Generalized mixed variational-like inequality for random fuzzy mappings

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ARTICLE INFO

Article history:
Received 19 October 2007
Received in revised form 3 March 2008

MSC:
47J40
47H10
49M05

Keywords:
Generalized mixed variational-like inequality
Auxiliary problem
Iterative algorithm
Convergence analysis

ABSTRACT

In this paper, we introduce and study a new class of generalized mixed variational-like inequality for random fuzzy mappings (GMVLIP). An existence theorem for auxiliary problem of the GMVLIP is established. Further, by exploiting the theorem, we construct and analyze a new iterative algorithm for finding the solution of the GMVLIP. Furthermore, we prove the existence of a unique solution of the GMVLIP and discuss the convergence analysis of iterative sequence generated by the iterative algorithm.

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1. Introduction

It is well known that variational inequality theories are very effective and powerful tools for studying a wide class of linear and nonlinear problems arising in many diverse fields of pure and applied sciences such as mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. In recent years, classical variational inequality theories have been generalized and applied in various directions, the readers are referred to the references therein. A useful and important generalization of variational inequalities is the mixed variational-like inequality. The generalized mixed variational-like inequalities have potential and significant applications in optimization theory [8, 17], structural analysis [19], and economics [8, 18]. It is noted that there are many effective numerical methods for finding approximate solutions of various variational inequalities. Among the most effective numerical technique is the projection method and its variant forms. However, the projection type techniques cannot be extended for constructing iterative algorithms for mixed variational-like inequalities, since it is not possible to find the projection of the solution. These facts motivated Glowinski et al. [9] to suggest another technique, which does not depend on the projection. The technique is called the auxiliary principle technique. Very recently, Huang et al. [10] and Ding [1] extend the auxiliary principle technique to study generalized nonlinear mixed variational-like inequalities.

On the other hand, in 1989, Chang and Zhu [11] introduced the concept of variational inequality for fuzzy mappings, which was extended in [12, 13]. In 1999, Huang [6] was the first to introduce and study a class of random set-valued nonlinear generalized variational inclusions with random fuzzy mappings in Hilbert spaces. For some related works, we refer to [2, 6, 14–16].

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doi:10.1016/j.cam.2008.03.049
Inspired and motivated by recent works [4–6,12,15,16], in this paper, we introduce and study a new class of generalized mixed variational-like inequality for random fuzzy mappings (GMVLIP). An existence theorem for auxiliary problem of the GMVLIP is established. Further, by exploiting the theorem, we construct and analyze a new iterative algorithm for finding the solution of the GMVLIP. Furthermore, we prove the existence of a unique solution of the GMVLIP and discuss the convergence analysis of iterative sequence generated by the iterative algorithm. Our results improve and generalize many known corresponding results presented in [1–3,6,10].

2. Preliminaries

Throughout this paper, let \( H \) be a real Hilbert space with inner product and norm denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively, and \( D \) be a nonempty closed convex subset of \( H \). We denote by \( 2^H \) and \( CB(H) \) the families of all the nonempty subsets and the families of the nonempty bounded closed subsets of \( H \), respectively. \( \bar{H}(\cdot, \cdot) \) represents the Hausdorff metric on \( CB(H) \).

Let \( (\Omega, \Sigma) \) be a measurable space, where \( \Omega \) is a set and \( \Sigma \) is \( \sigma \)-algebra of subsets of \( \Omega \). We denote by \( \beta(H) \) the class of Borel \( \sigma \)-fields in \( H \).

**Definition 2.1.** A mapping \( f : \Omega \rightarrow H \) is said to be measurable if for any \( C \in \beta(H) \) and
\[
\overline{f}^{-1}(C) = \{ t \in \Omega : f(t) \in C \} \in \Sigma.
\]

**Definition 2.2.** A mapping \( f : \Omega \times H \rightarrow H \) is called a random operator if for any \( w \in H, f(t, w) = w(t) \) is measurable. A random operator \( f : \Omega \times H \rightarrow H \) is said to be continuous if for any \( t \in \Omega \), the mapping \( f(t, \cdot) : H \rightarrow H \) is continuous.

**Definition 2.3.** A multivalued mapping \( A : \Omega \rightarrow CB(H) \) is said to be measurable if for any \( C \in \beta(H) \) and
\[
A^{-1}(C) = \{ t \in \Omega : A(t) \cap C \neq \emptyset \} \in \Sigma.
\]

**Definition 2.4.** A mapping \( u : \Omega \rightarrow H \) is called a measurable selection of the multivalued measurable mapping \( A : \Omega \rightarrow CB(H) \) if \( u \) is a measurable mapping and \( t \in \Omega, u(t) \in A(t) \).

**Definition 2.5.** A mapping \( T : \Omega \times H \rightarrow CB(H) \) is called a random multivalued mapping if for any \( w \in H, T(\cdot, w) \) is measurable. A random multivalued mapping \( T : \Omega \times H \rightarrow CB(H) \) is said to be \( H \)-continuous if for any \( t \in \Omega \), \( T(\cdot, \cdot) : H \rightarrow H \) is continuous in the Hausdorff metric.

Let \( F(H) \) be a collection of fuzzy sets over \( H \). A mapping \( \hat{F} \) from \( \Omega \) into \( F(H) \) is called a fuzzy mapping. If \( \hat{F} \) is a fuzzy mapping on \( H \), for any \( t \in \Omega \), \( \hat{F}(t) \) (denote it by \( \hat{F}_t \) in what follows) is a fuzzy set on \( H \) and \( \hat{F}_t(z) \) is the membership function of \( z \) in \( \hat{F}_t \). Let \( M \in F(H), q \in [0, 1] \), then the set \( (M)_q = \{ u \in H : M(u) \geq q \} \) is called a \( q \)-cut set of \( M \).

**Definition 2.6.** A fuzzy mapping \( \hat{F} : \Omega \rightarrow F(H) \) is called measurable if for any \( \alpha \in [0, 1], (\hat{F}(\cdot))_\alpha : \Omega \rightarrow 2^H \) is a measurable multivalued mapping.

**Definition 2.7.** A fuzzy mapping \( \hat{F} : \Omega \times H \rightarrow F(H) \) is called a random fuzzy mapping if for any \( w \in H, \hat{F}(\cdot, w) : \Omega \rightarrow F(H) \) is a measurable fuzzy mapping.

Clearly, the random fuzzy mapping includes multivalued mappings, random multivalued mappings and fuzzy mappings as the special cases.

Let \( A, \hat{T} : \Omega \times H \rightarrow F(H) \) be two random fuzzy mappings satisfying the following condition (I): if there exist two mappings \( a, c : H \rightarrow [0, 1] \) such that
\[
\forall(t, w) \in \Omega \times H, (\hat{A}_t)_w \subseteq CB(H), (\hat{T}_t)_w \subseteq CB(H).
\]

By using the random fuzzy mappings \( \hat{A} \) and \( \hat{T} \), we can define two random multivalued mappings \( A \) and \( T \) as follows:
\[
\forall(t, w) \in \Omega \times H, A : \Omega \times H \rightarrow CB(H), (t, w) \mapsto (\hat{A}_t)_w,
\]
\[
T : \Omega \times H \rightarrow CB(H), (t, w) \mapsto (\hat{T}_t)_w.
\]
So \( A \) and \( T \) are called the random multivalued mappings induced by the random fuzzy mappings \( \hat{A} \) and \( \hat{T} \), respectively.

Given mappings \( a, c : H \rightarrow [0, 1] \), the random fuzzy mappings \( \hat{A}, \hat{T} : \Omega \times H \rightarrow F(H) \) satisfy the condition (I). Let \( N, \eta : H \times H \rightarrow H \) be two mappings. Let \( b : H \times H \rightarrow (-\infty, +\infty) \) be a real-valued function. We shall study the following problem: For any measurable mapping \( v : \Omega \rightarrow H \), find measurable mappings \( u, x, y : \Omega \rightarrow H \) such that
\[
\begin{align*}
|\hat{A}_w(u(t))| & \geq a(u(t)), \\
|\hat{T}_w(u(t))| & \geq c(u(t)), \\
(N(x(t), y(t), \eta(v(t), u(t))) + b(u(t), v(t)) - b(u(t), u(t))) & \geq 0,
\end{align*}
\]
for all \( t \in \Omega \) and any measurable mapping \( v : \Omega \to H \), where the function \( b(\cdot, \cdot) \) is nondifferential and satisfies the following conditions:

(i) for any measurable mappings \( v : \Omega \to H \), \( b(\cdot, v(t)) \) is line;

(ii) for each measurable mapping \( w : \Omega \to H \), \( b(w(t), v(t)) \) is a convex function;

(iii) for any measurable mappings \( w, v : \Omega \to H \), \( b(w(t), v(t)) \) is bounded, that is, there exists a measurable functions \( y : \Omega \to [0, +\infty) \) such that \( b(w(t), v(t)) \leq y(t) \|w(t)\| \cdot \|v(t)\| \);

(iv) for any measurable mappings \( w, v, z : \Omega \to H \), \( b(w(t), v(t)) - b(w(t), z(t)) \leq b(w(t), v(t) - z(t)) \).

The inequality (2.1) is called generalized mixed variational-like inequality for random fuzzy mappings (GMVLIP).

Remark 2.1. (1) for any measurable mappings \( w, v : \Omega \to H \), \( b(\cdot, v(t)) = -b(\cdot, v(t)) \) and \( b(-\cdot, v(t)) \leq y(t) \|w(t)\| \cdot \|v(t)\| \) hold from condition (i) and (iii), respectively. So \( |b(w(t), v(t))| \leq y(t) \|w(t)\| \cdot \|v(t)\| \).

(2) for any measurable mappings \( w, v, z : \Omega \to H \), \( |b(w(t), v(t)) - b(w(t), z(t))| \leq y(t) \|w(t)\| \cdot \|v(t) - z(t)\| \) from condition (ii) and (iv). So \( b(w(t), v(t)) \) is continuous with respect to second argument.

Special cases:
1. Let \( \tilde{A}, \tilde{T} : D \to B^* \) and \( \eta : D \times D \to B \) be single-value mappings, then the problem (2.1) reduces to the following nonlinear mixed variational-like inequality: for a given \( w^* \in B^* \), find \( u \in D \) such that

\[
\langle N(Tu, Au) - w^*, \eta(v, u) \rangle + b(u, v) - b(u, u) \leq 0, \quad \forall v \in B.
\]

The problem (2.2) was considered in [1].

We note that for suitable choices of the mappings \( N, \tilde{A}, \tilde{T}, b \), GMVLIP (2.1) reduces to various classes of variational inequalities (e.g., [1–3,6,15] and the references therein). In brief, problem (2.1) is the most general and unifying one, which is also one of the main motivations of this paper.

Definition 2.8. Let \( D \) be a nonempty closed convex subset of \( H \), let \( \eta : D \times D \to D \) and \( N(\cdot, \cdot) : D \times D \to D \) be two measurable mappings.

(1) \( N(\cdot, \cdot) \) is said to be Lipschitz continuous in first argument, if there exists a measurable function \( k_{11} : \Omega \to (0, +\infty) \) such that

\[
\|N(u(t), \cdot) - N(v(t), \cdot)\| \leq k_{11}(t) \|u(t) - v(t)\|, \quad \forall u(t), v(t) \in D, t \in \Omega.
\]

(2) \( N(\cdot, \cdot) \) is said to be \( \eta \)-strongly monotone in first argument with respect to the random multivalued mapping \( A : \Omega \times H \to CB(H) \), if there exists a measurable function \( k_{21} : \Omega \to (0, +\infty) \) such that for any \( t \in \Omega \),

\[
\langle N(x_1(t), \cdot) - N(x_2(t), \cdot), \eta(u_1(t), u_2(t)) \rangle \geq k_{21}(t) \|u_1(t) - u_2(t)\|^2,
\]

\[
\forall u_1(t), u_2(t) \in H, x_1(t) \in A(t, u_1(t)), x_2(t) \in A(t, u_2(t)), t \in \Omega.
\]

Similarly, we can define Lipschitz continuity and the \( \eta \)-strongly monotonicity of the measurable mapping \( N(\cdot, \cdot) \) in second argument with respect to the random multivalued mapping \( T : \Omega \times H \to CB(H) \).

Definition 2.9. Let \( A, T : \Omega \times H \to CB(H) \) be two random multivalued mappings induced by the random fuzzy mappings \( \hat{A} \) and \( \hat{T} \), respectively, and \( \eta : D \times D \to D \) be mapping. For any \( t \in \Omega \), the mappings \( u(t) \to N(x(t), y(t)) \) and \( \eta \) are said to have 0-diagonally concave relation, if for any \( t \in \Omega \), the function \( \phi : \Omega \times D \to (-\infty, +\infty] \) defined by

\[
\phi(t, v(t), u(t)) = \langle N(x(t), y(t)), \eta(u(t), v(t)) \rangle, \quad (x(t) \in A(t, u(t)), y(t) \in T(t, u(t)))
\]

has 0-diagonally concave in \( v(t) \), i.e., for any \( t \in \Omega \), any finite set \( \{v_1(t), v_2(t), \ldots, v_m(t)\} \subset D \) and \( u(t) = \sum_{i=1}^{m} \alpha_i v_i(t)(\alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1) \),

\[
\sum_{i=1}^{m} \alpha_i \phi(t, v_i(t), u(t)) \leq 0.
\]

Definition 2.10. A random multivalued mapping \( A : \Omega \times H \to CB(H) \) is said to be \( \hat{H} \)-Lipschitz continuous, if there exists a measurable function \( \lambda : \Omega \to (0, +\infty) \) such that

\[
\hat{H}(A(t, u_1(t)), A(t, u_2(t))) \leq \lambda(t) \|u_1(t) - u_2(t)\|^2, \quad u_1(t), u_2(t) \in H.
\]

we give the following lemmas.

Lemma 2.1 ([7]). Let \( A : \Omega \times H \to CB(H) \) be a \( \hat{H} \)-continuous random multivalued mapping, then for measurable mapping \( u : \Omega \to H \), the multivalued mapping \( A(\cdot, u(\cdot)) : \Omega \to CB(H) \) is measurable.

Lemma 2.2 ([7]). Let \( A_1, A_2 : \Omega \to CB(H) \) be two measurable multivalued mappings, \( \varepsilon > 0 \) be a constant and \( x_1 : \Omega \to H \) be a measurable selection of \( A_1 \), then there exist a measurable selection \( x_2 : \Omega \to H \) of \( A_2 \) such that for all \( t \in \Omega \),

\[
\|x_1(t) - x_2(t)\| \leq (1 + \varepsilon) \|\hat{H}(A_1(t), A_2(t))\|.
\]
Lemma 2.3 ([3]). Let $(\Omega, \Sigma)$ be a measurable space, and $D$ be a nonempty convex subset of a topological vector space. Let 
\[ \varphi : \Omega \times D \times D \to (-\infty, +\infty) \] 
be a real-valued function such that

1. For each $(v, w) \in D \times D$, $t \mapsto \varphi(t, v, u)$ is measurable mapping;
2. For each $(t, v) \in \Omega \times D$, $u \mapsto \varphi(t, v, u)$ is continuous on each nonempty compact subset of $D$;
3. For each $(t, u) \in \Omega \times D$, $v \mapsto \varphi(t, v, u)$ is lower semicontinuous on each nonempty compact subset of $D$;
4. For each $t \in \Omega$, each nonempty finite set $\{v_1, v_2, \ldots, v_m\} \subset D$ and for each $u = \sum_{i=1}^{m} \alpha_i \varphi_i$ ($0 \leq \sum_{i=1}^{m} \alpha_i \varphi_i < 1$),
\[ \min_{1 \leq i \leq m} \varphi(t, v_i, u) \leq 0; \]
5. For each $t \in \Omega$, there exists a nonempty convex compact subset $D_t$ of $D$ and a nonempty compact subset $K_t$ of $D$ such that for all $u \in D \setminus K_t$, there is a $v \in D_t \cup \{u\}$ with $\varphi(t, v, u) > 0$.

Then there exists a measurable mapping $u : \Omega \to D$ such that $\varphi(t, v, u(t)) \leq 0$ for all $v \in D$ and $t \in \Omega$.

3. Auxiliary problem

Now, we consider the auxiliary problem related to GMVLIP (2.1) and establish an existence theorem for the auxiliary problem.

Auxiliary problem: Given a measurable mapping $u^* : \Omega \to D$, for any measurable mapping $v : \Omega \to D$, find measurable mappings $\hat{u} : \Omega \to D$, such that for all $t \in \Omega$, $\hat{x}(t) \in A(t, \hat{u}(t)), \hat{y}(t) \in T(t, \hat{u}(t))$, and
\[ (N(\hat{x}(t), \hat{y}(t)), \eta(\hat{v}(t), \hat{u}(t))) + b(u^*(t), v(t)) - b(u^*(t), \hat{u}(t)) \geq 0. \] (3.1)

Theorem 3.1. Let $(\Omega, \Sigma)$ be a measurable space, and $D$ be a nonempty convex subset of $H$. Let random fuzzy mappings $\hat{A}, \hat{T} : \Omega \times H \to F(H)$ satisfy the condition (I), and $A$ and $T$ be the random multivalued mappings induced by the random fuzzy mappings $\hat{A}$ and $\hat{T}$, respectively. Let $N, \eta : D \times D \to D$ be two mappings. Let $b : D \times D \to (-\infty, +\infty)$ be a real-valued function such that

1. For each $t \in \Omega$, the mappings $\hat{A}(\cdot, t), \hat{T}(\cdot, t)$ are $H$-continuous with the measurable functions $\lambda_1, \lambda_2 : \Omega \to (0, 1]$, respectively;
2. The measurable mapping $\eta$ is Lipschitz continuous with the measurable function $\sigma : \Omega \to (0, +\infty)$; the measurable mapping $\eta(t)$ is continuous in first argument and semicontinuous in second argument, and for all measurable mappings $u, v : \Omega \to D, \eta(u(t), v(t)) = -\sigma(t)\eta(v(t), u(t))$;
3. The measurable mapping $N(\cdot, \cdot)$ is Lipschitz continuous and $\eta$-strongly monotone with respect to the random multivalued mapping $A$ in first argument with the measurable functions $\lambda_{11}, \lambda_{21} : \Omega \to (0, +\infty)$ respectively. $N(\cdot, \cdot)$ is Lipschitz continuous and $\eta$-strongly monotone with respect to the random multivalued mapping $T$ in second argument with the measurable functions $\lambda_{12}, \lambda_{22} : \Omega \to (0, +\infty)$ respectively, too;
4. For each $t \in \Omega$, the mappings $u(t) \to N(x(t), y(t))$ and $\eta$ have the 0-diagonally concave relation;
5. The function $b(\cdot, \cdot)$ satisfies the conditions (i)–(iv).

Then the auxiliary problem (3.1) has a unique random solution.

Proof. For any fixed measurable mapping $u^* : \Omega \to D$, for any measurable mapping $u, v : \Omega \to D$, we define a function
\[ \varphi(t, v(t), u(t)) = \langle N(x(t), y(t)), \eta(u(t), v(t)) \rangle + b(u^*(t), v(t)) - b(u^*(t), \hat{u}(t)), \forall v(t), u(t) \in D, t \in \Omega \]
where $x(t) \in A(t, u(t)), y(t) \in T(t, u(t))$.

We will show that the mapping $\varphi$ satisfy all the conditions of Lemma 2.3. Indeed, since $A$ and $T$ are the random multivalued mappings induced by the random fuzzy mappings $\hat{A}$ and $\hat{T}$, respectively, i.e. for each $u(t) \in D, A(\cdot, u(t))$ and $T(\cdot, u(t))$ are measurable mappings, so for any fixed $(v(t), u(t)) \in D \times D$, $t \mapsto \varphi(t, v(t), u(t))$ is measurable.

For any $v : \Omega \to D$, the mapping $u(t) \mapsto N(x(t), y(t))$ is continuous. Then for each $v : \Omega \to D$ and any sequence $\{u_n(t)\} \subset D$ with $u_n(t) \to u(t)$, we have $\eta(u_n(t), v(t)) \to \eta(u(t), v(t)) (n \to \infty)$. Since for each $t \in \Omega$, the mappings $A(\cdot, t), T(\cdot, t)$ are $H$-continuous, it follows for any fixed $(t, v(t)) \in \Omega \times D$ that
\[ (N(x_n(t), y_n(t)), \eta(u_n(t), v(t))) - (N(x(t), y(t)), \eta(u(t), v(t))) \]
\[ \leq \max \left( \max \left( (N(x_n(t), y_n(t)) - N(x(t), y(t)), \eta(u_n(t), v(t))) \right) + \max \left( (N(x(t), y(t)), \eta(u(t), v(t))) - \eta(u_n(t), v(t)) \right) \right) \]
\[ \leq \max \left( \max \left( (N(x_n(t), y_n(t)) - N(x(t), y(t)), \eta(u_n(t), v(t))) \right) + \max \left( (N(x(t), y(t)) - N(x_n(t), y_n(t)), \eta(u(t), v(t))) \right) \right) \]
\[ \leq k_{11}(t) \| x_n(t) - x(t) \| \eta(u_n(t), v(t)) \]
\[ + k_{12}(t) \| y_n(t) - y(t) \| \eta(u_n(t), v(t)) \]
\[ \leq k_{11}(t) \lambda_1(t) \| u_n(t) - u(t) \| \eta(u_n(t), v(t)) \]
\[ + k_{12}(t) \lambda_2(t) \| u(t) - u_n(t) \| \eta(u_n(t), v(t)) \]
\[ \to 0 (n \to \infty). \]
Therefore for each fixed $(t, v(t)) \in \Omega \times D$, the function $u(t) \to \langle N(x(t), y(t), \eta(u(t), v(t))) \rangle$ is continuous on $D$, where $x(t) \in A(t, u(t))$, $y(t) \in T(t, u(t))$. Since the function $u(t) \to b(u^*(t), u(t))$ is continuous and convex on $D$ by the Remark 2.1 (2), so for each fixed $(t, v(t)) \in \Omega \times D$, $u(t) \to \varphi(t, v(t), u(t))$ is continuous on $D$. Since the function $v(t) \to b(u^*(t), v(t))$ is continuous on $D$ and for any measurable mappings $u(t) \in D$, $v(t) \to \eta(y(t), v(t))$ is semicontinuous, so for each fixed $(t, u(t)) \in \Omega \times D$, $v(t) \to \varphi(t, v(t), u(t))$ is semicontinuous on $D$. Thus, we can confirm that the function $\varphi(v(t), u(t))$ satisfies the conditions (i)-(iii) in Lemma 2.3.

Now we prove that the function $\varphi(v(t), u(t))$ satisfies the condition (iv) in Lemma 2.3. We suppose that the function $\varphi(v(t), u(t))$ satisfies the condition (iv) of Lemma 2.3. If it is not true, there exists $t_0 \in \Omega$, a finite set $\{v_1(t_0), v_2(t_0), \ldots, v_m(t_0)\} \subset D$ and $u(t_0) = \sum_{i=1}^{m} \alpha_i v_i(t_0)$ ($\alpha_i \geq 0$, $\sum_{i=1}^{m} \alpha_i = 1$), such that $\varphi(t_0, v_i(t_0), u(t_0)) > 0$ for all $i = 1, 2, \ldots, m$, that is

$$\langle N(x(t_0), y(t_0)), \eta(u(t_0), v_i(t_0)) \rangle + b(u^*(t_0), u(t_0)) - b(u^*(t_0), v_i(t_0)) > 0.$$  

It follows that

$$\sum_{i=1}^{m} \alpha_i \langle N(x(t_0), y(t_0)), \eta(u(t_0), v_i(t_0)) \rangle + b(u^*(t_0), u(t_0)) - \sum_{i=1}^{m} \alpha_i b(u^*(t_0), v_i(t_0)) > 0.$$

Noting that $b(\cdot, \cdot)$ is convex in the second argument, that is $\sum_{i=1}^{m} \alpha_i b(u^*(t_0), v_i(t_0)) \leq b(u^*(t_0), \sum_{i=1}^{m} \alpha_i v_i(t_0)) = b(u^*(t_0), u(t_0))$, we have

$$\sum_{i=1}^{m} \alpha_i \langle N(x(t_0), y(t_0)), \eta(u(t_0), v_i(t_0)) \rangle > 0. \quad (3.2)$$

Since for any $t \in \Omega$, the mappings $u(t) \to N(x(t), y(t))$ and $\eta$ have the 0-diagonally concave relation in $v(t)$, so for any $t \in \Omega$,

$$\sum_{i=1}^{m} \alpha_i \langle N(x(t), y(t)), \eta(u(t), v_i(t)) \rangle \leq 0,$$

which contradicts (3.2). Therefore, for any $t \in \Omega$, any finite set $\{v_1(t), v_2(t), \ldots, v_m(t)\} \subset D$ and $u(t) = \sum_{i=1}^{m} \alpha_i v_i(t_0)$ ($\alpha_i \geq 0$, $\sum_{i=1}^{m} \alpha_i = 1$), we have $\varphi(t, v_i(t), u(t)) \leq 0$ (i = 1, 2, ..., m). Thus condition (iv) of Lemma 2.3 holds.

For each $t \in \Omega$, let $\theta(t) = \frac{1}{2\sigma(t)} \{\|N(x^*(t), y^*(t))\| + \|\gamma(t)\|u^*(t)\|$}, $K = \{u(t) \in D : \|u(t) - u^*(t)\| \leq \theta(t)\}$, $D_0 = \{u^*(t)\}$, then $K$ and $D_0$ are both compact convex subsets of $D$. By (1)–(4) of the theorem, for each $u(t) \in D/K$, there exist $u^*(t) \in Co(D_0 \cap \{u(t)\})$, $x^*(t) \in A(t, u^*(t))$, $y^*(t) \in T(t, u^*(t))$, such that

$$\varphi(t, u^*(t), u(t)) = \langle N(x(t), y(t)), \eta(u(t), u^*(t)) \rangle + b(u^*(t), u(t)) - b(u^*(t), u^*(t))$$

$$\leq \langle N(x(t), y(t)) - N(x^*(t), y^*(t)), \eta(u(t), u^*(t)) \rangle + \langle N(x^*(t), y^*(t)) - N(x^*(t), y^*(t)), \eta(u(t), u^*(t)) \rangle$$

$$+ \langle N(x^*(t), y^*(t)), \eta(u(t), u^*(t)) \rangle + b(u^*(t), u(t)) - b(u^*(t), u^*(t))$$

$$\geq k_{21}(t) \lambda_1(t) \|u(t) - u^*(t)\| + k_{22}(t) \lambda_2(t) \|u(t) - u^*(t)\|^2 - \sigma(t) \|N(x^*(t), y^*(t))\| \cdot \|u(t) - u^*(t)\|$$

$$- \gamma(t) \|u^*(t)\| \cdot \|u(t) - u^*(t)\|$$

$$= \|u(t) - u^*(t)\| \{k_{21}(t) \lambda_1(t) + k_{22}(t) \lambda_2(t)\} \|u(t) - u^*(t)\| - \sigma(t) \|N(x^*(t), y^*(t))\| - \gamma(t) \|u^*(t)\|$$

$$> 0.$$  

Hence condition (5) of Lemma 3.2 is also satisfied. By Lemma 2.3, for any $t \in \Omega$ there exists a measurable mapping $\hat{u} : \Omega \to D$, such that $\varphi(t, v(t), \hat{u}(t)) \leq 0$.

We know the mapping $N(\cdot, \cdot)$ is Lipschitz continuous in first argument and in second argument, and the mappings $A(t, \cdot)$, $T(t, \cdot)$ are $H$-continuous. Based on Lemma 2.1, we obtain that for the measurable mapping $\hat{u} : \Omega \to D$, there exist $\hat{x}(t) \in A(t, \hat{u}(t))$, $\hat{y}(t) \in T(t, \hat{u}(t))$ such that

$$\langle N(\hat{x}(t), \hat{y}(t)), \eta(v(t), \hat{u}(t)) \rangle + b(u^*(t), v(t)) - b(u^*(t), \hat{u}(t)) \leq 0, \forall v : \Omega \to D, t \in \Omega.$$  

By $\eta(\hat{u}(t), v(t))$, we have

$$\langle N(\hat{x}(t), \hat{y}(t)), \eta(\hat{u}(t), v(t)) \rangle + b(u^*(t), \hat{u}(t)) - b(u^*(t), v(t)) \geq 0, \forall v : \Omega \to D, t \in \Omega.$$  

This implies that for any $t \in \Omega$ and for each fixed measurable mapping $u^* : \Omega \to D$, the measurable mapping $\hat{u} : \Omega \to D, \hat{x}(t) \in A(t, \hat{u}(t))$, $\hat{y}(t) \in T(t, \hat{u}(t))$ is the random solution of the Auxiliary problem (3.1). Now we prove that for any $t \in \Omega$, the measurable mapping $t \to \hat{u}(t)$, $\hat{x}(t) \in A(t, \hat{u}(t))$, $\hat{y}(t) \in T(t, \hat{u}(t))$ is a unique random solution of the auxiliary problem (3.1). Supposing the measurable mappings $u_1(t) \in D$, $x_1(t) \in A(t, u_1(t))$, $y_1(t) \in T(t, u_1(t))$ and $u_2(t) \in D$, $x_2(t) \in A(t, u_2(t))$, $y_2(t) \in T(t, u_2(t))$ are two random solutions of the auxiliary problem (3.1), we have the conclusion that for all $t \in \Omega$ and for each measurable mapping $u^* : \Omega \to D$,

$$\langle N(x_1(t), y_1(t)), \eta(v(t), u_1(t)) \rangle + b(u^*(t), v(t)) - b(u^*(t), u_1(t)) \geq 0, \quad (3.3)$$

$$\langle N(x_2(t), y_2(t)), \eta(v(t), u_2(t)) \rangle + b(u^*(t), v(t)) - b(u^*(t), u_2(t)) \geq 0. \quad (3.4)$$
Taking \( v(t) = u_2(t) \) in (3.3) and \( v(t) = u_1(t) \) in (3.4) and adding two inequalities, by the assumption on the function \( b \), we obtain
\[
\langle N(x_1(t), y_1(t)), \eta(u_2(t), u_1(t)) \rangle + \langle N(x_2(t), y_2(t)), \eta(u_1(t), u_2(t)) \rangle \geq 0.
\]
Since for all \( u(t), v(t) \in D \), \( \eta(u(t), v(t)) = -\eta(v(t), u(t)) \), we have
\[
\langle N(x_2(t), y_2(t)) - N(x_1(t), y_1(t)), \eta(u_1(t), u_2(t)) \rangle \geq 0.
\]

Noting that \( N(\cdot, \cdot) \) is \( \eta \)-strongly monotone with respect to the random multivalued mapping \( A \) in first argument with the measurable function \( k_{21} : \Omega \to (0, +\infty) \), and \( \eta \)-strongly monotone with respect to the random multivalued mapping \( T \) in second argument with \( k_{22} : \Omega \to (0, +\infty) \), we get
\[
\langle k_{21}(t) + k_{22}(t) \|u_2(t) - u_1(t)\|^2 \leq \langle N(x_2(t), y_2(t) - N(x_1(t), y_2(t), \eta(u_2(t), u_1(t)) \rangle
\]
\[
+ \langle N(x_1(t), y_1(t)), \eta(u_2(t), u_1(t)) \rangle \leq 0.
\]

Since \( k_{21}(t), k_{22}(t) > 0 \), we have \( u_1(t) = u_2(t) \).

Further, let \( x_1(t) \in A(t, u_1(t)), y_1(t) \in T(t, u_1(t)) \) and \( x_2(t) \in A(t, u_2(t), 2(t) \in T(t, u_2(t)) \), by Lemma 2.2, we have
\[
\|x_1(t) - x_2(t)\| \leq (1 + \varepsilon)\|A(t, u_1(t), A(t, u_2(t))\| \leq (1 + \varepsilon)\|u_1(t) - u_2(t)\|,
\]
\[
\|y_1(t) - y_2(t)\| \leq (1 + \varepsilon)\|T(t, u_1(t)), T(t, u_2(t))\| \leq (1 + \varepsilon)\|u_1(t) - u_2(t)\|.
\]

So we get \( x_1(t) = x_2(t) \) and \( y_1(t) = y_2(t) \), which imply that for any \( t \in \Omega \) and the measurable mapping \( u^* : \Omega \to D \), the measurable mappings \( \hat{u}, \hat{x}, \hat{y} : \Omega \to D \) such that \( t \in \Omega, \hat{x}(t) \in A(t, \hat{u}(t)), \hat{y}(t) \in T(t, \hat{u}(t)) \) is a unique random solution of the auxiliary problem (3.1).

By Theorem 3.1 again, we can construct the algorithm for GMVLIP (2.1) as follows:

**Algorithm 3.1.** For any given measurable mapping \( u_0 : \Omega \to D \), by Lemma 2.1, the multivalued mappings \( A(\cdot, u_0(\cdot)) \), \( T(\cdot, u_0(\cdot)) : \Omega \times H \to CB(H) \) are measurable, hence there exist measurable sections \( x_0 : \Omega \to D(\cdot, u_0(\cdot)) \) and \( y_0 : \Omega \to D \) of \( T(\cdot, u_0(\cdot)) \). From Theorem 3.1, there exist measurable mapping \( u_1 : \Omega \to D \), the measurable sections \( x_1 : \Omega \to D \) of \( A(\cdot, u_1(\cdot)) \) and \( y_1 : \Omega \to D \) of \( T(\cdot, u_1(\cdot)) \) such that \( \forall t \in \Omega \)
\[
\langle N(x_1(t), y_1(t), \eta(u_1(t), v(t))) + b(u_0(t), u_1(t)) - b(u_0(t), v(t)) \rangle \geq 0, \quad \forall v(t) \in D, t \in \Omega.
\]

and
\[
\|x_1(t) - x_0(t)\| \leq (1 + 1)\|A(t, u_1(t), A(t, u_0(t))\|,
\]
\[
\|y_1(t) - y_0(t)\| \leq (1 + 1)\|T(t, u_1(t)), T(t, u_0(t))\|.
\]

Continuing the above process inductively, we can define the following random iterative sequences \( \{u_n(t)\} \) and \( \{x_n(t)\} \) for solving problem (2.1) as follows:
\[
\langle N(x_n(t), y_n(t)), \eta(u_{n+1}(t), v(t)) \rangle + b(u_n(t), u_{n+1}(t)) - b(u_n(t), v(t)) \rangle \geq 0,
\]
\[
x_{n+1}(t) \in A(t, u_{n+1}(t)), x_n(t+1(t) \in T(t, u_n(t)),
\]
\[
\|x_{n+1}(t) - x_n(t)\| \leq (1 + (1 + n)^{-1})\|A(t, u_{n+1}(t)), A(t, u_n(t))\|,
\]
\[
\|y_{n+1}(t) - y_n(t)\| \leq (1 + (1 + n)^{-1})\|T(t, u_{n+1}(t)), T(t, u_n(t))\|.
\]

for any \( t \in \Omega \) and \( n = 0, 1, 2, \ldots \).

4. Convergence analysis

**Theorem 4.1.** If the conditions of the Theorem 3.1 are hold, and the function \( b(\cdot, \cdot) \) satisfies the conditions (i)–(iv) where for all \( t \in \Omega \), \( \gamma(t) \in \Omega, k_{21}(t) + k_{22}(t) \). Then the problem (2.1) has a unique random solution i.e. there exist the measurable mappings \( \hat{u}, \hat{x}, \hat{y} : \Omega \to D \) such that \( \forall t \in \Omega, \hat{x}(t) \in A(t, \hat{u}(t)), \hat{y}(t) \in T(t, \hat{u}(t)) \) and
\[
\langle N(\hat{x}(t), \hat{y}(t), \eta(v(t), \hat{u}(t))) + b(\hat{u}(t), v(t)) - b(\hat{u}(t), \hat{u}(t)) \rangle \geq 0, \quad \forall v(t) \in D, t \in \Omega.
\]

**Proof.** According to the conclusion of the Theorem 3.1, we know for each \( t \in \Omega \) and the measurable mapping \( u^* : \Omega \to D \), there exists a unique solution \( \hat{w}(i.e. \hat{u}(t) \in D, \hat{x}(t) \in A(t, \hat{u}(t)), \hat{y}(t) \in T(t, \hat{u}(t)) \) satisfying the auxiliary problem (3.1). Defining a mapping \( F : D \to D \) by \( u^* \to \hat{w}(u^*) \), we will prove that the mapping \( F \) is a contraction mapping. Indeed, for any \( u^*_1(t), u^*_2(t) \in D \), there exist unique \( \hat{w}_1 = F(u^*_1(t)), \hat{w}_2 = F(u^*_2(t)) \), for all \( v(t) \in D \) and \( t \in \Omega \) such that
\[
\langle N(x_1(t), y_1(t), \eta(v(t), u_1(t))) + b(u^*_1(t), v(t)) - b(u^*_1(t), u_1(t)) \rangle \geq 0, \quad (4.1)
\]
\[
\langle N(x_2(t), y_2(t), \eta(v(t), u_2(t))) + b(u^*_2(t), v(t)) - b(u^*_2(t), u_2(t)) \rangle \geq 0. \quad (4.2)
\]
Taking $v(t) = u_2(t)$ in (4.1) and $v(t) = u_1(t)$ in (4.2) and adding two inequalities, we have

$$N(x_1(t), y_1(t)), \eta(u_2(t), u_1(t))) + N(x_2(t), y_2(t)), \eta(u_1(t), u_2(t))) + b(u^*_1(t) - u^*_2(t), u_2(t)) - b(u^*_1(t) - u^*_2(t), u_1(t)) \geq 0.$$ 

By $\eta(u, v), v(t)) = -\eta(v(t), u(t))$ and the assumption on $b(\cdot, \cdot)$, we have

$$(k_{21} + k_{22})\|u_1(t) - u_2(t)\|^2 \leq \langle N(x_1(t), y_1(t)) - N(x_2(t), y_1(t)), \eta(u_1(t), u_2(t)) \rangle + \langle N(x_2(t), y_1(t)) - N(x_2(t), y_2(t)), \eta(u_1(t), u_2(t)) \rangle \leq b(u^*_1(t) - u^*_2(t), u_2(t)) - b(u^*_1(t) - u^*_2(t), u_1(t)) \leq \gamma(t)\|u^*_1(t) - u^*_2(t)\| \cdot \|u_1(t) - u_2(t)\|,$$

which derives

$$\|u_1(t) - u_2(t)\| \leq \frac{\gamma(t)}{k_{21}(t) + k_{22}(t)} \cdot \|u^*_1(t) - u^*_2(t)\|, \tag{4.3}$$

$$\|x_1(t) - x_2(t)\| \leq H(A(t, u_1(t)), A(t, u_2(t))) \leq \frac{\lambda_1(t)\gamma(t)}{k_{21}(t) + k_{22}(t)} \cdot \|u^*_1(t) - u^*_2(t)\|, \tag{4.4}$$

$$\|y_1(t) - y_2(t)\| \leq H(T(t, u_1(t)), T(t, u_2(t))) \leq \frac{\lambda_2(t)\gamma(t)}{k_{21}(t) + k_{22}(t)} \cdot \|u^*_1(t) - u^*_2(t)\|. \tag{4.5}$$

The inequalities (4.3)-(4.5) together with $\gamma(t) \in (0, k_{21}(t) + k_{22}(t))$ and $0 < \lambda_1(t), \lambda_2(t) \leq 1$ result in that $F$ is a contraction mapping. Hence, there exists a unique point $\hat{u}(t) \in D$ such that $\hat{u}(t) = F(\hat{u}(t))$ and

$$\langle N(\hat{x}(t), \hat{y}(t)), \eta(v(t), \hat{u}(t)) \rangle + b(\hat{u}(t), v) - b(\hat{u}(t), \hat{u}(t)) \geq 0, \forall v(t) \in D, t \in \Omega.$$ 

Now we know $\hat{u}(t) \in D, \hat{x}(t) \in A(t, \hat{t}(t)), \hat{y}(t) \in T(t, \hat{t}(t))$ is the unique random solution of the problem (2.1). $\square$

Hence, we discuss the convergence analysis of iterative sequence generated by the iterative Algorithm 3.1.

**Theorem 4.2.** Let $D$ be a nonempty closed convex subset of $H$ and the measurable mapping $\eta : D \times D \to D$ be strongly monotone and Lipschitz continuous with the measurable functions $\sigma, \delta : \Omega \to (0, +\infty)$, respectively, let the measurable function $b(\cdot, \cdot)$ satisfy the conditions (i)–(iv). The measurable mapping $N(\cdot, \cdot)$ is Lipschitz continuous and strongly monotone in first argument with respect to the random multivalued mapping $A$ with the measurable functions $k_{11}, k_{21} : \Omega \to (0, +\infty)$, respectively; $N(\cdot, \cdot)$ is Lipschitz continuous in second argument with respect to the random multivalued mapping $T$ with the measurable function $k_{12} : \Omega \to (0, +\infty)$. Let the measurable mappings $A$ and $T$ are $H$-continuous with the measurable functions $\lambda_1, \lambda_2 : \Omega \to (0, 1]$, respectively. If the following conditions hold for any $t \in \Omega$ and any measurable function $\rho : \Omega \to (0, +\infty)$:

$$\sigma(t) = \rho(t)\gamma(t) + \delta(t)\rho(t)\lambda_1(t)k_{11}(t)k_{12}(t) \geq 0,$$

Then the iterative sequence $\{u_0(t), \{x_0(t), y_0(t)\}$ obtained from Algorithm 3.1 strongly converges to $u(t), x(t), y(t)$ respectively, where $\{u(t), x(t), y(t)\}$ is a random solution of GMVLEP (2.1).

**Proof.** For any $v : \Omega \to D$, it follows from (3.5) that

$$\langle N(x_1(t), y_1(t)), \eta(u_2(t), v(t)) \rangle + b(u_1(t) - u_2(t), v(t)) \geq 0, \tag{4.7}$$

$$\langle N(x_1(t), y_1(t)), \eta(u_2(t), v(t)) \rangle + b(u_1(t), u_1(t)) - b(u_2(t), v(t)) \geq 0. \tag{4.8}$$

Taking $v(t) = u_{n+1}(t)$ in (4.7) and $v(t) = u_n(t)$ in (4.8), respectively, we get

$$\langle N(x_1(t), y_1(t)), \eta(u_{n+1}(t), u_{n+1}(t)) \rangle + b(u_n(t) - u_{n+1}(t), u_n(t)) - b(u_n(t), u_n(t)) \geq 0, \tag{4.9}$$

$$\langle N(x_1(t), y_1(t)), \eta(u_{n+1}(t), u_{n+1}(t)) \rangle + b(u_n(t), u_{n+1}(t)) - b(u_n(t), u_n(t)) \geq 0. \tag{4.10}$$

Adding two inequalities (4.9) and (4.10), by the assumption on the function $b$, we obtain

$$\langle N(x_1(t), y_1(t)) - N(x_{n+1}(t), y_{n+1}(t)), \eta(u_n(t), u_{n+1}(t)) \rangle + b(u_1(t) - u_n(t), u_n(t) - u_{n+1}(t)) \geq 0 \tag{4.11}$$

Further, we have

$$\langle u_n(t) - u_{n+1}(t), \eta(u_n(t), u_{n+1}(t)) \rangle \leq \langle u_n(t) - u_{n+1}(t) + \rho(t)(N(x_1(t), y_1(t)) - N(x_{n+1}(t), y_{n+1}(t))), \eta(u_n(t), u_{n+1}(t)) \rangle + \rho(t)b(u_{n-1}(t) - u_n(t), u_n(t) - u_{n+1}(t)), \tag{4.12}$$

where $\rho : \Omega \to (0, +\infty)$ is a measurable function.
Since $\eta$ is strongly monotone and Lipschitz continuous, from (4.12), we have
\[
\sigma(t)\|u_n(t) - u_{n+1}(t)\|^2 \leq \|u_n(t) - u_{n+1}(t) + \rho(t)(N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)))\| \cdot \|\eta(u_n(t), u_{n+1}(t))\|
\]
\[
+ \rho(t)|b(u_{n-1}(t) - u_n(t), u_n(t) - u_{n+1}(t))| \leq \delta(t)\|u_n(t) - u_{n+1}(t) + \rho(t)(N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)))\| \cdot \|u_n(t) - u_{n+1}(t)\|
\]
\[
+ \rho(t)\gamma(t)\|u_n(t) - u_{n-1}(t)\| \cdot \|u_n(t) - u_{n+1}(t)\|,
\]
which implies
\[
\sigma(t)\|u_n(t) - u_{n+1}(t)\| \leq \delta(t)\|u_n(t) - u_{n+1}(t) + \rho(t)(N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)))\| + \rho(t)\gamma(t)\|u_n(t) - u_{n-1}(t)\|
\]
\[
\leq \delta(t)\|u_n(t) - u_{n+1}(t) + \rho(t)(N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)))\| + \rho(t)\gamma(t)\|u_n(t) - u_{n-1}(t)\|
\]
\[
+ \delta(t)\|u_n(t) - u_{n+1}(t) + \rho(t)(N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)))\| + \rho(t)\gamma(t)\|u_n(t) - u_{n-1}(t)\|. \tag{4.13}
\]

Noting that the mapping $N(\cdot, \cdot)$ is Lipschitz continuous and $\eta$-strongly monotone with respect to the random multivalued mapping $A$ in first argument and Lipschitz continuous in second argument, and the random multivalued mapping $A, T$ are H-continuous, we obtain
\[
\|u_n(t) - u_{n+1}(t) + \rho(t)(N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)))\|^2
\]
\[
= \|u_n(t) - u_{n+1}(t)\|^2 + 2\rho(t)\langle N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t)), u_n(t) - u_{n+1}(t)\rangle
\]
\[
+ \rho^2(t)\|N(x_n(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t))\|^2
\]
\[
\leq (1 + 2\rho(t)\kappa_2(t) + \rho^2(t)\kappa_1^2(t)(1 + \lambda_1^2(t)))\|u_n(t) - u_{n+1}(t)\|^2, \tag{4.14}
\]
and
\[
\|N(x_{n+1}(t), y_n(t)) - N(x_{n+1}(t), y_{n+1}(t))\| \leq \kappa_2(t)\|y_n(t) - y_{n+1}(t)\|
\]
\[
\leq \lambda_2(t)\kappa_2(t)(1 + \frac{1}{n})\|u_n(t) - u_{n-1}(t)\|. \tag{4.15}
\]
From (4.13)–(4.15) it follows that
\[
\sigma(t)\|u_n(t) - u_{n+1}(t)\| \leq \delta(t)\left(1 + 2\rho(t)\kappa_2(t) + \rho^2(t)\kappa_1^2(t)(1 + \lambda_1^2(t))\right)\|u_n(t) - u_{n+1}(t)\|
\]
\[
+ \rho(t)\lambda_2(t)\kappa_2(t)(1 + \frac{1}{n})\|u_n(t) - u_{n+1}(t)\| + \rho(t)\gamma(t)\|u_n(t) - u_{n-1}(t)\|,
\]
which implies
\[
\|u_n(t) - u_{n+1}(t)\| \leq \theta_n(t)\|u_n(t) - u_{n-1}(t)\|, \tag{4.16}
\]
where $\theta_n(t) = \frac{\rho(t)\gamma(t)}{\sigma(t) - \delta(t)(1 + 2\rho(t)\kappa_2(t) + \rho^2(t)\kappa_1^2(t)(1 + \lambda_1^2(t)) + \rho(t)\lambda_2(t)\kappa_2(t)(1 + \frac{1}{n}))}$.

Letting
\[
\theta(t) = \frac{\rho(t)\gamma(t)}{\sigma(t) - \delta(t)(1 + 2\rho(t)\kappa_2(t) + \rho^2(t)\kappa_1^2(t)(1 + \lambda_1^2(t)) + \rho(t)\lambda_2(t)\kappa_2(t)(1 + \frac{1}{n}))}, \quad \forall t \in \Omega.
\]

We know that $\theta_n(t) \to \theta(t)$ for all $t \in \Omega$. By condition (4.6), it follows that $\theta(t) \in (0, 1)$ and hence (4.16) implies that $\{u_n(t)\}$ is a Cauchy sequence in $D$. Since $D$ is complete, there exists a measurable mapping $u : \Omega \to D$ such that $u_n(t) \to u(t)$ as $n \to \infty$. Further, from Algorithm 3.1, we have
\[
\|x_{n+1}(t) - x_n(t)\| \leq \lambda_1(t)\left(1 + \frac{1}{n}\right)\|u_{n+1}(t) - u_n(t)\|
\]
\[
\|y_{n+1}(t) - y_n(t)\| \leq \lambda_2(t)\left(1 + \frac{1}{n}\right)\|u_{n+1}(t) - u_n(t)\|
\]
which implies that $\{x_n(t), y_n(t)\}$ are also Cauchy sequences in $H$. Let $x_n(t) \to x(t), y_n(t) \to y(t), n \to \infty$. Since $\{u_n(t), x_n(t), y_n(t)\}$ are sequences of measurable mappings, we know that $u, x, y : \Omega \to H$ are measurable.

Now we prove that $x(t) \in A(t, u(t)), y(t) \in T(t, u(t))$, for any $t \in \Omega$, we have
\[
\text{d}(x(t), A(t, u(t))) = \inf\{\|x(t) - z\| : z \in A(t, u(t))\}
\]
\[
\leq \|x(t) - x_n(t)\| + \text{d}(x_n(t), A(t, u(t)))
\]
\[
\leq \|x(t) - x_n(t)\| + \tilde{H}(A(t, u(t)), A(t, u(t)))
\]
\[
\leq \|x(t) - x_n(t)\| + \lambda_1(t)\|u_n(t) - u(t)\| \to 0.
\]
hence $x(t) \in A(t, u(t))$, for all $t \in \Omega$. Similarly we can prove that $y(t) \in T(t, u(t))$. So we have

$$\langle N(x(t), y(t)), \eta(u(t), v(t)) \rangle + b(u(t), u(t)) - b(u(t), v(t)) \geq 0, \forall v(t) \in D, t \in \Omega.$$ 

This completes the proof. □

References