Exponentially many nonisomorphic genus embeddings of $K_{n,m}$

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We prove that for every $n, m \geq 6$, the complete bipartite graph $K_{n,m}$ has at least \[ \frac{1}{8} \frac{n m}{2} \left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{m-2}{4} \right\rfloor \] nonisomorphic orientable as well as nonorientable genus embeddings.

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1. Introduction

The orientable genus of a graph is the smallest integer $p$ such that the graph can be embedded in the surface $S_p$, the sphere with $p \geq 0$ handles attached; any such embedding is called an orientable genus embedding of the graph. The nonorientable genus of a graph is the smallest integer $q$ such that the graph can be embedded in the surface $N_q$, the sphere with $q > 0$ crosscaps attached; any such embedding is called a nonorientable genus embedding of the graph.

We can differentiate genus embeddings of a graph as labelled objects (in this case we speak about distinct embeddings, they have different face sets) and as unlabelled objects (in this case we speak about nonisomorphic embeddings).

Interest in investigating the number of nonisomorphic embeddings of a graph was motivated by the proof of the Map Color Theorem [13]. The most difficult part of the proof was to construct one orientable and one nonorientable genus embedding of every complete graph $K_n$. Having difficulties when constructing one genus embedding of every complete graph, the natural question arose of what is the rate of growth of the number of nonisomorphic genus embeddings of $K_n$?

Using current graphs, it was shown [5–9] that there are constants $M, c > 0, b \geq 1/2$ such that for every $n \geq M$, there are at least $c n^{\frac{2b}{3}}$ nonisomorphic orientable as well as nonorientable genus embeddings of $K_n$. An important class of genus embeddings of complete graphs is triangular embeddings. It is known (see [2]) that the number of triangular embeddings of $K_n$ cannot exceed $n^{\omega^2/3}$. Using recursive constructions and a cut-and-paste technique it was shown [1,3] that there are at least $2^{n^2/54-\alpha(n^2)}$ nonisomorphic face 2-colourable orientable (resp. nonorientable) triangular embeddings of $K_n$ for some families of $n$ such that $n \equiv 3$ or 7 (mod 12) (resp. $n \equiv 1$ or 3 (mod 6)). The paper [2] proves that, for a certain positive constant $a$ and for an infinite number of values of $n$, the number of nonisomorphic nonorientable triangular embeddings of $K_n$ is at least $n^{2^a}$, while [4] proves an analogous result for orientable surfaces. A similar lower bound is also given, for an infinite set of values of $n$, on the number of nonisomorphic triangular embeddings of the complete regular tripartite graph $K_{n,n,n}$ (see [2,4]). This is a place where the following question arises: Which of $n$-vertex graphs has the maximal number of nonisomorphic genus embeddings?
Two factors that can affect the number of nonisomorphic genus embeddings of a graph are the number of edges of the graph and the order of the group of automorphisms of the graph. The larger the number of edges, the larger the number of faces that a genus embedding of the graph can have, hence the larger the number of distinct (and maybe nonisomorphic) genus embeddings that the graph can have. On the other hand, the more symmetric a graph is, the more distinct genus embeddings of the graph can be isomorphic.

One can suggest that the graph $K_n$ has the maximum number of nonisomorphic genus embeddings among all $n$-vertex graphs, but then one must bear in mind that although $K_n$ has the maximum number of edges, the graph has also the maximum number of automorphisms, namely, $n!$. For comparison, the complete bipartite graph $K_{n/2,n/2}$ (for even $n$) has roughly half as many edges as $K_n$ does, but the group of automorphisms of the bipartite graph has order 2 $((n/2)!)^2$ which is much smaller than $n!$ as $n$ increases.

In the present paper we investigate the number of nonisomorphic genus embeddings of $K_{n,m}$. Here it is worth noting that at the moment we do not know an upper bound on the number such that the bound can be claimed to be sharp or tight enough. Moreover, it seems that the methods used to construct exponentially many nonisomorphic genus embeddings of $K_n$ do not work for the graph $K_{n,m}$. The only result that we know on the number of nonisomorphic genus embeddings of $K_{n,m}$ is Theorem 2.6 of [14] which can be interpreted as constructing at least \( \frac{1}{2(n-2)} \left( \frac{3(n-2)}{n-1} \right)^{n-2} 2^{n-4} \) nonisomorphic orientable genus embeddings of $K_{4t+2,n}$, where $n \geq 4t + 2$ and $t \geq 1$. However, as remarked by the referees, their construction is incomplete. Note also that the approach of the paper [14] does not seem to apply for the nonorientable case.

In Section 3 we apply a simpler approach (as compared to [14]) to prove the following theorem that covers both orientable and nonorientable cases.

**Theorem 1.** For every $n$, $m \geq 6$, the complete bipartite graph $K_{n,m}$ has at least $\frac{1}{8nm} 2^{(n-1)/3} (m-2)/4$ nonisomorphic orientable as well as nonorientable genus embeddings.

The proof of the theorem uses a construction similar to the “diamond construction” (see [10, page 118]) used to find the genus of complete bipartite graphs.

In the present paper we give a new method of constructing exponentially many nonisomorphic genus embeddings of a graph with high symmetry. As a byproduct, the paper gives a new construction for determining the genus of bipartite graphs. Starting with genus embeddings of $K_{n,m}$ for every $n$, $m \leq 5$, the method used in the paper yields genus embeddings of $K_{n,m}$ for every $n$, $m \geq 6$.

2. Preliminaries

Ringel [11, 12] proved that the orientable genus of $K_{n,m}$ for every $n$, $m \geq 2$ is $\lceil (n-2)(m-2)/4 \rceil$, and that the nonorientable genus of $K_{n,m}$ for every $n$, $m \geq 3$ is $\lceil (n-2)(m-2)/2 \rceil$. During the proof, genus embeddings of $K_{n,m}$ were constructed. The embeddings are cellular and either are quadrangulations or almost all faces are quadrangles.

In what follows by an embedding of a graph we mean a cellular embedding of the graph. We will consider graphs without loops and multiple edges. Cyclic permutations are written as $(\tau_1, \tau_2, \ldots, \tau_m)$.

The neighborhood $N(v)$ of a vertex $v$ of a graph $G$ is the set of vertices adjacent to $v$. We call a cyclic permutation $\rho[v]$ of $N(v)$ a rotation at $v$ and the collection $\{\rho[v] : v \in V(G)\}$ a rotation of $G$. If $D$ denotes the collection $\{\rho[v] : v \in V(G)\}$, then $D_v$ denotes the rotation $\rho[v]$. Two rotations $D$ and $D'$ of $G$ are equivalent if for every vertex $v$ of $G$, $(D')_v$ is either $D_v$ or $(D_v)^{-1}$.

Consider an embedding of $G$. The boundary of a face of the embedding is a closed walk of $G$ called the boundary walk of the face. Combinatorially, by the face set of the embedding we mean the set of boundary walks of the faces of the embedding. Two embeddings of $G$ are distinct if they have different face sets. A face will be designated as a cyclic sequence $[v_1, v_2, \ldots, v_m]$ of vertices (for convenience, we enclose the sequence in brackets) obtained by listing the incident vertices when traversing the boundary cycle of the face in some chosen direction. The sequences $[v_1, v_2, \ldots, v_m]$ and $[v_m, \ldots, v_2, v_1]$ designate the same face.

Combinatorially, two embeddings $f$ and $f'$ of a graph $G$ are isomorphic if there is an automorphism $\psi$ of $G$ such that for every face of $f$, if the face is $[v_1, v_2, \ldots, v_m]$, then $[\psi(v_1), \psi(v_2), \ldots, \psi(v_m)]$ is a face of $f'$.

An orientation at a vertex $v$ of an embedded graph induces a rotation of $G$; the rotation of $v$ represents the circular order of the vertices of $N(v)$ around $v$ on the surface. Reversing the orientation at $v$ reverses the induced rotation of $v$. If we know the faces of the embedding, then the rotations of $v$ induced by orientations at $v$ can be obtained taking into account the following:

(A) A rotation $(u_1, u_2, \ldots, u_m)$ of $v$ is induced by an orientation at $v$ if and only if the embedding has faces $[u_1, v, u_2, \ldots, u_m]$, $[u_2, v, u_3, \ldots, u_m]$, ..., $[u_{m-1}, v, u_m, \ldots, u_1]$, and $[u_m, v, u_1, \ldots]$.

Arbitrarily fix an orientation at every vertex of the embedded graph $G$. This collection of local rotations determines a rotation $D$ of $G$, where for every vertex $v$, the rotation $D_v$ is induced by the orientation at $v$. We say that the rotation $D$ of $G$ is induced by the embedding of $G$. By (A), the same (not distinct) embeddings of $G$ induce equivalent rotations. Hence we obtain the following:

(B) If rotations $D$ and $D'$ of $G$ induced by embeddings $f$ and $f'$, respectively, of $G$ are nonequivalent, then the embeddings $f$ and $f'$ are distinct.
The graph $K_{n,m}$ have bipartite classes $V = \{v(1), v(2), \ldots, v(n)\}$ and $W = \{w(1), w(2), \ldots, w(m)\}$. Suppose that there are $N$ distinct orientable (resp. nonorientable) genus embeddings of $K_{n,m}$ such that each of the embeddings induces the same rotations of $v(1)$ and induces the same rotations of $w(1)$. Then there are at least $\frac{N}{8nm}$ nonisomorphic orientable (resp. nonorientable) genus embeddings of $K_{n,m}$.

**Proof.** Denote by $\mathcal{D}$ the set of $N$ nonequivalent rotations of $K_{n,m}$ induced by the $N$ distinct orientable (resp. nonorientable) genus embeddings of $K_{n,m}$. Without loss of generality, we may assume that the vertices $v(1)$ and $w(1)$ have rotations $(w(1), w(2), \ldots, w(m))$ and $(v(1), v(2), \ldots, v(n))$, respectively, in all rotations of $\mathcal{D}$. Denote by $\mathcal{A}$ the set of all $(n−1)!(m−1)!$ bijections $\varphi$ between the vertices of $K_{n,m}$ such that: $\varphi : V \rightarrow V; \varphi : W \rightarrow W; \varphi(v(1)) = v(1); \varphi(w(1)) = w(1)$.

For every $\varphi \in \mathcal{A}$ and $D \in \mathcal{D}$, the rotation $\varphi(D)$ of $K_{n,m}$ is defined as follows: for every vertex $u$ of $K_{n,m}$, if $D_u = (b(1), b(2), \ldots, b(\ell))$, then $\varphi(D_u)(\varphi(u)) = (\varphi(b(1)), \varphi(b(2)), \ldots, \varphi(b(\ell)))$. We claim that for every $\varphi \in \mathcal{A}$ and $D \in \mathcal{D}$, the rotation $\varphi(D)$ is induced by an orientable (resp. nonorientable) genus embedding of $K_{n,m}$. Indeed, consider the orientable (resp. nonorientable) genus embedding of $K_{n,m}$ induced by $\varphi$. Relabel the vertices of the embedding in such a way that every vertex $u$ of the embedding becomes the vertex $\varphi(u)$. We obtain an orientable (resp. nonorientable) genus embedding of $K_{n,m}$ inducing the rotation $\varphi(D)$.

That the rotations of $\mathcal{D}$ are nonequivalent means that for any two rotations $D, D' \in \mathcal{D}$, there is a vertex having nonequivalent rotations in $D$ and $D'$. If a vertex $u$ has nonequivalent rotations in $D$ and $D'$, then the vertex $\varphi(u)$ has nonequivalent rotations in $\varphi(D)$ and $\varphi(D')$. Hence, given $\varphi \in \mathcal{A}$, all rotations $\varphi(D), D \in \mathcal{D}$, are nonequivalent.

Given $\varphi \in \mathcal{A}$, the vertices $v(1)$ and $w(1)$ have rotations $(\varphi_v(w(1)), \varphi_v(w(2)), \ldots, \varphi_v(w(m)))$ and $(\varphi_v(v(1)), \varphi_v(v(2)), \ldots, \varphi_v(v(n)))$, respectively, in the rotation $\varphi(D)$ for every $D \in \mathcal{D}$. Two bijections $\varphi, \varphi' \in \mathcal{A}$ are $r$-equivalent if and only if $(\varphi(D)_u)$ and $(\varphi'(D)_u)$ are equivalent for $u = v(1), w(1)$. It is easy to see that the $r$-equivalence is an equivalence relation and that every equivalence class consists of 4 bijections (as an example, $\varphi'$ and $\varphi''$ are $r$-equivalent if $\varphi'(w(i)) = \varphi(w(i))$ for $i = 1, 2, \ldots, m$ and $\varphi'(v(j)) = \varphi(v(n + 2 - j))$ for $j = 2, 3, \ldots, n$). Choose one element from each equivalence class. The chosen elements form the set $\mathcal{A} \in \mathcal{A}$, $|\mathcal{A}| = \frac{1}{4} (n − 1)! (m − 1)!$, such that any two $\varphi, \varphi' \in \mathcal{A}$ are not $r$-equivalent.

Now we see that for any two $\varphi, \varphi' \in \mathcal{A}$, at least one of the vertices $v(1)$ and $w(1)$ has nonequivalent rotations in $\varphi(D)$ and $\varphi'(D)$ for every $D \in \mathcal{D}$. Hence, the $|\mathcal{A}| \cdot |D|$ rotations $\varphi(D)$, where $\varphi \in \mathcal{A}$ and $D \in \mathcal{D}$, are nonequivalent and induced by orientable (resp. nonorientable) genus embeddings of $K_{n,m}$. It follows from (B) that all the $|\mathcal{A}| \cdot |D|$ genus embeddings of $K_{n,m}$ are distinct.

Since for $n = m$ (resp. $n \neq m$) the group of automorphisms of $K_{n,m}$ consists of $2(n!)^2$ (resp. $n!m!$) elements, it follows that each isomorphism class of the genus embeddings contains at most $2(n!)^2$ (resp. $n!m!$) embeddings. Hence, among the $|\mathcal{A}| \cdot |D|$ distinct genus embeddings of $K_{n,m}$ there are at least $\frac{1}{2(n!)^2} |\mathcal{A}| \cdot |D| = \frac{N}{8nm}$ for $n = m$ (resp. $\frac{1}{n!m} |\mathcal{A}| \cdot |D| = \frac{N}{8nm} > \frac{N}{8nm}$ for $n \neq m$) nonisomorphic genus embeddings. □

In what follows, given a face of an embedded graph, we place 5 new vertices inside the face and choose a set $\mathcal{B}$ of distinct boundary vertices. Then we attach handles inside the face so that in the face with the attached handles we can embed $5|\mathcal{B}|$ mutually noncrossing edges joining the 5 new vertices with all vertices of $\mathcal{B}$. We will attach every handle and embed the edges as shown in Fig. 1(a). In the figure, the interiors of the 2-cells denoted by $H$ are deleted and then the two boundaries of the 2-cells are identified in such a way that the indicated directions of the boundaries coincide. A thick oriented line labelled by a new vertex $u$ (in Fig. 1(a) the new vertices are denoted by 1, 2, 3, 4, and 5) designates a collection of “parallel” edges incident with the vertex $u$ and oriented away from $u$ (see Fig. 1(c)). Fig. 1(b) is a schematic designation of Fig. 1(a) (here the dashed quadrangle designates a region of the face in which the handle is attached).
3. Proof of Theorem 1

Let $u$ be a $k$-valent vertex of an embedded graph and $H \subseteq N(u)$, $|H| = h$. By an $H$-rotation of $u$ induced by the embedding we mean a cyclic sequence $(b(1), b(2), \ldots, b(h))$ of all elements of $H$ such that there is a rotation $(c(1), c(2), \ldots, c(k))$ of $u$ induced by the embedding such that $c(i_1) = b(1), c(i_2) = b(2), \ldots, c(i_h) = b(h)$ for $1 \leq i_1 < i_2 < \cdots < i_h \leq k$. In other words, the $H$-rotation of $u$ is the cyclic order on $H$ induced by the rotation of $u$.

If two embeddings of a graph induce nonequivalent $H$-rotations of a vertex $u$, $H \subseteq N(u)$, then the embeddings induce nonequivalent rotations of $u$, hence, by (B), the embeddings are distinct.

Lemma 2. Let the graph $K_{n,t}$, where $n, t \geq 4$, have bipartite classes $V = \{v(1), v(2), \ldots, v(n)\}$ and $W = \{w(1), w(2), \ldots, w(t)\}$. Suppose that there is an orientable (resp. nonorientable) genus embedding $f$ of $K_{n,t}$ such that the embedding induces the rotation $(w(1), w(2), \ldots, w(t))$ of $v(1)$. Then there are $2^{\lceil (n-1)/3 \rceil}$ distinct orientable (resp. nonorientable) genus embeddings of the graph $K_{n,t+4}$ with bipartite classes $V$ and $W = \{w(\ell + 1), w(\ell + 2), w(\ell + 3), w(\ell + 4)\}$ such that:

(i) each of the embeddings induces the rotation $(w(1), w(2), \ldots, w(\ell + 4))$ of $v(1)$;

(ii) for any two of the embeddings, there is a vertex $v \in V$ such that the two embeddings induce nonequivalent $\{w(\ell), w(\ell + 1), w(\ell + 2), w(\ell + 3)\}$-rotations of $v$.

Proof. All faces of the cellular embedding $f$ incident with $w(\ell)$ form a 2-cell. Delete $w(\ell)$ and the incident edges from the 2-cell. Now the 2-cell becomes a face $F$ of the obtained embedding of the graph $K_{n,t} - \{w(\ell)\}$ shown at the right top of Fig. 2, where $w(\ell - 1), v(1), w(1)$ are consecutive vertices on the boundary of the face $F$ (an induced rotation of $v(1)$ is indicated). All vertices of $V$ lie on the boundary of the face. Now we place the vertices $w(\ell), w(\ell + 1), w(\ell + 2), w(\ell + 3), w(\ell + 4)$ inside the face and then attach $n - 2$ handles as shown at the bottom of Fig. 2, where $\{v(1), b(1), b(2), \ldots, b(n - 1)\} = V$ (since the embedding $f$ is not necessary quadrangular, some vertices of $V$ can appear more than once on the boundary of $F$).

We obtain an embedding of $K_{n,t+4}$ in the orientable (resp. nonorientable) surface of genus $\lceil (n-2)(\ell-2)/4 \rceil + (n-2) = \lceil (n-2)((\ell+4)-2)/4 \rceil$ (resp. $\lceil (n-2)(\ell-2)/2 \rceil + (n-2) = \lceil (n-2)((\ell+4)-2)/2 \rceil$), hence the embedding is a genus embedding.

Denote $H = \{w(\ell), w(\ell + 1), w(\ell + 2), w(\ell + 3)\}$.

By the $j$-flip, $j = 1, 2, \ldots, \lceil (n-1)/3 \rceil$, we mean the transformation shown in Fig. 3. Every $j$-flip transforms an orientable (resp. nonorientable) genus embedding of $K_{n,t+4}$ into an orientable (resp. nonorientable) genus embedding of $K_{n,t+4}$. Now we claim that the two embeddings induce nonequivalent $H$-rotations of one of the vertices $b(3j-2), b(3j-1)$,
and \(b(3j)\). Consider Fig. 3. Taking Figs. 1 and 2 into account, we see that \(\{\varepsilon_2, \varepsilon_4\} = \{w(\ell + 1), w(\ell + 3)\}\), hence \(\{w(\ell), w(\ell + 2), w(\ell + 4)\} = \{\varepsilon_1, \varepsilon_3, \varepsilon_5\}\). The vertex \(b(3j - 2)\) in Fig. 3(a) and (b) has the same \(H\)-rotation \((\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)\) for \(w(\ell + 4) = \varepsilon_1\), has nonequivalent \(H\)-rotations \((\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5)\) and \((\varepsilon_1, \varepsilon_4, \varepsilon_5, \varepsilon_2)\), respectively, for \(w(\ell + 4) = \varepsilon_3\), and has nonequivalent \(H\)-rotations \((\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\) and \((\varepsilon_1, \varepsilon_4, \varepsilon_2, \varepsilon_3)\), respectively, for \(w(\ell + 4) = \varepsilon_5\). Analogously, one can check that the vertex \(b(3j - 1)\) (resp. \(b(3j)\)) has equivalent rotations in Fig. 3(a) and (b) only for \(w(\ell + 4) = \varepsilon_3\) (resp. \(w(\ell + 4) = \varepsilon_5\)). This proves our claim.

Using the \([n - 1]/3\) flips, we obtain \(2^{(n-1)/3}\) orientable (resp. nonorientable) genus embeddings of \(K_{n+4}\) such that (i) and (ii) hold. It follows from (ii) that the embeddings are distinct. \(\Box\)

**Lemma 3.** Let the graph \(K_{n,m}\), where \(n, m \geq 6\), have bipartite classes \(V = \{v(1), v(2), \ldots, v(n)\}\) and \(W = \{w(1), w(2), \ldots, w(m)\}\). Then there are \(2^{(n-1)/3}\) (resp. \(n-2)/4\)) distinct orientable (resp. nonorientable) genus embeddings of \(K_{n,m}\) such that each of the embeddings induces the rotation \((w(1), w(2), \ldots, w(m))\) of \(v(1)\) and the rotation \((v(1), v(2), \ldots, v(n))\) of \(w(1)\).

**Proof.** Denote \(\delta = m - 4 \cdot (m - 2)/4\). We have \(2 \leq \delta \leq 5\). Denote \(H_0 = \{w(\delta + 4i), w(\delta + 4i + 1), w(\delta + 4i + 2), w(\delta + 4i + 3)\}\), \(i = 0, 1, \ldots, \lfloor (m - 2)/4 \rfloor - 1\).

Take an orientable (resp. nonorientable) genus embedding of \(K_{n,\delta}\) with vertex partition classes \(V = \{v(1), v(2), \ldots, v(n)\}\) and \(W = \{w(1), w(2), \ldots, w(\delta)\}\). Without loss of generality, we may assume that the embedding induces the rotation \((w(1), w(2), \ldots, w(\delta))\) of \(v(1)\) and the rotation \((v(1), v(2), \ldots, v(n))\) of \(w(1)\). Adding new vertices \(w(\delta + 1), w(\delta + 2), w(\delta + 3), w(\delta + 4)\) and applying Lemma 2 we obtain \(2^{(n-1)/3}\) distinct orientable (resp. nonorientable) genus embeddings of the graph \(K_{n,\delta+4}\) with bipartite classes \(V = \{v(1), v(2), \ldots, v(n)\}\) and \(W = \{w(1), w(2), \ldots, w(\delta + 4)\}\) such that:

(a1) each of the embeddings induces the rotation \((w(1), w(2), \ldots, w(\delta + 4))\) of \(v(1)\) and the rotation \((v(1), v(2), \ldots, v(n))\) of \(w(1)\) (note that in the proof of Lemma 2 adding the vertices \(w(\delta), w(\delta + 1), w(\delta + 2), w(\delta + 3), w(\delta + 4)\) and \(w(\delta + 4)\) does not change the induced rotations of vertices \(v(1), w(1), \ldots, v(n), w(1)\);

(a2) for any two of the embeddings, there is a vertex \(v \in V\) such that the two embeddings induce nonequivalent \(H_0\)-rotations of \(v\).

Now for each of the \(2^{(n-1)/3}\) distinct genus embeddings of \(K_{n,\delta+4}\), add new vertices \(w(\delta + 5), w(\delta + 6), w(\delta + 7), w(\delta + 8)\) and apply Lemma 2. We obtain \(2^{(n-1)/3}\) orientable (resp. nonorientable) genus embeddings of the graph \(K_{n,\delta+8}\) with bipartite classes \(V = \{v(1), v(2), \ldots, v(n)\}\) and \(W = \{w(1), w(2), \ldots, w(\delta + 8)\}\) such that:

(b1) each of the embeddings induces the rotation \((w(1), w(2), \ldots, w(\delta + 8))\) of \(v(1)\) and the rotation \((v(1), v(2), \ldots, v(n))\) of \(w(1)\);

(b2) for any two of the embeddings, there is a vertex \(v \in V\) such that the two embeddings induce nonequivalent \(H_0\)-rotations of \(v\) for some \(j \in \{0, 1, \ldots, (m - 2)/4\} - 1\) (note that in the proof of Lemma 2 adding the vertices \(w(\delta + 4), w(\delta + 5), w(\delta + 6), w(\delta + 7), w(\delta + 8)\) does not change the induced \(H_0\)-rotations of vertices of \(V\)).

Hence, the \(2^{(n-1)/3}\) orientable (resp. nonorientable) genus embeddings of \(K_{n,m}\) are distinct.

Proceeding this way (adding new vertices \(w(\delta + 9), \ldots, w(\delta + 12), \ldots\), etc.), we finally obtain \(2^{(n-1)/3}\) orientable (resp. nonorientable) genus embeddings of \(K_{n,m}\) such that each of the embeddings induces the rotation \((w(1), w(2), \ldots, w(m))\) of \(v(1)\) and the rotation \((v(1), v(2), \ldots, v(n))\) of \(w(1)\), and for any two of the embeddings, there is a vertex \(v \in V\) such that the two embeddings induce nonequivalent \(H_0\)-rotations of \(v\) for some \(j \in \{0, 1, \ldots, (m - 2)/4\} - 1\). Hence, the \(2^{(n-1)/3}\) embeddings of \(K_{n,m}\) are distinct. \(\Box\)

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