v.

(2) *Splitting.* Let u be a simplicial vertex of L and let $v\in[u]_L$. If v is in G_i for $i=1$ or 2, there is nothing more to prove. So assume, without loss of generality, that $u\in V(G_1)$ and v is in the base of L. Let $z\in V(G_2)$ be a simplicial vertex adjacent to v. Observing that $[u]_L\cap[z]_L$ is contained in the base completes the proof.

(3) *Escape property.* Let u,v be adjacent vertices of L with $0<\rho(u)<\rho(v)$. The only non-trivial case to consider is that of v in the base and, without loss of generality, $u\in V(G_1)$. Let w be a simplicial vertex of G_1 - and, hence, of L - adjacent to v. By the definition of L (part 4), $u\in[w]_{G_1}$. Since G_1 is splitting, there is a simplicial vertex $z\in V(G_1)$ with $u\in[z]_{G_1}$ and $\rho([w]_{G_1}\cap[z]_{G_1})=\rho(u)$. Let $z=x_0\dots x_l=u$ be a ranked path with $l=\rho(u)$. Now $x_{l-1}\notin[w]_{G_1}$ and so x_{l-1} is not adjacent to v. Also, $\rho(x_{l-1})=l-1$ as desired. □

Lemma 20. *Let G be a finite chordal graph which is rank connected, splitting and has the escape property. If, on the robber's move, the robber occupies a vertex u of rank l and the cop a vertex v of rank greater than l, then the robber can make at least $l+1$ more moves.*

Proof. By induction on l. If $l=0$, the robber may simply remain at u, thanks to Lemma 8. Suppose $l>0$. By Lemma 17, the robber may remain at u on each of his moves until the cop arrives at a vertex w adjacent to u and with $\rho(w)>l$. The escape property now ensures the robber of a move to a vertex z not adjacent to w, with $\rho(z)=l-1$. The robber is thus assured of at least l moves by the induction hypothesis plus the (at least) one just made to get to z. □

Lemma 21. *If a connected chordal graph G is rank connected, splitting, has the escape property and $R(G)=k\geqslant 2$ then the robber has a strategy that guarantees his making at least k moves.*

Proof. If $k=0$ or 1, the robber can make at least the first move in any case. Assume, therefore, $k\geqslant 2$. We shall consider two cases.

(1) The cop's first move is to a vertex u of rank k. By Lemma 6 and because G is rank connected, there is a vertex z of rank $k-1$ adjacent to u. By the escape property, the robber's first move can be made to a vertex w not adjacent to u and with $\rho(w)=k-2$. As in the proof of Lemma 20, the robber can now wait until the cop moves to an adjacent vertex of rank at least $k-1$ guaranteed by Lemma 17. Then we apply Lemma 20.

(2) The cop's first move is to a vertex u of rank $\rho(u)<k$. Since G is uniformly deep, there is a vertex $z\in[u]$ with $\rho(z)=k$. Let v be a simplicial vertex such that $u\in[v]$. Since G is splitting, there is a simplicial vertex w such that $\rho([v]\cap[w])=k$. From a ranked path from w to z we obtain a vertex x such that $\rho(x)=k-1$ and $x\notin[v]$. The robber's first move is to x. Again by Lemmas 17 and 8, the robber may wait at x until the cop moves to an adjacent vertex y with $\rho(y)=k$. Then Lemma 20 applies. □

Theorem 22. *For every $k\geqslant 1$ there is a finite diameter two chordal graph G_k on which the robber can survive for at least k moves.*

Proof. Let G_1 be a finite diameter two chordal graph of rank one which is rank connected, splitting and has the escape property. In particular, take G_1 to be K_n-e (the complete graph on n vertices less an edge) for $n=3$ or 4. Having defined G_k, let G_{k+1} be a strict composite of two disjoint copies of G_k. Then G_{k+1} has all the properties listed and $R(G_{k+1})=k+1$. The result now follows by Lemma 21. □

4. Infinite chordal graphs that are not cop-win

The characterization of [12] of cop-win graphs that applies also to infinite graphs goes as follows. Given a graph G, define relations $\leqslant_\alpha$ on its vertex set inductively for all ordinals α as follows.

- $u\leqslant_0 u$ for all $u\in G$.
- $u\leqslant_\alpha v$ for $\alpha>0$ if for every $z\in N[u]$ there is a $w\in N[v]$ and $\beta<\alpha$ such that $z\leqslant_\beta w$.

Observe that $\leqslant_\alpha\subseteq\leqslant_\beta$ for $\alpha<\beta$. As there are no more than $|V(G)|$ different such relations for infinite graphs and only a finite number for finite ones, there is a least α such that $\leqslant_\alpha\equiv\leqslant_{\alpha+1}$. Let $\leqslant$ be this $\leqslant_\alpha$.

Theorem 23 (Nowakowski and Winkler [12]). *A graph G is cop-win if and only if the relation $\leqslant$ is trivial, that is, if and only if $u\leqslant v$ for all $u,v\in G$.*

The preceeding theorem allows for optimal strategies for the players. Depending on whether the graph in question is or is not cop-win, the robber can prolong the game as much as possible and the cop can find a fastest way to catch the robber. See the proof of the theorem in [12].

Applying Theorem 23 to decide whether an infinite graph is cop-win does not seem to be easy, even in the case of chordal graphs. Therefore, we need to prove our results directly.

4.1. Existence

We begin by proving the existence of chordal graphs of diameter two that are not cop-win by compactness, using the graphs of Section 3.

Theorem 24. *Suppose that for every k there is a finite chordal graph of diameter two on which the cop needs at least k moves to win. Then there is an infinite chordal graph of diameter two on which the cop cannot win.*

Proof. For each $n<\omega$ fix G_n, a finite chordal graph of diameter two on which the cop needs at least $n+1$ moves to win. We show, using logical compactness, that there exists an infinite, diameter two, chordal graph which is not cop-win. Let L be the language for graphs. Add to it infinitely many binary predicate symbols $C_k(x,y)$, $(k<\omega)$. The axioms for the theory will be those asserting that the graph is diameter two and chordal along with the following:

(1) $C_k(x,y)\rightarrow y\notin N[x]$,
(2) $C_k(x,y)\rightarrow(\forall x_1\in N[x]\exists y_1\in N[y])(C_{k+1}(x_1,y_1))$, $(k<\omega)$,
(3) $\forall x\exists y C_0(x,y)$.

We assert that the resulting theory is consistent. To see this suppose that we only take the axioms involving C_k for $k\leqslant N$. Expand G_{N+1} to the new language by setting $C_k(x,y)=\emptyset$ for $k>N$, and for $k\leqslant N$ let $C_k(x,y)=\{(x,y)\colon\ y\notin N[x]$ and if the cop is at x and the robber at y then the robber can survive $N-k$ more moves$\}$. Then for each x there is a y so that with the cop at x and the robber at y the robber can survive at least N moves. That $C_k(x,y)$ and $x_1\in N[x]$ implies there is a $y_1\in N[y]$ such that $C_{k+1}(x_1,y_1)$ for $k<N$ follows because the robber can survive for $N-k$ steps.

Thus the theory is consistent. Let H be any model. For any move to x of the cop there is a move to y for the robber with $C_0(x,y)$. The axioms 2. provide the means of continuing for the robber, and he never loses (in finitely many steps) because for each k, $C_k(x,y)$ implies $y\notin N[x]$. □

We can, of course, give a direct construction of such a graph. In fact, below we describe a very general construction of infinite chordal graphs of diameter two which are not cop-win; the graphs so constructed can have any infinite cardinality and allow a simple construction of such graphs of any diameter (at least two). The first construction is an example and an outline of the general method.

4.2. A countable example

We first explicitly construct a countable, diameter two, chordal, robber-win (i.e. not cop-win) graph. We will then use it to construct infinite chordal graphs of diameter two of any cardinality. Let $B=\{1,2\}$ and let $B^n=\{b_0b_1\dots b_{n-1}\colon\ b_i\in B,\ i<n\}$. Let $B^*=\bigcup_{n<\omega}B^n$, the set of words on B. The empty word over B, the only element of B^0, is denoted by ε. Define the *length* of the word $b_0b_1\dots b_{n-1}=\mathbf{b}$ to be n. For $\mathbf{u},\mathbf{v}\in B^*$ write $\mathbf{uv}$ for the concatenation of $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{v}$ is of length one, that is, $\mathbf{v}=k$, we write simply $\mathbf{u}k$. We will need a partial order on B^*: define $\mathbf{u}\prec\mathbf{v}$ just in case $\mathbf{u}$ is an initial segment of $\mathbf{v}$. For $\mathbf{u},\ \mathbf{v}\in B^*$ denote by $\mathbf{u}\wedge\mathbf{v}$ the longest common initial segment of the two words. The diagram of $(B^*,\prec)$ naturally forms a rooted oriented tree with root ε. It is this facet which will permit the robber to win. Call the natural order induced on B^*—that is, shorter words precede longer ones and words of the same length are ordered lexicographically—*length-lexicographic*.

The tree itself is, of course, chordal, but far from diameter two. In order to reduce the diameter we need to add edges (and many vertices) to the tree and to remove the directions of the edges, all the while avoiding chordless cycles and still allowing the robber to win. We do this as follows. For each $\mathbf{u}\in B^*$ let $B_{\mathbf{u}}=\{\mathbf{u}\}\times(B^*\times B^*)$, that is, replace each vertex of B^* by an infinite set (a copy of $B^*\times B^*$). For $x=(\mathbf{u},\mathbf{v},\mathbf{w})$ set $\pi_1(x)=\mathbf{v},\ \pi_2(x)=\mathbf{w}$.

Definition 25. The directed graph G_0 is the graph defined by

(1) $V(G_0)=B^*\times(B^*)^2=\bigcup_{\mathbf{u}\in B^*}B_{\mathbf{u}}$.
(2) $E(G_0)=E_1\cup E_2\cup E_3$ with

- $E_1=\bigcup_{\mathbf{u}\in B^*}(B_{\mathbf{u}}^2\setminus\{(x,x)\colon\ x\in B_{\mathbf{u}}\})$
- $E_2=\{xy\colon\ x\in B_{\mathbf{u}},\ y\in B_{\mathbf{v}},\ \mathbf{v}=\mathbf{u}k \text{ for some } k\in B\}$
- $E_3=\{xy\colon\ x\in B_{\mathbf{u}},\ y\in B_{\mathbf{v}},\ \mathbf{v}=\mathbf{u}k\mathbf{z} \text{ for some } k\in B \text{ and some } \mathbf{z}\in B^*\setminus\{\varepsilon\}, \text{ and } \mathbf{z}\prec\pi_k(x)\}$.

The graph G is obtained from G_0 by forgetting the orientation of the edges and identifying double edges.

The edges in E_1 give each $B_{\mathbf{u}}$ the structure of a complete graph. The edges in E_2 make a complete bipartite graph of each pair $(B_{\mathbf{u}},B_{\mathbf{u}k}),\ k\in B$. This preserves enough of the structure of the tree given by B^* to permit the robber to escape. Finally, the edges in E_3 reduce the diameter to two—but because we carefully exploit the natural family of pairs of orthogonal partitions arising from the projections, and the notion of extension of $\prec$ together with the structure of B^*, the resulting graph is still chordal and robber-win. More formally, we have the following (we will treat $B^*\times(B^*)^2$ as $(B^*)^3$).

Lemma 26. *Let G be the graph defined in Definition 25. For each $\mathbf{x}\in B^*$ the function $\phi_{\mathbf{x}}:V\to V$ defined by $\phi_{\mathbf{x}}((\mathbf{u},\mathbf{v},\mathbf{w}))=(\mathbf{xu},\mathbf{v},\mathbf{w})$ is an isomorphism of G with the graph $G_{\mathbf{x}}$ induced by $\bigcup_{\mathbf{x}\prec\mathbf{u}}B_{\mathbf{u}}$.*

Proof. Trivial. □

Theorem 27. *The graph G defined in Definition* 25

(1) *has diameter two*,
(2) *is chordal*,
(3) *is robber-win*.

Proof. (1) Let $x \in B_{\mathbf{u}}$ and $y \in B_{\mathbf{v}}$ be two non-adjacent vertices of G. Let $\mathbf{w} = \mathbf{u} \wedge \mathbf{v}$. If $\mathbf{w} = \mathbf{u}$ or $\mathbf{w} = \mathbf{v}$, we may suppose—without loss of generality—that $\mathbf{w} = \mathbf{u}$ and that $\mathbf{v} = \mathbf{u}k\mathbf{z}$ for some $k \in B$, $\mathbf{z} \in B^*$. Since x and y are not adjacent, $\mathbf{z} \neq \varepsilon$. Now both x and y are adjacent to $(\mathbf{u}, \mathbf{z}, \mathbf{z})$. We may thus suppose, without loss of generality, that $\mathbf{u} = \mathbf{w}1\mathbf{z}$ and $\mathbf{v} = \mathbf{w}2\mathbf{t}$, with $\mathbf{z}, \mathbf{t} \in B^*$. Thanks to the edges in E_2, the only non-trivial case arises when both $\mathbf{z}$ and $\mathbf{t}$ are not ε. But then both x and y are adjacent to $(\mathbf{w}, \mathbf{z}, \mathbf{t})$.

(2) Define a relation $\sim$ on $V(G)$ by putting $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \sim (\mathbf{u}', \mathbf{v}', \mathbf{w}')$ if either $\mathbf{u} \prec \mathbf{u}'$ or $\mathbf{u}' \prec \mathbf{u}$. It follows from the definition of $E(G)$ that if xy is an edge of G then $x \sim y$. Let now $C = x_0 \ldots x_{n-1}$, $x_i = (\mathbf{u_i}, \mathbf{v_i}, \mathbf{w_i})$, be a cycle in G and assume, without loss of generality, that u_0 is the minimum of the x_i, $i = 0, \ldots, n-1$, in the length-lexicographic order. Since $x_i \sim x_{i+1}$ (addition modulo n), we can say that an edge of C is *up* if $\mathbf{u_i} \prec \mathbf{u_{i+1}}$ and is *down* if $\mathbf{u_{i+1}} \prec \mathbf{u_i}$ (an edge could be both up and down in case $\mathbf{u_i} = \mathbf{u_{i+1}}$).

Observe first that if C contains two non-adjacent vertices of some $B_{\mathbf{u}}$, it has a chord. Thus if $n \geqslant 4$, C has a vertex x_i with $\mathbf{u_i} > \mathbf{u_0}$ lexicographically. We conclude that C must contain an up edge $x_i x_{i+1}$ followed by a down edge $x_{i+1} x_{i+2}$. We claim that $x_i x_{i+2}$ is an edge. To see this, notice that since $x_i x_{i+1}$ and $x_{i+1} x_{i+2}$ are edges, there are $k, k' \in B$ and $\mathbf{s}, \mathbf{s}' \in B^*$ such that $\mathbf{u_i}k\mathbf{s} = \mathbf{u_{i+1}} = \mathbf{u_{i+2}}\mathbf{k}'\mathbf{s}'$ and so $x_i \sim x_{i+2}$. Without loss of generality assume that $u_i \prec u_{i+2}$. If $\mathbf{u_{i+2}} = \mathbf{u_i}$ or $\mathbf{u_{i+1}}\mathbf{k}$ then $x_i x_{i+2} \in E_1 \cup E_2$. Otherwise $\mathbf{u_{i+2}} = \mathbf{u_i}\mathbf{k}\mathbf{t}$, $t \in B^* \setminus \{\varepsilon\}$ and $\mathbf{s}' = \mathbf{t}\mathbf{t}'$. This means that $x_i x_{i+1} \in E_3$ and so $\mathbf{t}\mathbf{t}' \prec \pi_k(x_i)$. But then $\mathbf{t} \prec \pi_k(x_i)$ and $x_i x_{i+2} \in E_3$ as well.

(3) In view of Lemma 26 we can assume that the cop begins at some vertex in B_ε. Suppose that just before the cop's move the robber is at $y \in B_{\mathbf{u}}$ at distance two from the cop's vertex x. We shall show that the robber can maintain this distance after each of his moves. We claim that if $z \in B_{\mathbf{v}}$ is adjacent to y then there is a w adjacent to y at distance two from z. To see this, suppose that $z \in B_{\mathbf{v}}$ is adjacent to y. Then $\mathbf{u} \prec \mathbf{v}$ or $\mathbf{v} \prec \mathbf{u}$ and $\mathbf{u} \neq \mathbf{v}$, or $\mathbf{u} = \mathbf{v}$. We treat the first two possibilities leaving the third to the reader. Assume $\mathbf{u} \neq \mathbf{v}$.

Suppose first that $\mathbf{u} \prec \mathbf{v}$ and let $\mathbf{v} = \mathbf{u}k\mathbf{s}$, $k \in B$, $\mathbf{s} \in B^*$. Now any vertex in $B_{\mathbf{u}(3-k)}$ is adjacent to y but not to z. Next assume $\mathbf{v} \prec \mathbf{u}$ and $\mathbf{u} = \mathbf{v}k\mathbf{s}$, $k \in B$, $\mathbf{s} \in B^*$. Observe that with either $t = 1$ or $t = 2$ we have $\mathbf{s}t \not\prec \pi_k(z)$ and that any vertex of $B_{\mathbf{u}t}$ is therefore adjacent to y but not to z. □

4.3. Examples of all cardinalities

We can use the graph G to construct chordal graphs of diameter two which are not cop-win and that have any cardinality $\kappa \geqslant \omega$. This is very easy. Let K_κ be a complete graph on κ. Let $G_\kappa = G[K_\kappa]$ be the lexicographic product of G around K_κ (i.e., replace each vertex of G by a copy of K_κ and add all edges between two copies of K_κ if and only if the corresponding vertices in G were adjacent. It is easy to see that G_κ has the required properties. In fact, we have the following.

Lemma 28. *Let G be a graph, H a complete graph. Then G has each of the following properties if and only if $G[H]$ does.*

(1) *it is bridged*;
(2) *it is chordal*;
(3) *it has diameter two*;
(4) *it is cop-win.*

The method used in the explicit construction of the countable example generalizes to give directly examples of all cardinalities, but as it is tedious to describe and of not sufficiently broad interest, we omit its description.

5. Finite induced subgraphs

We now return to the question of whether every finite subset of vertices of an infinite bridged graph of diameter k lies in a finite induced bridged subgraph of diameter at most k. Observe that if *bridged* is replaced by *chordal* and the diameter requirement is dropped, the answer is a trivial *yes*. In general, however, all we can prove is the following.

Theorem 29. *Let G be a bridged graph of diameter two. Then any finite set of vertices of G lies in a finite induced subgraph of G which is bridged and has diameter two.*

In order to prove the theorem we need few lemmas. We shall assume throughout this section that *G is a bridged graph of diameter two*. If $H=(V(H),E(H))$ is a graph we will write $u\in H$ for $u\in V(H)$ and similarly for $G\backslash H, N(u)\cap H, H\cup\{u\}$, etc.

Lemma 30. *In any cycle of length four or five in G there is a vertex adjacent to all the other vertices of the cycle.*

Proof. For a four-cycle the result is an immediate consequence of the fact that G is bridged. If $v_0v_1v_2v_3v_4$ is a five-cycle, it has a chord v_0v_2 (without loss of generality). In the four-cycle $v_0v_2v_3v_4$ either v_0v_3 or v_2v_4 is an edge. In the first case v_0 and in the second v_2 is the vertex wanted. □

Lemma 31. *If H is a diameter two induced subgraph of G then H is bridged.*

Proof. Let C be a shortest cycle of H without a bridge and let k be its length. Thanks to Lemma 30 we can assume that $k>5$. Let u,v be vertices of C of distance at least three in C. In H, either u is adjacent to v or there is a z adjacent to both u and v. In either case we have a bridge, contradicting the choice of C. □

Lemma 32. *Let H be a finite induced subgraph of G of diameter at most two and let $u \in G \setminus H$ be such that $N(u) \cap H \neq \emptyset$. Then there is a finite induced subgraph $\hat{H}$ of G of diameter at most two containing both H and u.*

Proof. We induct on the number n of vertices of distance at least three from u in the graph induced by $V(H) \cup \{u\}$. As induction hypothesis we strengthen the conclusion by requiring that for $z \in \hat{H} \setminus (H \cup \{u\})$ we have $N(z) \cap H \supseteq N(u) \cap H$.

If $n = 0$ there is nothing to prove as $H \cup \{u\}$ induces a graph of diameter two. Let $n > 0$ and let $v_1, \ldots, v_n$ enumerate the vertices of distance at least three from u in the graph induced by $H \cup \{u\}$. For each such v_i there is a $z_i \in G \setminus H$ adjacent to both u and v_i. Fix an i, $1 \leqslant i \leqslant n$ and let $x \in N(u) \cap H$. Since xv_i is not an edge, we can fix a w in $H \setminus N(u)$ adjacent to both x and v_i. Now $uxwv_iz_i$ is a five-cycle and Lemma 30 applies; in fact z_i is adjacent to all of u, x, w, v_i. Since x was arbitrary, $N(z_i) \cap H \supseteq N(u) \cap H$. It follows that the number of vertices at distance at least three from z_i in the graph induced by $H \cup \{z_i\}$ is strictly less than n. Applying the induction hypothesis to H and z_i, let H_i be an induced subgraph of diameter two of G containing H and z_i and such that for any $y \in H_i \setminus (H \cup \{z_i\})$, $N(y) \cap H \supseteq N(z_i) \cap H$. Since $N(z_i) \cap H \supseteq N(u) \cap H$, we have that for any $y \in H_i \setminus H$, $N(y) \cap H \supseteq N(u) \cap H$. Now let $\hat{H} = \{u\} \cup \bigcup_{i=1}^{n} H_i$. Clearly $\hat{H}$ is finite and if $y \in \hat{H} \setminus H$ we have $N(y) \cap H \supseteq N(u) \cap H$. It remains to check that the diameter is two. The choice of z_i in the construction of H_i ensures that u is, in $\hat{H}$, at distance at most two from each vertex of H. For each $y \in \hat{H} \setminus H$ distance at most two from u is guaranteed by the fact that $N(y) \cap H \supseteq N(u) \cap H \neq \emptyset$. Each H_i has diameter at most two and so the only remaining case to consider is that of $u_i \in H_i \setminus H$, $u_j \in H_j \setminus H$, $i \neq j$. But here $N(u_i) \cap N(u_j) \supseteq N(u) \cap H \neq \emptyset$ and so u_i and u_j have distance at most two in $\hat{H}$. □

We can now prove the theorem.

Proof of Theorem 29. Since the case of G finite is trivial, assume that it is infinite. Let S be a finite set of vertices of G. By adding finitely many vertices if necessary, we may assume that the subgraph induced by S in G is connected. Fix an enumeration $v_1, \ldots, v_k$ of S so that for each i, $1 < i \leqslant k$ there is an edge v_iv_j for some $j < i$. Now H_1, the graph induced by v_1 trivially has diameter at most two. Having constructed H_i, apply Lemma 32 to H_i and v_{i+1} to obtain H_{i+1}. The desired finite induced subgraph of G of diameter two containing S is H_k. □

6. Comments and open problems

A careful reader may have looked at the paper [12] by Nowakowski and Winkler and concluded that too much work is being done here in view of the remark of those authors that *regular incomplete graphs are robber-win*. This is a minor error: while incomplete regular *finite* graphs are never cop-win if they are not complete, infinite ones can be. For an example, consider the complete graph on ω to which we add a new vertex adjacent to all even members of ω. This is clearly a regular graph and is cop-win in one move.

The same reader may ask why we did not exploit the results in [15,11] on representations of chordal graphs as intersection graphs of subtrees of a tree. The reason is that we are interested in cases where compactness fails and so wished to explore this direction, which also gave us more insight into the structure of the graphs under consideration.

Several questions are left open. The first and most obvious is that of characterizing those infinite graphs for which the Nowakowski–Winkler relation is trivial, by some other, simpler, means. In other words, can we *tell* which infinite graphs are cop-win?

The second group of question concerns the relationship between the length of the game and various parameters of graphs. Given a cop-win graph, what is the maximum number of moves the cop needs to win? Is there a good (i.e. achievable) bound in terms of some known parameter? We know from Section 3 that the diameter does not qualify. It is easy to see that neither do the length of the longest path (consider the complete graph), or the length of the longest chordless path (consider a graph obtained from a path by adding a new vertex and all edges from it to the vertices of the path). One might hope to get a bound in terms of the least l such that $v_l,\dots,v_n$ induce a complete graph in a finite cop-win graph G with an enumeration of its vertices as guaranteed by Theorem 1. The last-mentioned counterexample works here as well. So, in spite of the optimum strategy suggested by Theorem 23 and described in [12], we still do not know what the optimal number of moves is. The question is more interesting—and as open—for infinite graphs. Let S be the graph obtained from $(\omega,\{0i\colon\ 0<i<\omega\})$ by replacing each edge $0i$ by a path $0v_i^1v_i^2\dots v_i^i$ of length i. Clearly the robber determines the length of the game by his choice of his starting vertex. Let *SP* be the graph obtained from the ray $v_0v_1v_2\dots$ by the addition of a new vertex v adjacent to all the vertices of the ray (call *SDP* the graph obtained from the double ray by the same method). The two graphs *SP* and *SDP* seem to provide counterexamples to hypotheses as to the bounds on the number of moves in terms of other parameters.

The third set of questions concerns generalizations of Theorem 29. In general, one considers a class of graphs $\mathscr{G}$ such that if G is an infinite element of $\mathscr{G}$ then any finite set S of vertices of G can be extended to a finite induced subgraph H of G which is in $\mathscr{G}$. Chastand, Laviolette and Polat call such a class *dually compact closed* and they have recently shown in [4] that the class of bridged graphs is dually compact closed (see also [13]).

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