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Embeddings of Grassmann graphs

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ABSTRACT

Let V and V' be vector spaces of dimension n and n', respectively. Let $k \in \{2, \dots, n-2\}$ and $k' \in \{2, \dots, n'-2\}$. We describe all isometric and l-rigid isometric embeddings of the Grassmann graph $\Gamma_k(V)$ in the Grassmann graph $\Gamma_{k'}(V')$.

1. Introduction

Let V be an n-dimensional left vector space over a division ring R. Denote by $\mathcal{G}_k(V)$ the Grassmannian consisting of k-dimensional subspaces of V. Two elements of $\mathcal{G}_k(V)$ are *adjacent* if their intersection is $(k-1)$-dimensional. The *Grassmann graph* $\Gamma_k(V)$ is the graph whose vertex set is $\mathcal{G}_k(V)$ and whose edges are pairs of adjacent k-dimensional subspaces. By Chow's theorem [3], if $1 < k < n-1$ then every automorphism of $\Gamma_k(V)$ is induced by a semilinear automorphism of V or a semilinear isomorphism of V to the dual vector space V^* and the second possibility can be realized only in the case when $n = 2k$ (if $k = 1, n-1$ then any two distinct vertices of $\Gamma_k(V)$ are adjacent and every bijective transformation of $\mathcal{G}_k(V)$ is an automorphism of $\Gamma_k(V)$). Some results closely related with Chow's theorem can be found in [2,7–9,11,12] and we refer [13] for a survey.

Let V' be an n'-dimensional left vector space over a division ring R'. We investigate isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ under assumption that $1 < k < n-1$ and $1 < k' < n'-1$ (the case $k = k'$ was considered in [9]). Then $n, n' \geq 4$ and the existence of such embeddings implies that the

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diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_{k'}(V')$, i.e.

$$\min\{k, n-k\} \leq \min\{k', n'-k'\}.$$

In the case when $k \leq n-k$, we show that every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is induced by a semilinear $(2k)$-embedding (a semilinear injection such that the image of every independent $(2k)$-element subset is independent) of V in V'/S, where S is a $(k'-k)$-dimensional subspace of V', or a semilinear $(2k)$-embedding of V in U^*, where U is a $(k'+k)$-dimensional subspace of V' (Theorem 4.1). If $k > n-k$ then there are isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ which cannot be induced by semilinear mappings of V to some vector spaces.

Our second result (Theorem 6.1) is related with so-called l-rigid embeddings. An embedding f of a graph Γ in a graph Γ' is *rigid* if for every automorphism g of Γ there is an automorphism g' of Γ' such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & \Gamma' \\ \downarrow g & & \downarrow g' \\ \Gamma & \xrightarrow{f} & \Gamma' \end{array}$$

is commutative, roughly speaking, every automorphism of Γ can be extended to an automorphism of Γ'. We say that an embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is *l-rigid* if every automorphism of $\Gamma_k(V)$ induced by a linear automorphism of V can be extended to the automorphism of $\Gamma_{k'}(V')$ induced by a linear automorphism of V'.

In the case when $n = 2k$, every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is l-rigid. In general case (we do not require that $k \leq n-k$), every l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is induced by a semilinear embedding (a semilinear injection transferring independent subsets to independent subsets) of V in V'/S, where S is a $(k'-k)$-dimensional subspace of V', or a semilinear embedding of V in U^*, where U is a $(k'+k)$-dimensional subspace of V'. The proof of this result is based on a characterization of semilinear embeddings (Theorem 5.1).

Using [10], we establish the existence of isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ which are not l-rigid (Example 6.4).

2. Basic facts and definitions

2.1.

Let Γ be a connected graph. A subset in the vertex set of Γ formed by mutually adjacent vertices is called a *clique*. Using Zorn lemma, we can show that every clique is contained in a maximal clique. The *distance* $d(v, w)$ between two vertices v and w of Γ is defined as the smallest number i such that there exists a path of length i (a path consisting of i edges) connecting v and w. The *diameter* of Γ is the maximum of all distances $d(v, w)$.

An injective mapping of the vertex set of Γ to the vertex set of a graph Γ' is called an *embedding* of Γ in Γ' if vertices of Γ are adjacent only in the case when their images are adjacent vertices of Γ'. Every surjective embedding is an isomorphism. An embedding is said to be *isometric* if it preserves the distance between any two vertices.

2.2.

Let $k \in \{1, \ldots, n-1\}$. Consider incident subspaces $S, U \subset V$ such that

$$\dim S < k < \dim U$$

and denote by $[S, U]_k$ the set formed by all $X \in \mathcal{G}_k(V)$ satisfying $S \subset X \subset U$. In the case when $S = 0$ or $U = V$, this set will be denoted by $\langle U]_k$ or $[S\rangle_k$, respectively. The *Grassmann space* $\mathfrak{G}_k(V)$ is the partial linear space whose point set is $\mathcal{G}_k(V)$ and whose lines are subsets of type

$$[S, U]_k, \quad S \in \mathcal{G}_{k-1}(V), \quad U \in \mathcal{G}_{k+1}(V).$$

It is clear that $\mathfrak{G}_1(V) = \Pi_V$ and $\mathfrak{G}_{n-1}(V) = \Pi^*_V$ (we denote by Π_V the projective space associated with V and write Π^*_V for the corresponding dual projective space). Two distinct points of $\mathfrak{G}_k(V)$ are collinear (joined by a line) if and only if they are adjacent vertices of the Grassmann graph $\Gamma_k(V)$.

If $1 < k < n-1$ then there are precisely the following two types of maximal cliques of $\Gamma_k(V)$:

(1) the *top* $\langle U]_k, U \in \mathcal{G}_{k+1}(V)$,
(2) the *star* $[S\rangle_k, S \in \mathcal{G}_{k-1}(V)$.

The top $\langle U]_k$ and the star $[S\rangle_k$ together with the lines contained in them are projective spaces. The first projective space is Π^*_U and the second can be identified with $\Pi_{V/S}$.

The distance $d(X, Y)$ between $X, Y \in \mathcal{G}_k(V)$ in the graph $\Gamma_k(V)$ is equal to

$$k - \dim(X \cap Y)$$

and the diameter of $\Gamma_k(V)$ is equal to $\min\{k, n-k\}$.

2.3.

All linear functionals of V form an n-dimensional right vector space over R. The associated left vector space over the opposite division ring R^* is called the *dual* vector space and denoted by V^*. The division rings R and R^* have the same set of elements and the same additive operation; the multiplicative operation of R^* is defined as $a * b := ba$ and we have $R = R^*$ only in the case when R is a field. The second dual space V^{**} can be canonically identified with V.

For subspaces $X \subset V$ and $Y \subset V^*$ the subspaces

$$X^0 := \{\, x^* \in V^* \ : \ x^*(x) = 0 \quad \forall\, x \in X \,\},$$

$$Y^0 := \{\, x \in V \ : \ x^*(x) = 0 \quad \forall\, x^* \in Y \,\}$$

are called the *annihilators* of X and Y, respectively. The annihilator mapping (which transfers every subspace $S \subset V$ to the annihilator $S^0 \subset V^*$) induces an isomorphism between $\Gamma_k(V)$ and $\Gamma_{n-k}(V^*)$ for every $k \in \{1, \dots, n-1\}$.

2.4.

An additive mapping $l : V \to V'$ is said to be *semilinear* if there exists a homomorphism $\sigma : R \to R'$ such that

$$l(ax) = \sigma(a)l(x)$$

for all $x \in V$ and all $a \in R$. If l is non-zero then there is only one homomorphism satisfying this condition. Every non-zero homomorphism of R to R' is injective.

Every semilinear injection of V to V' induces a mapping of $\mathcal{G}_1(V)$ to $\mathcal{G}_1(V')$ which transfers lines of Π_V to subsets in lines of $\Pi_{V'}$ (note that this mapping is not necessarily injective). We will use the following version of the Fundamental Theorem of Projective Geometry [4–6], see also [13, Theorem 1.4].

Theorem 2.1 (C.A. Faure, A. Frölicher, H. Havlicek). *Let f be a mapping of $\mathcal{G}_1(V)$ to $\mathcal{G}_1(V')$ transferring lines of Π_V to subsets in lines of $\Pi_{V'}$. If the image of f is not contained in a line then f is induced by a semilinear injection of V to V'.*

A semilinear mapping of V to V' is called a *semilinear isomorphism* if it is bijective and the associated homomorphism of R to R' is an isomorphism. If u is a semilinear automorphism of V then the mapping u_k sending every $X \in \mathcal{G}_k(V)$ to $u(X)$ is an automorphism of $\Gamma_k(V)$. If $n = 2k$ and $v : V \to V^*$ is a semilinear isomorphism then the bijection transferring every $X \in \mathcal{G}_k(V)$ to the annihilator of $v(X)$ is an automorphism of $\Gamma_k(V)$.

Theorem 2.2 (W.L. Chows [3]). *Every automorphism of $\Gamma_k(V)$, $1 < k < n-1$ is induced by a semilinear automorphism of V or a semilinear isomorphism of V to V^*; the second possibility can be realized only in the case when $n = 2k$.*

3. General properties of embeddings

Let $k \in \{2, \dots, n-2\}$ and $k' \in \{2, \dots, n'-2\}$. Every embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ transfers maximal cliques of $\Gamma_k(V)$ to subsets in maximal cliques of $\Gamma_{k'}(V')$; moreover, every maximal clique of $\Gamma_{k'}(V')$ contains at most one image of a maximal clique of $\Gamma_k(V)$ (otherwise, the preimages of some adjacent vertices of $\Gamma_{k'}(V')$ are non-adjacent which is impossible).

Proposition 3.1. *For any embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ the image of every maximal clique of $\Gamma_k(V)$ is contained in precisely one maximal clique of $\Gamma_{k'}(V')$.*

Proof. Let f be an embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. Suppose that $\mathcal{X}$ is a maximal clique of $\Gamma_k(V)$ and $f(\mathcal{X})$ is contained in two distinct maximal cliques of $\Gamma_{k'}(V')$. Since the intersection of two distinct maximal cliques is empty or a one-element set or a line, there exist $S \in \mathcal{G}_{k'-1}(V')$ and $U \in \mathcal{G}_{k'+1}(V')$ such that

$$f(\mathcal{X}) \subset [S, U]_{k'}. \tag{3.1}$$

We take any maximal clique $\mathcal{Y} \neq \mathcal{X}$ of $\Gamma_k(V)$ which intersects $\mathcal{X}$ in a line and consider a maximal clique $\mathcal{Y}'$ of $\Gamma_{k'}(V')$ containing $f(\mathcal{Y})$. The inclusion (3.1) guarantees that the line $[S, U]_{k'}$ intersects $f(\mathcal{Y}) \subset \mathcal{Y}'$ in a set containing more than one element. Then $[S, U]_{k'} \subset \mathcal{Y}'$ (a line is contained in a maximal clique or intersects it in at most one element). So, the maximal clique $\mathcal{Y}'$ contains the images of both $\mathcal{X}$ and $\mathcal{Y}$ which are distinct maximal cliques of $\Gamma_k(V)$, a contradiction. □

It was noted above that the intersection of two distinct maximal cliques of $\Gamma_k(V)$ is empty or a one-element set or a line. The latter possibility can be realized only in the case when the maximal cliques are of different types — one of them is a star and the other is a top. For any distinct maximal cliques $\mathcal{X}$, $\mathcal{Y}$ of $\Gamma_k(V)$ there is a sequence of maximal cliques of $\Gamma_k(V)$

$$\mathcal{X} = \mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_i = \mathcal{Y}$$

such that $\mathcal{X}_{j-1} \cap \mathcal{X}_j$ is a line for every $j \in \{1, \dots, i\}$. This implies the following.

Proposition 3.2. *For every embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ one of the following possibilities is realized:*

(A) *stars go to subsets of stars and tops go to subsets of tops,*
(B) *stars go to subsets of tops and tops go to subsets of stars.*

We say that an embedding is of *type* (A) or (B) if the corresponding possibility is realized.

If an embedding f of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is of type (A) then the embedding of $\Gamma_k(V)$ in $\Gamma_{n'-k'}(V'^*)$ sending every $X \in \mathcal{G}_k(V)$ to the annihilator of $f(X)$ is of type (B).

4. Isometric embeddings

A semilinear injection of V to V' is said to be a *semilinear m-embedding* if the image of every independent m-element subset is independent. The existence of such mappings implies that $m \leq \min\{n, n'\}$. In the case when $n \leq n'$, semilinear n-embeddings of V in V' will be called *semilinear embeddings*.

Remark 4.1. By [10], there exist fields F and F' such that for any natural numbers p and q there is a semilinear p-embedding of F^{p+q} in F'^{p}. It is clear that such semilinear p-embeddings cannot be $(p+1)$-embeddings.

Let $l : V \to V'$ be a semilinear m-embedding. For every $p \in \{1, \dots, m\}$ and every p-dimensional subspace $X \subset V$ the dimension of the subspace spanned by $l(X)$ is equal to p. So, we have the mapping

$$l_p : \mathcal{G}_p(V) \to \mathcal{G}_p(V'),$$
$$X \to \langle l(X) \rangle.$$

By [9, Proposition 2.2], if $2p \leq \min\{n, n'\}$ and $l : V \to V'$ is a $(2p)$-embedding then l_p is an isometric embedding of $\Gamma_p(V)$ in $\Gamma_p(V')$.

Remark 4.2. By [9, Proposition 2.1], every semilinear $(k+1)$-embedding of V in V' induces an injection of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$ sending adjacent subspaces to adjacent subspaces; but this mapping is not necessarily an embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$. So, the following problem is open: construct non-isometric embeddings of $\Gamma_k(V)$ in $\Gamma_k(V')$?

Let $k \in \{2, \dots, n-2\}$ and $k' \in \{2, \dots, n'-2\}$. The existence of isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ implies that

$$\min\{k, n-k\} \leq \min\{k', n'-k'\}$$

(the diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_{k'}(V')$). In the next three examples we suppose that $k \leq n-k$, i.e.

$$k \leq \min\{k', n-k, n'-k'\}. \tag{4.1}$$

Example 4.1. Let $S \in \mathcal{G}_{k'-k}(V')$. By (4.1),

$$\dim(V'/S) = n' - k' + k \geq 2k.$$

If $l : V \to V'/S$ is a semilinear $(2k)$-embedding then l_k is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V'/S)$. Let π be the natural isometric embedding of $\Gamma_k(V'/S)$ in $\Gamma_{k'}(V')$ (which transfers every k-dimensional subspace of V'/S to the corresponding k'-dimensional subspace of V'). Then πl_k is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ of type (A).

Example 4.2. Let $U \in \mathcal{G}_{k'+k}(V')$ (by (4.1), we have $k' + k \leq n'$). If $v : V \to U^*$ is a semilinear $(2k)$-embedding then v_k is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U^*)$. By duality, it can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(U)$. Since U is contained in V', we get an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ of type (B).

Example 4.3. Suppose that $n = 2k$ and $S \in \mathcal{G}_{k'-k}(V')$, $U \in \mathcal{G}_{k'+k}(V')$ are incident. Then

$$\dim(U/S) = 2k = n.$$

By Example 4.1, every semilinear embedding of V in $U/S \subset V'/S$ induces an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. If $w : V \to (U/S)^*$ is a semilinear embedding then w_k is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k((U/S)^*)$ and, by duality, it can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U/S)$. As in Example 4.1, we construct an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

Theorem 4.1. *Let f be an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. If $k \leq n-k$ then one of the following possibilities is realized:*

- *there is $S \in \mathcal{G}_{k'-k}(V')$ such that f is induced by a semilinear $(2k)$-embedding of V in V'/S, see Example 4.1;*

- *there is $U \in \mathcal{G}_{k'+k}(V')$ such that f is induced by a semilinear $(2k)$-embedding of V in U^*, see Example 4.2.*

In the case when $n = 2k$, there are incident $S \in \mathcal{G}_{k'-k}(V')$ and $U \in \mathcal{G}_{k'+k}(V')$ such that f is induced by a semilinear embedding of V in U/S or a semilinear embedding of V in $(U/S)^*$, see Example 4.3.

Remark 4.3. Suppose that f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ and $k > n-k$. Consider the mapping which sends every $X \in \mathcal{G}_{n-k}(V^*)$ to $f(X^0)$. This is an isometric embedding of $\Gamma_{n-k}(V^*)$ in $\Gamma_{k'}(V')$. Since $n-k < n-(n-k)$, it is induced by a semilinear $2(n-k)$-embedding of V^* in one of vector spaces described above.

Proof of Theorem 4.1. Suppose that f is an embedding of type (A). By Section 3, there exists an injective mapping

$$f_{k-1} : \mathcal{G}_{k-1}(V) \to \mathcal{G}_{k'-1}(V')$$

such that

$$f([X\rangle_k) \subset [f_{k-1}(X)\rangle_{k'} \quad \forall\, X \in \mathcal{G}_{k-1}(V).$$

Then

$$f_{k-1}(\langle Y]_{k-1}) \subset \langle f(Y)]_{k'-1} \quad \forall\, Y \in \mathcal{G}_k(V).$$

Since for any two adjacent vertices there is a top containing them, the latter inclusion implies that f_{k-1} is adjacency preserving. Thus for any $X, Y \in \mathcal{G}_{k-1}(V)$ we have

$$d(X, Y) \geq d(f_{k-1}(X), f_{k-1}(Y)).$$

We prove the inverse inequality.

The condition $2k \leq n$ implies the existence of $X', Y' \in \mathcal{G}_k(V)$ such that $X \subset X'$, $Y \subset Y'$ and

$$X \cap Y = X' \cap Y'.$$

Then

$$d(X, Y) = d(X', Y') - 1 \tag{4.2}$$

(indeed, $d(X, Y) = k - 1 - \dim(X \cap Y) = k - 1 - \dim(X' \cap Y') = d(X', Y') - 1$). Since f_{k-1} is induced by f, we have

$$f_{k-1}(X) \subset f(X') \quad \text{and} \quad f_{k-1}(Y) \subset f(Y')$$

which guarantees that

$$\dim(f_{k-1}(X) \cap f_{k-1}(Y)) \leq \dim(f(X') \cap f(Y')). \tag{4.3}$$

Using (4.2) and (4.3), we get the following

$$\begin{aligned} d(X, Y) &= d(X', Y') - 1 = d(f(X'), f(Y')) - 1 = k' - 1 - \dim(f(X') \cap f(Y')) \\ &\leq k' - 1 - \dim(f_{k-1}(X) \cap f_{k-1}(Y)) = d(f_{k-1}(X), f_{k-1}(Y)). \end{aligned}$$

So, f_{k-1} is an isometric embedding of $\Gamma_{k-1}(V)$ in $\Gamma_{k'-1}(V')$. This is an embedding of type (A) (it was established above that f_{k-1} sends tops to subsets of tops). Step by step, we construct a sequence of isometric embeddings

$$f_i : \mathcal{G}_i(V) \to \mathcal{G}_{k'-k+i}(V), \quad i = k, \ldots, 1$$

of $\Gamma_i(V)$ in $\Gamma_{k-k'+i}(V')$ such that $f_k = f$ and we have

$$f_i([X\rangle_i) \subset [f_{i-1}(X)\rangle_{k'-k+i} \quad \forall\, X \in \mathcal{G}_{i-1}(V)$$

and

$$f_{i-1}(\langle Y]_{i-1}) \subset \langle f_i(Y)]_{k'-k+i-1} \quad \forall\, Y \in \mathcal{G}_i(V) \tag{4.4}$$

if $i > 1$.

The image of f_1 is a clique of $\Gamma_{k'-k+1}(V')$. This clique cannot be contained in any top (otherwise, there is $X' \in \mathcal{G}_{k'-k+2}(V')$ such that $f_2(X) = X'$ for every $X \in \mathcal{G}_2(V)$ and f_2 is not injective). Therefore, there is $S \in \mathcal{G}_{k'-k}(V')$ such that the image of f_1 is contained in the star $[S\rangle_{k'-k+1}$.

By (4.4), f_1 transfers lines of Π_V to subsets of lines contained in $[S\rangle_{k'-k+1}$. It was noted in Section 2.2 that the star $[S\rangle_{k'-k+1}$ (together with all lines contained in it) can be identified with the projective space $\Pi_{V'/S}$. Theorem 2.1 shows that f_1 is induced by a semilinear injection $l : V \to V'/S$.

Using (4.4), we establish that

$$f_1(\langle X]_1) \subset \langle f(X)]_{k'-k+1} \quad \forall\, X \in \mathcal{G}_k(V).$$

On the other hand,

$$f_1(\langle X]_1) \subset \langle \pi(\langle l(X)\rangle)]_{k'-k+1} \quad \forall\, X \in \mathcal{G}_k(V),$$

where π is the mapping which transfers every subspace of V'/S to the corresponding subspace of V'. Since the intersection of two distinct tops contains at most one element, we get

$$f(X) = \pi(\langle l(X)\rangle) \quad \forall\, X \in \mathcal{G}_k(V)$$

which means that $f = \pi l_k$.

Every independent $(2k)$-element subset $A \subset V$ can be presented as the disjoint union of two independent k-element subsets A_1 and A_2. Then

$$d(f(\langle A_1\rangle), f(\langle A_2\rangle)) = d(\langle A_1\rangle, \langle A_2\rangle) = k$$

which means that $\langle l(A_1)\rangle$ and $\langle l(A_2)\rangle$ are k-dimensional subspaces of V'/S intersecting in 0. Hence the subspace spanned by $l(A_1 \cup A_2) = l(A)$ is $(2k)$-dimensional.

So, l is a semilinear $(2k)$-embedding of V in V'/S and f is as in Example 4.1. In the case when $n = 2k$, l is a semilinear embedding and the image of l is contained in U/S, where $U \in \mathcal{G}_{k'+k}(V')$.

Now suppose that f is an embedding of type (B). By duality, f can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{n'-k'}(V'^*)$ (this embedding sends every $X \in \mathcal{G}_k(V)$ to the annihilator of $f(X)$). We get an embedding of type (A) and its image is contained in

$$[S'\rangle_{n'-k'}, \quad S' \in \mathcal{G}_{n'-k'-k}(V'^*);$$

in the case when $n = 2k$, the image is contained in

$$[S', U']_{n'-k'}, \quad S' \in \mathcal{G}_{n'-k'-k}(V'^*), \; U' \in \mathcal{G}_{n'-k'+k}(V'^*).$$

The image of f is contained in $\langle U]_{k'}$, where $U \in \mathcal{G}_{k'+k}(V')$ is the annihilator of S'. Thus f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(U)$. By duality, f can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U^*)$ of type (A). Hence it is induced by a semilinear $(2k)$-embedding of V in U^*, i.e. f is as in Example 4.2.

If $n = 2k$ then the image of f is contained in $[S, U]_{k'}$, where $S \in \mathcal{G}_{k'-k}(V')$ and $U \in \mathcal{G}_{k'+k}(V')$ are the annihilators of U' and S', respectively. This means that $f = \pi f'$, where f' is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U/S)$ of type (B) and π is the mapping which transfers every subspace of V'/S to the corresponding subspace of V'. Since

$$\dim(U/S) = 2k,$$

f' can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k((U/S)^*)$ of type (A). The latter embedding is induced by a semilinear embedding of V in $(U/S)^*$ and f is as in Example 4.3. □

Remark 4.4. Using the same idea, the author describes the images of isometric embeddings of Johnson graphs in Grassmann graphs [14, Theorem 4].

5. Characterization of semilinear embeddings

Theorem 5.1. *Let $l : V \to V'$ be a semilinear injection. Then l is a semilinear embedding if and only if for every linear automorphism $u \in \mathrm{GL}(V)$ there is a linear automorphism $u' \in \mathrm{GL}(V')$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{l} & V' \\ \downarrow u & & \downarrow u' \\ V & \xrightarrow{l} & V' \end{array} \tag{5.1}$$

is commutative.

To prove Theorem 5.1 we use the following result.

Theorem 5.2 (M. Pankov [14])**.** *For a finite subset $\mathcal{X} \subset \mathcal{G}_1(V)$ the following conditions are equivalent:*

- *every permutation on $\mathcal{X}$ is induced by a semilinear automorphism of V,*
- *$\mathcal{X}$ is a simplex or an independent subset of Π_V.*

Recall that $Q_1, \ldots, Q_m \in \mathcal{G}_1(V)$ form an independent subset of Π_V if non-zero vectors $x_1 \in Q_1, \ldots, x_m \in Q_m$ form an independent subset of V. An $(m+1)$-element subset $\mathcal{X} \subset \mathcal{G}_1(V)$ is called an *m-simplex* of Π_V if it is not independent and every m-element subset of $\mathcal{X}$ is independent [1, Section III.3].

Remark 5.1. If $x_1, \ldots, x_m \in V$ and $\langle x_1 \rangle, \ldots, \langle x_m \rangle$ form an $(m-1)$-simplex then $x_1, \ldots, x_{m-1}$ are linearly independent and $x_m = \sum_{i=1}^{m-1} a_i x_i$, where every scalar a_i is non-zero.

Proof of Theorem 5.1. Suppose that $l : V \to V'$ is a semilinear embedding. Let $\{x_i\}_{i=1}^n$ be a base of V. For every vector $x = \sum_{i=1}^n a_i x_i$ and every linear automorphism $u \in \mathrm{GL}(V)$ we have

$$l(x) = \sum_{i=1}^n \sigma(a_i) l(x_i) \quad \text{and} \quad l(u(x)) = \sum_{i=1}^n \sigma(a_i) l(u(x_i)),$$

where $\sigma : R \to R'$ is the homomorphism associated with l. Since $\{l(x_i)\}_{i=1}^n$ and $\{l(u(x_i))\}_{i=1}^n$ both are independent subsets of V', the diagram (5.1) is commutative for any linear automorphism $u' \in \mathrm{GL}(V')$ transferring every $l(x_i)$ to $l(u(x_i))$.

Conversely, suppose that for every linear automorphism $u \in \mathrm{GL}(V)$ there is a linear automorphism $u' \in \mathrm{GL}(V')$ such that the diagram (5.1) is commutative. Let $B = \{x_i\}_{i=1}^n$ be a base of V. Every permutation on the associated base of Π_V is induced by a linear automorphism of V. Then, by our assumption, every permutation on the set

$$\mathcal{X}(B) := \{\langle l(x_i) \rangle\}_{i=1}^n$$

is induced by a linear automorphism of V'. Theorem 5.2 implies that $\mathcal{X}(B)$ is an $(n-1)$-simplex or an independent subset of $\Pi_{V'}$. In the second case, $l(x_1), \ldots, l(x_n)$ are linearly independent and l is a semilinear embedding.

If $\mathcal{X}(B)$ is an $(n-1)$-simplex then $\mathcal{X}(B')$ is an $(n-1)$-simplex for every base $B' \subset V$ (indeed, if there is a base $B' \subset V$ such that $\mathcal{X}(B')$ is an independent subset of $\Pi_{V'}$ then l is a semilinear embedding and $\mathcal{X}(B')$ is an independent subset for every base $B' \subset V$). We need to show that this possibility cannot be realized.

So, suppose that $\mathcal{X}(B)$ is an $(n-1)$-simplex. By Remark 5.1, $\{l(x_i)\}_{i=1}^{n-1}$ is an independent subset of V' and

$$l(x_n) = \sum_{i=1}^{n-1} a_i l(x_i), \tag{5.2}$$

where every scalar a_i is non-zero. Let u be an automorphism of V. Then $u(B)$ is a base of V and $\mathcal{X}(u(B))$ is an $(n-1)$-simplex. The latter implies that $\{l(u(x_i))\}_{i=1}^{n-1}$ is an independent subset of V' and

$$l(u(x_n)) = \sum_{i=1}^{n-1} b_i l(u(x_i)), \tag{5.3}$$

where every scalar b_i is non-zero. If u' is a linear automorphism of V' such that the diagram (5.1) is commutative then it transfers every $l(x_i)$ to $l(u(x_i))$ and (5.2), (5.3) show that $a_i = b_i$ for every i, i.e.

$$l(u(x_n)) = \sum_{i=1}^{n-1} a_i l(u(x_i)).$$

Consider a linear automorphism $v \in \mathrm{GL}(V)$ such that $u(x_i) = v(x_i)$ if $i \le n-1$ and $u(x_n) \ne v(x_n)$. By the arguments given above,

$$l(v(x_n)) = \sum_{i=1}^{n-1} a_i l(v(x_i)) = \sum_{i=1}^{n-1} a_i l(u(x_i)) = l(u(x_n))$$

which contradicts the injectivity of l. □

6. *l*-Rigid isometric embeddings

As above, we suppose that $k \in \{2, \dots, n-2\}$ and $k' \in \{2, \dots, n'-2\}$. Let f be an embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. We say that f is *l-rigid* if for every linear automorphism $u \in \mathrm{GL}(V)$ there is a linear automorphism $u' \in \mathrm{GL}(V')$ such that the diagram

$$\begin{array}{ccc} \Gamma_k(V) & \xrightarrow{f} & \Gamma_{k'}(V') \\ \downarrow u_k & & \downarrow u'_{k'} \\ \Gamma_k(V) & \xrightarrow{f} & \Gamma_{k'}(V') \end{array}$$

is commutative, i.e. every automorphism of $\Gamma_k(V)$ induced by a linear automorphism of V can be extended to the automorphism of $\Gamma_{k'}(V')$ induced by a linear automorphism of V'.

Example 6.1. Suppose that $n = 2k$ and V is a subspace of V'. We also require that the associated division ring is isomorphic to the opposite division ring. This implies the existence of semilinear isomorphisms of V to V^*. Then the natural embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$ is not rigid, since every automorphism of $\Gamma_k(V)$ induced by a semilinear isomorphism of V to V^* cannot be extended to an automorphism of $\Gamma_k(V')$. It is clear that this embedding is l-rigid.

Let u be a linear automorphism of V and let u^* be the adjoint linear automorphism of V^*. The linear automorphism $\check{u} := (u^*)^{-1}$ is called the *contragradient* of u. It transfers the annihilator of every subspace $S \subset V$ to the annihilator of $u(S)$ [13, Section 1.3.3].

Lemma 6.1. *If f is an l-rigid embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ then the same holds for the following two embeddings:*

- *the embedding of $\Gamma_k(V)$ in $\Gamma_{n'-k'}(V'^*)$ transferring every $X \in \mathcal{G}_k(V)$ to the annihilator of $f(X)$,*

- *the embedding of* $\Gamma_{n-k}(V^*)$ *in* $\Gamma_{k'}(V')$ *transferring every* $X \in \mathcal{G}_{n-k}(V^*)$ *to* $f(X^0)$.

Proof. Let u be a linear automorphism of V. Then u_k is an automorphism of $\Gamma_k(V)$. Suppose that it is extendable to the automorphism of $\Gamma_{k'}(V')$ induced by a linear automorphism $v \in \mathrm{GL}(V')$. Then $\check{v}$ defines the extension of u_k to an automorphism of $\Gamma_{n'-k'}(V'^*)$.

Consider the second embedding. The contragradient $\check{u}$ is a linear automorphism of V^* and $\check{u}_{n-k}$ is an automorphism of $\Gamma_{n-k}(V^*)$. It is clear that $\check{u}_{n-k}$ is extendable to the automorphism of $\Gamma_{k'}(V')$ induced by v. This completes our proof, since every linear automorphism of V^* is the contragradient of a linear automorphism of V. $\square$

We give two examples of l-rigid isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. In contrast with the previous section, we do not require that $k \le n-k$.

Example 6.2. Suppose that

$$k \le k' \quad \text{and} \quad n-k \le n'-k'.$$

If $S \in \mathcal{G}_{k'-k}(V')$ then

$$\dim V'/S = n' - k' + k \ge n.$$

For every semilinear embedding $l : V \to V'/S$ the mapping l_k is an l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V'/S)$. As in the previous section, we denote by π the mapping which transfers every subspace of V'/S to the corresponding subspace of V'. Then πl_k is an l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ of type (A).

Example 6.3. Suppose that

$$n \le k + k' \le n'$$

and $U \in \mathcal{G}_{k+k'}(V')$. Let $v : V \to U^*$ be a semilinear embedding. Then v_k is an l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U^*)$. By duality, it can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(U)$. Lemma 6.1 guarantees that the latter embedding is l-rigid. Since U is contained in V', we get an l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ of type (B).

It follows from Theorem 4.1 and the examples given above that *every isometric embedding of* $\Gamma_k(V)$ *in* $\Gamma_{k'}(V')$ *is* l*-rigid if* $n = 2k$. Now, we describe all l-rigid isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ in the case when $k \ne n-k$.

Theorem 6.1. *If* f *is an* l*-rigid isometric embedding of* $\Gamma_k(V)$ *in* $\Gamma_{k'}(V')$ *then one of the following possibilities is realized:*

- $k \le k'$, $n-k \le n'-k'$ *and there is* $S \in \mathcal{G}_{k'-k}(V')$ *such that* f *is induced by a semilinear embedding of* V *in* V'/S*, see Example 6.2;*
- $n \le k+k' \le n'$ *and there is* $U \in \mathcal{G}_{k'+k}(V')$ *such that* f *is induced by a semilinear embedding of* V *in* U^**, see Example 6.3.*

The following example shows that there are isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ which are not l-rigid.

Example 6.4. Suppose that $R = F$ and $R' = F'$, where F and F' are the fields from Remark 4.1, and $n > n'$. Let $l : V \to V'$ be a semilinear n'-embedding. If $n' \ge 2k$ then l_k is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$. By Theorem 6.1, this embedding is not l-rigid.

Proof of Theorem 6.1. Let f be an l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

Case $k < n-k$.

If f is an embedding of type (A) then, by Theorem 4.1, there exist $S \in \mathcal{G}_{k'-k}(V')$ and a semilinear $(2k)$-embedding $l : V \to V'/S$ such that f transfers every $X \in \mathcal{G}_k(V)$ to the subspace of V' corresponding to $\langle l(X) \rangle$. Since f is l-rigid, l_k is l-rigid and Theorem 5.1 implies that l is a semilinear embedding.

In the case when f is an embedding of type (B), Theorem 4.1 implies the existence of $U \in \mathcal{G}_{k'+k}(V')$ and a semilinear $(2k)$-embedding $v : V \to U^*$ such that f transfers every $X \in \mathcal{G}_k(V)$ to the annihilator of $\langle v(X) \rangle$ in U. Since f is an l-rigid embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(U)$, Lemma 6.1 implies that v_k is l-rigid. By Theorem 5.1, v is a semilinear embedding.

Case $k > n - k$.

Let f be an embedding of type (A). Consider the embedding of $\Gamma_{n-k}(V^*)$ in $\Gamma_{k'}(V')$ transferring every $X \in \mathcal{G}_{n-k}(V^*)$ to $f(X^0)$. By Lemma 6.1, this embedding is l-rigid; moreover, it is an embedding of type (B). Therefore, there exist $U \in \mathcal{G}_{k'+n-k}(V')$ and a semilinear embedding $v : V^* \to U^*$ such that f transfers every $X \in \mathcal{G}_k(V)$ to the annihilator of $\langle v(X^0) \rangle$ in U. Since v is a semilinear embedding, the image of v spans an n-dimensional subspace of U^*. Denote by S the annihilator of this subspace in U. Then $S \in \mathcal{G}_{k'-k}(V')$ and the image of f is contained in $[S\rangle_k$.

Consider the mapping g which sends every subspace $X \subset V$ to the annihilator of $\langle v(X^0) \rangle$ in U. This is an injection of the set of all subspaces of V to the set of all subspaces of U containing S. The image of every $\mathcal{G}_i(V)$ is contained in $[S\rangle_{k'-k+i}$ and the restriction of g to $\mathcal{G}_k(V)$ coincides with f. The mapping g is inclusions preserving: for any subspaces $X, Y \subset V$

$$X \subset Y \implies g(X) \subset g(Y).$$

This implies that the restriction of g to $\mathcal{G}_1(V)$ transfers every line of Π_V to a subset in a line of the projective space $[S\rangle_{k'-k+1}$. By Theorem 2.1, this restriction is induced by a semilinear injection $l : V \to V'/S$. We need to show that for every $X \in \mathcal{G}_k(V)$

$$g(X) = \pi(\langle l(X) \rangle),$$

where π transfers every subspace of V'/S to the corresponding subspace of V'.

Since v is a semilinear embedding, g sends every base of Π_V to an independent subset of the projective space $[S\rangle_{k'-k+1}$. This implies that l is a semilinear embedding. Let $X \in \mathcal{G}_k(V)$. We take $P_1, \dots, P_k \in \mathcal{G}_1(V)$ such that

$$X = P_1 + \cdots + P_k.$$

Then

$$\pi(\langle l(X) \rangle) = g(P_1) + \cdots + g(P_k) \subset g(X)$$

and we get the required equality (since our subspaces both are k'-dimensional).

Now, let f be an embedding of type (B). We consider f as an embedding of $\Gamma_k(V)$ in $\Gamma_{n'-k'}(V'^*)$. This is an embedding of type (A) and, as in the proof of Theorem 4.1, we establish that the image of f is contained in

$$\langle U]_{k'}, \quad U \in \mathcal{G}_{k'+k}(V').$$

So, f is an l-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(U)$. By duality, it can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U^*)$ of type (A). Lemma 6.1 implies that the latter embedding is l-rigid. Thus it is induced by a semilinear embedding of V in U^*. □

References

[1] R. Baer, Linear Algebra and Projective Geometry, Academic Press, New York, 1952.
[2] A. Blunck, H. Havlicek, On bijections that preserve complementarity of subspaces, Discrete Math. 301 (2005) 46–56.
[3] W.L. Chow, On the geometry of algebraic homogeneous spaces, Ann. of Math. 50 (1949) 32–67.
[4] C.A. Faure, A. Frölicher, Morphisms of projective geometries and semilinear maps, Geom. Dedicata 53 (1994) 237–262.
[5] C.A. Faure, An elementary proof of the fundamental theorem of projective geometry, Geom. Dedicata 90 (2002) 145–151.
[6] H. Havlicek, A generalization of Brauner's theorem on linear mappings, Mitt. Math. Sem. Univ. Giessen 215 (1994) 27–41.
[7] W.-l. Huang, Adjacency preserving transformations of Grassmann spaces, Abh. Math. Sem. Univ. Hamburg 68 (1998) 65–77.
[8] W.-l. Huang, H. Havlicek, Diameter preserving surjections in the geometry of matrices, Linear Algebra Appl. 429 (2008) 376–386.

[9] J. Kosiorek, A. Matras, M. Pankov, Distance preserving mappings of Grassmann graphs, Beiträge Algebra Geom. 49 (2008) 233–242.
[10] A. Kreuzer, Projective embeddings of projective spaces, Bull. Belg. Math. Soc. Simon Stevin 5 (1998) 363–372.
[11] M.H. Lim, Surjections on Grassmannians preserving pairs of elements with bounded distance, Linear Algebra Appl. 432 (2010) 1703–1707.
[12] M. Pankov, Chows theorem and projective polarities, Geom. Dedicata 107 (2004) 17–24.
[13] M. Pankov, Grassmannians of classical buildings, in: Algebra and Discrete Math. Series, vol. 2, World Scientific, 2010
[14] M. Pankov, Isometric embeddings of Johnson graphs in Grassmann graphs, J. Algebraic Combin. 33 (2011) 555–570.