Prym varieties associated to graphs

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Abstract

We present a Prym construction which associates abelian varieties to vertex-transitive strongly regular graphs. As an application we construct Prym–Tyurin varieties of arbitrary exponent \( \geq 3 \), generalizing a result by Lange, Recillas and Rojas.

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1. Introduction

We describe a Prym construction which associates abelian varieties to certain graphs. More precisely, given the adjacency matrix \( A = (a_{ij})_{i,j=1}^d \) of a vertex-transitive strongly regular graph \( G \) along with a covering of curves \( p : C \to \mathbb{P}^1 \) of degree \( d \) and a labeling \( \{x_1, \ldots, x_d\} \) of an unramified fiber such that the induced monodromy group of \( p \) is represented as a subgroup of the automorphism group of \( G \), we construct a symmetric divisor correspondence \( D \) on \( C \times C \) which then serves to define complementary subvarieties \( P_+ \) and \( P_- \) of the Jacobian \( J(C) \). The correspondence \( D \) is defined in such a way that the point \((x_i, x_j)\) appears in \( D \) with multiplicity \( a_{ij} \), analogous to Kanev’s construction in [7]. The varieties \( P_\pm \) are given by \( P_\pm = \ker(\gamma - r_\pm \text{id}_{J(C)})_0 \), where \( r_\pm \) are special eigenvalues of \( A \) and \( \gamma \) is the endomorphism on \( J(C) \) canonically associated to \( D \) (i.e., sending the divisor class \([x - x_0]\) to the class...
\[ D(x) - D(x_0) \]. It is easy to show that
\[
(\gamma - r_+ \text{id}_{J(C)})(\gamma - r_- \text{id}_{J(C)}) = 0
\]
and \( P_\pm = \text{im}(\gamma - r_\pm \text{id}_{J(C)}) \). In particular, if \( D \) is fixed point free and \( r_+ = 1 \), then \( P_+ \) is a Prym–Tyurin variety of exponent \( 1 - r_- \) for \( C \). Given the ramification of \( p \) it is not hard to compute the dimension of \( P_\pm \).

For a thorough definition of \( D \) we consider the Galois closure \( \pi: X \to \mathbb{P}^1 \) of \( p \) and use the induced representation \( \text{Gal}(\pi) \to \text{Aut}(\mathcal{G}) \) to construct symmetric correspondences \( D_\pm \) on \( X \times X \) (much the way Mérindol does in [11], or Donagi in [5]). With \( C \) being a quotient curve of \( X \), the correspondence \( D \) is derived from \( D_\pm \) taking quotients and adding \( r_\pm \Delta_C \), where \( \Delta_C \) is the diagonal of \( C \times C \); see Section 4. Given the endomorphisms \( \gamma_\pm \) on \( J(X) \) canonically associated to \( D_\pm \), we show that \( \text{im} \gamma_\pm \) and \( P_\pm \) are isogenous.

The lattice graphs \( L_2(n), n \geq 3 \), and their complements \( \overline{L}_2(n) \) offer important examples. For instance, applying the method to \( L_2(n) \) and appropriate coverings \( C \to \mathbb{P}^1 \) of degree \( n^2 \) with branch loci of cardinality \( 2(l + 2n - 2) \) for \( l \geq 1 \), we obtain \( l \)-dimensional Prym–Tyurin varieties of exponent \( n \) for the curves \( C \); see Section 7. We give a characterization of these varieties and show that for \( n = 3 \) they coincide with the non-trivial Prym–Tyurin varieties of exponent 3 described by Lange, Recillas and Rojas in [9].

**Conventions and notations.** The ground field is assumed to be the field \( \mathbb{C} \) of complex numbers. By a covering of curves we mean a non-constant morphism of irreducible smooth projective curves. The symbol \( S_n \) denotes the symmetric group acting on \( n \) letters, with \( n \in \mathbb{N} \).

### 2. Strongly regular graphs and matrices

We start our discussion with the definition of a strongly regular graph and its adjacency matrix and collect some properties of such graphs. For additional information we refer to [13].

By definition the set of **strongly regular** graphs \( \text{SRG}(d,k,\lambda,\mu) \), \( k > 0 \), consists of the graphs \( G \) with vertex set \( \{v_1, \ldots, v_d\} \) such that

(a) the set \( \Gamma(v_i) \) of vertices adjacent to \( v_i \) has exactly \( k \) elements and \( v_i \notin \Gamma(v_i) \);

(b) for any two adjacent vertices \( v_i, v_j \) there are exactly \( \lambda \) vertices adjacent to both \( v_i \) and \( v_j \);

(c) for any two distinct non-adjacent vertices \( v_i, v_j \) there are exactly \( \mu \) vertices adjacent to both \( v_i \) and \( v_j \).

Let \( A = (a_{ij}) \in \{0,1\}^{d \times d} \) be the adjacency matrix of such a strongly regular graph \( G \), i.e., \( a_{ij} = 1 \) if and only if \( v_i \) is adjacent to \( v_j \). Then \( A \) is symmetric and \( (1, \ldots, 1) \in \mathbb{R}^d \) is an eigenvector of \( A \) with eigenvalue \( k \). The set of eigenvalues of \( A \) is \( \{k, r_+, r_-\} \) with \( r_- < 0 \leq r_+ \leq k \) and

\[
r_\pm = \frac{1}{2} \left[ \lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right],
\]

implying that

\[
(A - r_+ I_d)(A - r_- I_d) = \frac{(k - r_+)(k - r_-)}{d} J_d,
\]

where \( J_d \) is the \( d \times d \) identity matrix.
where \( J_d \) is the \( d \times d \) matrix whose entries are equal to 1. Given parameters \((d, k, \lambda, \mu)\) such that \((d, k, \lambda, \mu) \neq (4\mu + 1, 2\mu, \mu - 1, \mu)\), one can show that \( r_\pm \in \mathbb{Z} \) (cf. [13, Theorem 21.1]). In fact, if \((d, k, \lambda, \mu) = (4\mu + 1, 2\mu, \mu - 1, \mu)\), then non-integral values of \( r_\pm \) can occur; for instance, the Paley graph \( P(5) \) (see [13, Example 21.3]) has parameters \((5, 2, 0, 1)\) and eigenvalues \( r_\pm = -\frac{1}{2} \pm \frac{1}{2} \sqrt{5} \). However, the Paley graph \( P(4\mu + 1) \in \text{SRG}(4\mu + 1, 2\mu, \mu - 1, \mu)\), where \( 4\mu + 1 = p^n \) with \( p \) an odd prime and \( n \in \mathbb{N} \), has integer eigenvalues \( r_\pm = \frac{1}{2}(1 \pm p^n) \).

Many strongly regular graphs stem from geometry. The most classical example is offered by the configuration of 27 lines on a cubic surface.

**Example 2.1.** Given a non-singular cubic surface \( X \subset \mathbb{P}^3 \), let \( \mathcal{L} \) be the intersection graph of the 27 lines on \( X \). The configuration of the 27 lines is completely described by the 36 Schl"{a}fli double-sixes, i.e., pairs \( M := (\{a_1, \ldots, a_6\}, \{b_1, \ldots, b_6\}) \) of sets of 6 skew lines on \( X \) such that each line from one set is skew to a unique line from the other set. Fix a double-six \( M \); in matrix notation we may write

\[
M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}
\]

such that two lines meet if and only if they are in different rows and columns. The remaining 15 lines on \( X \) are the \( c_{ij} := a_i b_j \cap a_j b_i \) with \( i \neq j \), where \( a_i b_j \) is the plane spanned by the lines \( a_i \) and \( b_j \). In this notation the 36 double-sixes are \( M \), the 15 \( M_{i,j} \)’s and the 20 \( M_{i,j,k} \)’s, where the double-sixes \( M_{i,j} \) and \( M_{i,j,k} \) are respectively given by

\[
\begin{pmatrix} a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} \end{pmatrix}, \quad \begin{pmatrix} a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\ c_{jk} & c_{ik} & c_{ij} & b_i & b_m & b_n \end{pmatrix}.
\]

It follows that the Schl"{a}fli graph \( \mathcal{L} \) is in \( \text{SRG}(27, 10, 1, 5) \) (its only element). Clearly, the stabilizer of a double-six is a subgroup of \( \text{Aut}(\mathcal{L}) \) of index 36, isomorphic to \( S_6 \times \mathbb{Z}/2\mathbb{Z} \). Thus \( \#(\text{Aut}(\mathcal{L})) = 6! \cdot 2 \cdot 36 \). In fact, \( \text{Aut}(\mathcal{L}) \) is isomorphic to the Weyl group \( W(E_6) \): Consider the Dynkin diagram for \( E_6 \).

\[
\begin{array}{cccccc}
& x_1 & x_2 & x_3 & x_4 & x_5 \\
\circ & & & & & \circ \\
o & & & & & \\
\circ & y & & & & \\
\end{array}
\]

By definition \( W(E_6) \) is generated by the reflections \( s_1, \ldots, s_5, s \) associated to the simple roots \( x_1, \ldots, x_5, y \). One shows that there is a surjective homomorphism \( W(E_6) \to \text{Aut}(\mathcal{L}) \) sending \( s_i \) (respectively \( s \)) to the transformation that interchanges the rows of the double-six \( M_{i,i+1} \) (respectively \( M_{1,2,3} \)) (cf. [10, Sections 25, 26]). Then recall that \( W(E_6) \) is of order \( 6! \cdot 2 \cdot 36 \). Under the apparent isomorphism the 27 lines on \( X \) correspond to the 27 weights of the minimal representation of \( E_6 \), which is one of the two minuscule representations that \( E_6 \) has, cf. [7,8]. Hence \( W(E_6) \) acts transitively on the weights, i.e., \( \text{Aut}(\mathcal{L}) \) is a transitive subgroup of \( S_{27} \).

**Additional properties** (of strongly regular graphs):

1. If \( G \) is disconnected, then \( G \) is the disjoint union of \( m > 1 \) copies of the complete graph \( K_{k+1} \) with adjacency matrix \( J_{k+1} - I_{k+1} \), where \( I_{k+1} \) is the identity matrix. So \((d, k, \lambda, \mu) =

(m(k + 1), k, k − 1, 0) and A has exactly two distinct eigenvalues: k (= r+) with multiplicity m and r− = −1 with multiplicity d − m;
(2) if G is connected and G ≠ Kk+1, then k ≠ r± and k is a simple eigenvalue;
(3) G and its complement G ∈ SRG(d, d − k − 1, d − 2k + μ − 2, d − 2k + λ) are connected if and only if 0 < μ < k < d − 1, in which case G is said to be non-trivial.

Let Stab(A) be the stabilizer of A under the natural operation of $S_d$ on $d \times d$ integer matrices by $(a_{ij}) \mapsto (a_{σ(i)σ(j)})$ and observe that Stab(A) coincides with Aut(G). For disconnected G it is easily seen that Stab(A) is a transitive subgroup of $S_d$, implying that disconnected strongly regular graphs are vertex-transitive. In practice it turns out that vertex-transitivity is quite rare among non-trivial strongly regular graphs, although most of the sets $SRG(d, k, λ, μ)$ of non-trivial strongly regular graphs have a vertex-transitive element.

**Definition 2.2.** Let $A ∈ \mathbb{Z}^{d×d}$ be a symmetric matrix with transitive stabilizer group Stab(A). Then A is a Prym matrix if there exist integers $k, r+, r−$ with $r+ > r−$ such that Eq. (2.2) holds.

**Remarks.** Suppose that $A ∈ \mathbb{Z}^{d×d}$ is a symmetric matrix for which there exist integers $k, r+, r−$ with $r+ > r−$ such that Eq. (2.2) holds. Decomposing $\mathbb{R}^d$ into eigenspaces of A we may assume that A takes diagonal form. Then $J_d$ simultaneously transforms into the diagonal matrix $\text{diag}(d, 0, \ldots, 0)$, implying that $(1, \ldots, 1) ∈ \mathbb{R}^d$ is an eigenvector of A with, say, eigenvalue $η$. Hence A has eigenvalues $η, r+, r−$ and by (2.2) we have $(η − r+) (η − r−) = (k − r+) (k − r−)$, that is, $η = k$ or $η = r+ + r− − k$. Further, if $η ≠ r±$, then $η$ is simple. A $d × d$ Prym matrix A therefore has integer eigenvalues $k, r+, r−$ with $r+ > r−$ such that (2.2) holds and k belongs to the eigenvector $(1, \ldots, 1)$ of A. Clearly, if A is such a matrix, then for any $m ∈ \mathbb{N}$ the matrix $A^{⊗ m} := \bigoplus_{i=1}^{m} A$ has the same eigenvalues $k, r+, r−$ and satisfies the equation

$$
(A^{⊗ m} − r+ I_{md})(A^{⊗ m} − r− I_{md}) = \frac{(k − r+)(k − r−)}{d} J_d^{⊗ m}.
$$

Moreover, it is immediately seen that $A^{⊗ m}$ has transitive stabilizer group Stab$(A^{⊗ m}) ⊂ S_{md}$. Throughout the paper the eigenvalues of a Prym matrix A will be denoted by $k, r+, r−$, where k belongs to the eigenvector $(1, \ldots, 1)$ of A and $r+ > r−$.

Many of the known constructions for Prym varieties rely on the definition of a reduced symmetric divisor correspondence for a curve. These correspondences can often be related to Prym matrices whose entries are in the set $\{0, 1\}$. Hence it is useful to have a characterization for such matrices.

**Proposition 2.3.** Assume that $A = (a_{ij}) ∈ \{0, 1\}^{d×d}$ is a Prym matrix with $a_{11} = 0$. Then A is the adjacency matrix of a graph $G ∈ SRG(d, k, λ, μ)$ with $λ = k + r+ r− + r+ + r−$ and $μ = k + r+ r−$.

**Proof.** By transitivity we have $a_{ii} = 0$ for all $i = 1, \ldots, d$, hence A is the adjacency matrix of a regular graph G of degree k with d vertices, i.e., each vertex is of degree k. If indeed we have $G ∈ SRG(d, k, λ, μ)$ for some integers λ and μ, then solving Eq. (2.1) for λ, μ yields $λ = k + r+ r− + r+ + r−$ and $μ = k + r+ r−$. To show that G is strongly regular we may assume by transitivity that G is the disjoint union of finitely many copies of a connected graph $G_c$ with adjacency matrix $A_c$. If $k ≠ r±$, then $k$ is simple, so $A = A_c$ and it follows that G is strongly.
regular (cf. [13, p. 265]). Hence assume that $A$ has just two distinct eigenvalues, i.e., $k = r_+$. The complementary graph $\overline{G}$ has adjacency matrix $J_d - I_d - A$ which is Prym and has eigenvalues $k' = d - k - 1$, $r'_+ = -r_- - 1$ and $r'_- = -k - 1$. Thus if $r_- \neq k - d$, then $k' \neq r'_\pm$, that is, $\overline{G}$ is a non-complete connected strongly regular graph and so $G_c = K_{k+1}$. Finally, suppose that $r_- = k - d$. By Eq. (2.2) we have $A^2 = (2k - d)A - k(k - d)I_d$, hence if $v_h, v_i, v_j$ are three distinct vertices of $G$ such that $v_h$ is adjacent to $v_i$ and $v_j$ is adjacent to $v_j$, then the relation $1 = a_{hi} \leq \sum_{l=1}^d a_{hl}a_{lj} = (2k - d)a_{hj}$ implies that $v_h$ is adjacent to $v_j$. Consequently $\overline{G} = K_{k+1}$. □

Example 2.4. Among the set of strongly regular graphs there are some distinguished families of non-trivial vertex transitive graphs. One such family is that of lattice graphs; for $n \geq 3$, the lattice graph $L_2(n)$ is the graph with vertex set $\{1, \ldots, n\}^2$ such that two distinct vertices $(i, j)$ and $(l, m)$ are adjacent if and only if $i = l$ or $j = m$. Clearly, $S_n \times S_n$ is a transitive subgroup of $\text{Aut}(L_2(n))$, hence the adjacency matrix of $L_2(n)$ and that of its complement $\overline{L_2(n)}$ are Prym.

Another subgroup of $\text{Aut}(L_2(n))$ is $S_2$; it permutes the coordinates of the vertices of $L_2(n)$. In fact, it is well known that the automorphism group of $L_2(n)$ is $\text{SRG}(n^2, 2(n - 1), n - 2, 2)$ is equal to the semi-direct product $S_2 \ltimes (S_n \times S_n)$. With an eye on future examples (e.g., Examples 5.2 and 5.3) we observe that those $\varphi \in \text{Aut}(L_2(n))$ for which no vertex $(i, j)$ of $L_2(n)$ is adjacent to $\varphi(i, j)$ are exactly the $\varphi \in \langle (1) \times S_n \rangle \cup (S_n \times (1))$. Similarly, for $\varphi \in \text{Aut}(L_2(n))$ of order 2, no vertex $(i, j)$ of $L_2(n)$ is adjacent to $\varphi(i, j)$ if and only if

(i) for $n$ odd: $\varphi = (\sigma, \sigma^{-1}) \circ t$ with $\sigma \in S_n$ and $t \in S_2$ is the transposition of coordinates;

(ii) for $n$ even: $\varphi$ is as in (i) or $\varphi = (\sigma, \tau)$, where $\sigma, \tau \in S_n$ decompose into $\frac{1}{2} n$ mutually disjoint transpositions.

Note moreover that $L_2(3)$ and $\overline{L_2(3)}$ are isomorphic; if we identify the set $\{1, 2, 3\}$ with $\mathbb{Z}/3\mathbb{Z}$, then the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ defines an isomorphism of graphs $L_2(3) \xrightarrow{\sim} L_2(3)$.

Example 2.5. Another non-trivial family is that of Latin square graphs; for $n \geq 3$, the Latin square graph $L_3(n)$ is $\text{SRG}(n^2, 3(n - 1), n, 6)$ is the graph with vertex set $(\mathbb{Z}/n\mathbb{Z})^2$ such that two distinct vertices $(i, j)$ and $(l, m)$ are adjacent if and only if $i = l$ or $j = m$ or $i + j = l + m$. We identify three subgroups of $\text{Aut}(L_3(n))$; to begin with, the diagonal action of $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$ on $(\mathbb{Z}/n\mathbb{Z})^2$ induces an embedding $\text{Aut}(\mathbb{Z}/n\mathbb{Z}) \hookrightarrow \text{Aut}(L_3(n))$. Similarly, $(\mathbb{Z}/n\mathbb{Z})^2$ induces a subgroup of $\text{Aut}(L_3(n))$ by translation, thus implying that $L_3(n)$ is vertex-transitive. Finally, the subgroup $S := \langle s, t \rangle \subset \text{Aut}((\mathbb{Z}/n\mathbb{Z})^2)$ with $s$ and $t$ respectively given by the matrices $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, is immediately seen to be a subgroup of $\text{Aut}(L_3(n))$. Note that sending $s \mapsto (1 2)$ and $t \mapsto (1 3)$ induces an isomorphism $S \xrightarrow{\sim} S_3$. Clearly, the actions of $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$ and $S$ commute. It can be shown that $\text{Aut}(L_3(n))$ coincides with the semi-direct product $(\mathbb{Z}/n\mathbb{Z})^2 \rtimes (S \times \text{Aut}(\mathbb{Z}/n\mathbb{Z}))$.

3. Prym varieties of a Galois covering

Given a Prym matrix $A$, we describe a method that associates certain abelian varieties to finite Galois coverings of $\mathbb{P}^1$ whose Galois group is represented as a transitive subgroup of $\text{Stab}(A^\oplus m)$, $m \geq 1$. We assume from now on that $A$ is a $d \times d$ Prym matrix with eigenvalues $k, r_+, r_-$. 

Definition 3.1. Consider $\mathcal{P} = (A^\oplus m, r, \pi, \phi)$ for $m \geq 1$, where $r \in \{r_+, r_-\}$, $\pi$ is a finite Galois covering of $\mathbb{P}^1$ and $\phi: \text{Gal}(\pi) \to S_{md}$ is a transitive representation. Then $\mathcal{P}$ is said to represent Prym data if $\phi(\text{Gal}(\pi))$ is a subgroup of $\text{Stab}(A^\oplus m)$. 

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For instance, assume that we have a finite subset $B = \{b_1, \ldots, b_n\}$ of $\mathbb{P}^1$ along with non-trivial permutations $\sigma_1, \ldots, \sigma_n \in \text{Stab}(\mathbb{P}^1)$ such that $\sigma_1 \cdot \ldots \cdot \sigma_n = (1)$ and $G := \langle \sigma_1, \ldots, \sigma_n \rangle$ is a transitive subgroup of $S_{md}$. Let $\Sigma_i$ be the $G$-conjugacy class of $\sigma_i$. According to Riemann’s Existence Theorem (RET), the number of equivalence classes of Galois coverings of $\mathbb{P}^1$ of ramification type $\mathcal{R} := [G, B, \{\Sigma_i\}_{i=1}^n]$ is equal to the number of sets $\{(g \tau_1 g^{-1}, \ldots, g \tau_n g^{-1}) | g \in G\}$ with $\tau_i \in \Sigma_i$ such that $\tau_1 \cdot \ldots \cdot \tau_n$ is trivial (cf. [14, p. 37]). Hence, let $\pi$ be a Galois covering of type $\mathcal{R}$. Clearly, if $\phi : \text{Gal}(\pi) \rightarrow G$ is a group isomorphism, then $(A^{\oplus m}, r, \pi, \phi)$ represents Prym data. In this way, varying the continuous parameters $b_1, \ldots, b_n$, we obtain finitely many smooth $n$-dimensional families of Galois coverings with associated Prym data.

**The construction.** Let $\mathcal{P} = (A^{\oplus m}, r, \pi, \phi)$ be Prym data for a given Galois covering $\pi : X \rightarrow \mathbb{P}^1$. We are going to define two symmetric divisor correspondences, one for $X$ and one for the quotient curve $C := X/H$, where $H \subset G := \text{Gal}(\pi)$ is the stabilizer of the letter 1 with respect to $\phi$. Let $(\cdot, \cdot) : \mathbb{R}^{md} \times \mathbb{R}^{md} \rightarrow \mathbb{R}$ be the symmetric bilinear form canonically associated to the matrix $A^{\oplus m} - rI_{md}$. Given the standard basis $e_1, \ldots, e_{md}$ of $\mathbb{R}^{md}$, consider the permutation representation of $G$ on $\mathbb{R}^{md}$ induced by $g : e_i \mapsto e_{g(i)}$. For $g \in G$, denote $\hat{g} = HgH$. Then $(\cdot, \cdot)_r$ is immediately seen to be $G$-invariant and $(g_1 e_1, e_1)_r = (g_2 e_1, e_1)_r$ for all $g_1, g_2 \in G$ such that $\hat{g}_1 = \hat{g}_2$. Let $\alpha : X \rightarrow C$ and $p : C \rightarrow \mathbb{P}^1$ be the canonical mappings. For each $g \in G$ take the graph $\Gamma_g = (id_X, g)(X)$ of $g$ and let $\hat{\Gamma}_g = (\alpha \times \alpha)(\Gamma_g)$ be its reduced image in $C \times C$. Assume that $B$ is the branch locus of $\pi$ and put $C_0 = p^{-1}(\mathbb{P}^1 \setminus B)$. Observe that

$$\hat{\Gamma}_{g_1} \cap \hat{\Gamma}_{g_2} \cap (C_0 \times C_0) \neq \emptyset \iff \hat{g}_1 = \hat{g}_2 \iff \hat{\Gamma}_{g_1} = \hat{\Gamma}_{g_2},$$

for all $g_1, g_2 \in G$. Hence we have divisor correspondences $D_{\mathcal{P}}$ on $X \times X$ and $\hat{D}_{\mathcal{P}}$ on $C \times C$, given by

$$D_{\mathcal{P}} = \sum_{g \in G} (ge_1, e_1)_r \Gamma_g, \quad \hat{D}_{\mathcal{P}} = \sum_{\hat{g} \in \hat{G}} (ge_1, e_1)_r \hat{\Gamma}_g$$

with $\hat{G} = H \setminus G/H$. Note that $D_{\mathcal{P}}$ and $\hat{D}_{\mathcal{P}}$ are symmetric as $(ge_1, e_1)_r = (g^{-1} e_1, e_1)_r$ for all $g \in G$.

**Definition 3.2.** Let $\gamma_{\mathcal{P}}$ on $J(X)$ (respectively $\hat{\gamma}_{\mathcal{P}}$ on $J(C)$) be the endomorphism canonically associated to $D_{\mathcal{P}}$ (respectively $\hat{D}_{\mathcal{P}}$). Then we call $Z_{\mathcal{P}} = \text{im} \gamma_{\mathcal{P}}$ (respectively $\hat{Z}_{\mathcal{P}} = \text{im} \hat{\gamma}_{\mathcal{P}}$) the Prym variety of $X$ (respectively $C$) associated to $\mathcal{P}$.

**Remark.** Prym data can be seen as an ‘ornamented’ covering (i.e., a covering with additional data), where the ornamentation is such that we can define divisorial correspondences and Prym varieties.

For $q \in \mathbb{P}^1 \setminus B$ an identification $\pi^{-1}(q) \leftrightarrow G$ is called a Galois labeling of the fiber $\pi^{-1}(q)$ if the action of $G$ on the fiber is compatible with its action on itself via multiplication on the left. Moreover, a Galois labeling of $\pi^{-1}(q)$ induces a Galois labeling $p^{-1}(q) \leftrightarrow H \setminus G$ of the fiber of $p : C \rightarrow \mathbb{P}^1$ over $q$. Denoting $\bar{g} = Hg$ for $g \in G$, we have:
Lemma 3.3. Given \( q \in \mathbb{P}^1 \setminus B \), take a Galois labeling \( p^{-1}(q) \leftrightarrow H \setminus G \). Then, for any \( \sigma \in G \), the restriction of \( \hat{D}_P \) to \( \{\bar{\sigma}\} \times C \) is given by the identity

\[
\hat{D}_P(\bar{\sigma}) = \sum_{g \in H \setminus G} (ge_1, e_1)_r g\bar{\sigma}.
\]

**Proof.** Because \((\bar{\sigma}, g\bar{\sigma}) \in \hat{\Gamma}_f \Leftrightarrow \hat{f} = \hat{g}\), for all \( f, g \in G \), the point \((\bar{\sigma}, g\bar{\sigma})\) appears in \( \hat{D}_P \) with multiplicity \((ge_1, e_1)_r\).

Since \( H \) is the stabilizer of a point, the transitivity of the representation \( \phi : G \to S_{md} \) implies that there exists a bijection \( H \setminus G \to \{1, \ldots, md\} \). Hence the Galois labeling \( p^{-1}(q) \leftrightarrow H \setminus G \) induces a labeling \( \{y_1, \ldots, y_{md}\} \) of \( p^{-1}(q) \) such that \( \bar{g} \) corresponds to \( y_{g^{-1}(1)} \). The formula in the preceding lemma now turns into

\[
\hat{D}_P(y_i) = \sum_{j=1}^{md} (e_i, e_j)_r y_j = -r y_i + \sum_{j=1}^{md} (e_j A^{\oplus m} e_i)_r y_j,
\]

for all \( i = 1, \ldots, md \).

**Proposition 3.4.** Let \( \mathcal{P} = (A^{\oplus m}, r, \pi, \phi) \) be Prym data with \( \pi : X \to \mathbb{P}^1 \). Then the Prym varieties \( Z_{\mathcal{P}} \) of \( X \) and \( \hat{Z}_{\mathcal{P}} \) of \( C := X/H \) are isogenous. Using the notation \( \mathcal{P}' = (J_{\mathcal{P}}^{\oplus m}, 0, \pi, \phi) \) and \( s = r_+ + r_- \), we have the following quadratic equations for the endomorphisms \( \gamma_{\mathcal{P}} \) on \( J(C) \) and \( \gamma_{\mathcal{P}} \) on \( J(X) \):

\[
\gamma_{\mathcal{P}}(\gamma_{\mathcal{P}} + (2r - s) \text{id}_{J(C)}) = \frac{(k - r_+)(k - r_-)}{d} \gamma_{\mathcal{P}}'
\]

and

\[
\gamma_{\mathcal{P}}(\gamma_{\mathcal{P}} + (H)(2r - s) \text{id}_{J(X)}) = \frac{(k - r_+)(k - r_-)}{d} \gamma_{\mathcal{P}}'.
\]

**Proof.** Write \( \alpha^* : J(C) \to J(X) \) for the map induced by \( \alpha \) and write \( N_{\alpha} : J(X) \to J(C) \) for the norm map. Abusing notation, we define homomorphisms \( \alpha^* : \mathbb{Z}[H \setminus G] \to \mathbb{Z}[G] \) and \( N_{\alpha} : \mathbb{Z}[G] \to \mathbb{Z}[H \setminus G] \), respectively induced by \( g \mapsto \sum_{h \in H} hg \) and \( g \mapsto \bar{g} \). As a direct consequence of Lemma 3.3 we have \( N_{\alpha} D_{\mathcal{P}} = \#(H) \hat{D}_{\mathcal{P}} N_{\alpha} \). Hence \( N_{\alpha} \gamma_{\mathcal{P}} = \#(H) \hat{\gamma}_{\mathcal{P}} N_{\alpha} \) and \( \gamma_{\mathcal{P}} = \alpha^* \hat{\gamma}_{\mathcal{P}} N_{\alpha} \). The first identity implies that the restricted mapping \( N_{\alpha} : Z_{\mathcal{P}} \to \hat{Z}_{\mathcal{P}} \) is surjective, while the two identities combined imply that \( \dim \text{im} Z_{\mathcal{P}} = \dim \text{im} \hat{Z}_{\mathcal{P}} \). Therefore the restriction of \( N_{\alpha} \) to \( Z_{\mathcal{P}} \) is an isogeny.

Applying Eq. (3.1) to \( \mathcal{P}' \) we obtain \( \hat{D}_{\mathcal{P}'}(y_i) = \sum_{j=1}^{md} (e_j A^{\oplus m} e_i)_y y_j \), for all \( i = 1, \ldots, md \). Hence Eqs. (2.3) and (3.1) imply

\[
\hat{D}_{\mathcal{P}}(\hat{D}_{\mathcal{P}}(y)) + (2r - s) \hat{D}_{\mathcal{P}}(y) = \frac{(k - r_+)(k - r_-)}{d} \hat{D}_{\mathcal{P}'}(y)
\]

for all \( y \) in a fiber of \( p : C \to \mathbb{P}^1 \) over a point outside the branch locus of \( \pi \). Let \( \hat{\delta} \) (respectively \( \delta \)) denote the difference between the expressions on the left- and the right-hand side of the first
(respectively second) identity of the proposition. We already know that \( \hat{\delta} = 0 \). Hence, as \( \gamma \pi^* = \alpha^* \hat{\gamma} \pi^* N_\alpha \), we have \( \delta = \#(H) \alpha^* \hat{\delta} N_\alpha = 0 \). \( \square \)

**Proposition 3.5.** In the special case \( m = 1 \) the endomorphisms \( \hat{\gamma} \pi^* \) and \( \gamma \pi^* \) vanish and, using the notation \( \mathcal{P}^\pm = (A, r_\pm, \pi, \phi) \), we find that \( \hat{\mathcal{Z}}_{p^+} \) and \( Z_{p^-} \) are complementary subvarieties of \( J(C) \). Moreover, \( \hat{\mathcal{Z}}_{p^\pm} = \ker(\hat{\gamma} \pi^* \pm 0) \) and \( Z_{p^\pm} = \ker(\gamma \pi^* \pm \#(H)(r_+ - r_-) \text{id} J(X) \_0). \)

**Proof.** Let \( q \in \mathbb{P}^1 \setminus B \). By definition of \( D_{\mathcal{P}} \) we have \( D_{\mathcal{P}^*}(x) = \pi^* \pi(x) \) for all \( x \in \pi^{-1}(q) \), while Lemma 3.3 implies that \( \hat{D}_{\mathcal{P}^*}(y) = p^* p(y) \) for all \( y \in p^{-1}(q) \). Hence \( \gamma \pi^* \) and \( \hat{\gamma} \pi^* \) vanish. As \( \hat{\gamma} \pi^*_x = \hat{\gamma} \pi^*_x + (r_+ - r_-) \text{id} J(C) \), we note that \( \hat{\mathcal{Z}}_{p^\pm} = \text{im}(\hat{\gamma} \pi^*_x + (r_+ - r_-) \text{id} J(C)) \). Using standard arguments one shows that \( \varepsilon_+ := \frac{1}{r_+ - r_-}(\hat{\gamma} \pi^*_x + (r_+ - r_-) \text{id} J(C)) \) are symmetric idempotents in \( \text{End}_Q(J(C)) \). Since \( \varepsilon_+ = \text{id} J(C) \_\varepsilon_\_ \), it follows that \( Z_{p^\pm} \) are complementary subvarieties of \( J(C) \) and \( \text{dim} Z_{p^\pm} + \text{dim} Z_{p^-} = g(C) \). As a consequence of Proposition 3.4 we have \( \hat{\mathcal{Z}}_{p^\pm} \subset \ker(\hat{\gamma} \pi^* \pm 0) \). Moreover, by [4, Proposition 5.1.1] the analytic representation \( q_a(\hat{\gamma} \pi^*) \in \text{End}(H^0(C, \omega C)^*) \) of \( \hat{\gamma} \pi^* \) is self-adjoint with eigenvalues \( r_- - r_+ \) and 0 with respect to the Riemann form \( \psi(C) \), where \( \psi(C) \) is the canonical polarization of \( J(C) \).

Finally, we note that the Prym data \( \mathcal{P} = (A, r, \pi, \phi) \) associated to the adjacency matrix \( A \), a strongly regular graph \( \mathcal{G} \) and the Prym data \( \mathcal{P} = (J_d - I_d - A, -r - 1, \pi, \phi) \) associated to the adjacency matrix \( J_d - I_d - A \) of the complementary graph \( \mathcal{G} \) yield the same Prym varieties because of the identities \( \gamma \pi^* = \gamma \pi^* = 0 \) and \( \hat{\gamma} \pi^* = \hat{\gamma} \pi^* = 0 \).

4. **Prym varieties of a non-Galois covering**

We now shift our attention to non-Galois coverings of \( \mathbb{P}^1 \). Given a \( d \times d \) Prym matrix \( A \), a number \( m \in \mathbb{N} \) and a special covering \( p : C \rightarrow \mathbb{P}^1 \) of degree \( md \) along with an appropriate labeling class (to be defined below), we shall define a symmetric divisor correspondence on \( C \times C \) which then serves to obtain a pair of Prym varieties in \( J(C) \).

To define labeling classes, let \( p : C \rightarrow \mathbb{P}^1 \) be a covering of degree \( n \) with branch locus \( B \). Given a point \( q \in \mathbb{P}^1 \setminus B \), a labeling \( \{x_1, \ldots, x_n\} \) of the fiber \( p^{-1}(q) \) induces a bijection \( \nu : p^{-1}(q) \rightarrow \{1, \ldots, n\} \) sending \( x_i \) to \( i \). For \( q_j \in \mathbb{P}^1 \setminus B \) with \( j = 1, 2 \), let \( \{x_{j1}, \ldots, x_{jn}\} \) be a labeling of the fiber above \( q_j \) inducing a bijection \( \nu_j \). Then the bijections \( \nu_1 \) and \( \nu_2 \) are said to be equivalent if there exists a path \( \mu \subset \mathbb{P}^1 \setminus B \) running from \( q_1 \) to \( q_2 \) such that the lift of \( \mu \) to \( C \) with initial point \( x_{1i} \) has end point \( x_{2i} \), for \( i = 1, \ldots, n \). The equivalence class \( [\nu] \) of an induced bijection \( \nu \) is called a labeling class for \( p \).

**Definition 4.1.** Consider the triple \( T = (A^m, p, [\nu]) \) for \( m \geq 1 \), where \( p : C \rightarrow \mathbb{P}^1 \) is a covering of degree \( md \) and \( [\nu] \) is a labeling class for \( p \). We say that \( T \) is a Prym triple if \( \text{Stab}(A^m) \) contains the monodromy group of \( p \) with respect to \( [\nu] \).

Let \( B = \{b_1, \ldots, b_n\} \) be a finite subset of \( \mathbb{P}^1 \) and take non-trivial permutations \( \sigma_1, \ldots, \sigma_n \in \text{Stab}(A^m) \) such that \( \sigma_n \ldots \sigma_1 = (1) \) and \( G = \{\sigma_1, \ldots, \sigma_n\} \) is a transitive subgroup of \( S_{md} \). Recalling the monodromy version of RET (cf. [12, p. 92]), we may assume that \( p : C \rightarrow \mathbb{P}^1 \) is a
covering of degree \(md\) with branch locus \(B\) and labeling class \([v]\) such that the ramification of \(p\) above \(b_i\) is induced by \(\sigma_i\), for \(i = 1, \ldots, n\). Then \((A^{\oplus m}, p, [v])\) is a Prym triple.

**A symmetric correspondence.** Assume that \(\mathcal{T} = (A^{\oplus m}, p, [v])\) is a Prym triple with \(p : C \to \mathbb{P}^1\) and \(v : p^{-1}(q) \to \{1, \ldots, md\}\). We shall define a symmetric correspondence on \(C \times C\). Take the Galois closure \(\pi : X \to \mathbb{P}^1\) of \(p\) and let \(H\) be the Galois group of the covering \(X \to C\). Denote \(G = \text{Gal}(\pi)\) and choose a Galois labeling \(\pi^{-1}(q) \leftrightarrow G\) such that the induced labeling \(p^{-1}(q) \leftrightarrow H \setminus G\) yields \(v(H) = 1\). Define the group \(\Sigma = \{\sigma_g \in S_{md} \mid g \in G\}\), where \(\sigma_g\) is the permutation sending \(\nu(Hf) \mapsto v(Hfg^{-1})\) and consider the canonical homomorphism \(\phi : G \to \Sigma, g \mapsto \sigma_g\). Since \(\ker(\phi)\) is a normal subgroup of \(G\) contained in \(H\), the minimality of \(\pi\) dictates that \(\ker(\phi)\) is trivial, implying that \(\phi\) is an isomorphism. Noticing that, with respect to the Galois labeling, the monodromy group of \(\pi\) acts on \(G\) via multiplication on the right with the elements of \(G\), we conclude that \(\Sigma\) is the monodromy group of \(p\) with respect to \([v]\). Hence \(\phi : G \leftrightarrow \text{Stab}(A^{\oplus m})\) is a transitive representation. The fact that \(H\) is the stabilizer of the letter 1 with respect to \(\phi\) thus implies that \(\mathcal{P} = (A^{\oplus m}, r, \pi, \phi)\) for \(r \in \{r_+, r_-\}\) represents Prym data. Therefore

\[
D_T := \hat{\nu} \mathcal{P} + r\Delta_C
\]

is a well-defined symmetric correspondence on \(C \times C\). Recall that \(k\) is the eigenvalue of the eigenvector \((1, \ldots, 1)\) of \(A\). Because \(\nu(Hg) = (\phi(g))^{-1}(1)\) for all \(g \in G\), Eq. (3.1) gives the following interpretation of \(D_T\).

**Lemma 4.2.** Let \(\{x_1, \ldots, x_{md}\}\) be a labeling in the class \([v]\) and denote \(A^{\oplus m} = (s_{ij})_{i,j=1}^{md}\). Then the point \((x_i, x_j)\) appears in \(D_T\) with multiplicity \(s_{ij}\). In particular, \(D_T\) is of bidegree \((k, k)\).

In fact, if \(S\) denotes the set of non-zero entries of \(A\) and for each \(s \in S\) we define a set \(\hat{G}_s = \{\hat{g} \in H \setminus G/H \mid s(\phi({\hat{g}}))(1), 1 = s\}\), then we find reduced divisors \(D_s = \sum_{\hat{g} \in \hat{G}_s} \hat{\nu}_s\) on \(C \times C\) such that \(D_T = \sum_{s \in S} s D_s\).

**Definition 4.3.** Let \(\gamma_T = \hat{\nu} \mathcal{P} + r \text{id}_{J(C)}\). Then \(P_{\pm}(T) = \ker(\gamma_T - r_{\mp} \text{id}_{J(C)})\) are the Prym varieties associated to \(T\).

**Remark.** Given a second labeling \(\nu'\) of the fiber \(p^{-1}(q)\), according to RET there exists an \(f \in \text{Aut}(p)\) such that \(v = \nu' \circ f\) if and only if \(v\) and \(\nu'\) yield the same monodromy representation for \(p\). If such an \(f\) exists, then \(f\) induces an isomorphism of the Prym varieties.

By Proposition 3.5, if \(m = 1\), then \(P_+(T)\) and \(P_-(T)\) are complementary subvarieties of \(J(C)\) defined by \(P_{\pm}(T) = \text{im}(\gamma_T - r_{\mp} \text{id}_{J(C)})\). If in addition \(D_T\) is fixed point free and \(r_+ = 1\), then a theorem of Kanev (cf. [6, Theorem 3.1]) states that \(P_+(T)\) is a Prym–Tyurin variety of exponent \(1 - r_-\) for \(C\).

We now try to compute the dimension of \(P_{\pm}(T)\). Let \(\eta \in \{0, 1\}\) be such that \(\eta = 1\) if \(k \neq r_+\) and \(\eta = 0\) else.

**Proposition 4.4.** Let \(\mathcal{T} = (A^{\oplus m}, p, [v])\) be a Prym triple with \(p : C \to \mathbb{P}^1\) and assume that \(A\) has diagonal \((s, \ldots, s)\). Denote \(\mathcal{T}' = (A^{\oplus m} - sI_{md}, p, [v])\) and \(\mathcal{T}_0 = (J^{\oplus m}_d, p, [v])\). Using the
notation \( d_\pm = \dim P_\pm(T) \) and \( d_0 = \dim P_-(T_0) \), we have the following identity for the dimension of \( P_\pm(T) \):

\[
\pm(r_+ - r_-)d_\pm = (k - r_\pm)\eta d_0 + (r_\pm - s)g(C) - k + s + \frac{1}{2}(DT \cdot \Delta_C).
\]

**Proof.** Denote \( \gamma_0 = \gamma_{T_0} \) and define an endomorphism \( \varepsilon \) on \( J(C) \) such that \( \varepsilon = \gamma_T - k \text{id}_J(C) \) if \( \eta = 1 \) and \( \varepsilon = \text{id}_J(C) \) else. Then Lemma 4.2 implies that \( \varepsilon (\gamma_T - r_\pm \text{id}_J(C))(\gamma_T - r_- \text{id}_J(C)) = 0 \). Recall that \( \varrho_a(\gamma_T) \) and \( \varrho_a(\gamma_0) \) are self-adjoint w.r.t. the form \( c_1(\Theta_C) \). Hence by direct consequence of Proposition 3.4, if \( \eta = 1 \) (respectively \( \eta = 0 \)), then \( \varrho_a(\gamma_T) \) has eigenvalues \( k, r_+, r_- \) (respectively \( r_+, r_- \)) with respective multiplicities \( d_0, d_-, d_+ \) (respectively \( d_-, d_+ \)). Since \( \text{Tr}(\varrho_a(\gamma_T)) = \text{Tr}(\varrho_a(\gamma_T')) + \sg(C) \), we obtain

\[
\begin{cases}
  d_+ + d_- + \eta d_0 = g(C), \\
  r_-d_+ + r_+d_- + k\eta d_0 = \text{Tr}(\varrho_a(\gamma_T')) + \sg(C).
\end{cases}
\]

(4.1)

Let \( \text{Tr}_r(\gamma_T') \) be the rational trace of \( \gamma_T' \), i.e., \( \text{Tr}_r(\gamma_T') \) is the trace of the extended rational representation \((\varrho_r \otimes 1)(\gamma_T')\) of \( H_1(C, \mathbb{Z}) \otimes \mathbb{C} \). As \( \varrho_r \otimes 1 \) is equivalent to \( \varrho_a \oplus \varrho_a' \), it follows that \( \text{Tr}_r(\gamma_T') = 2 \text{Tr}(\varrho_a(\gamma_T')) \). With \( DT' \) being of bidegree \((k-s, k-s)\), Proposition 11.5.2. of [4] implies

\[
\text{Tr}(\varrho_a(\gamma_T')) = \frac{1}{2} \text{Tr}_r(\gamma_T') = k - s - \frac{1}{2}(DT' \cdot \Delta_C).
\]

Solving (4.1) for \( d_\pm \) we obtain the desired result. \( \square \)

Proposition 3.5 implies that for \( m = 1 \) we have \( d_0 = 0 \). To compute \( \dim P_\pm(T) \) we need to determine the intersection number \( (DT' \cdot \Delta_C) \). We shall do this for a Prym triple \( T = (A^{\otimes m}, p, [v]) \), where \( A^{\otimes m} = (s_i, j)_{i,j=1}^{md} \) has zero diagonal. Denoting the set of non-zero entries of \( A \) by \( S \), we recall that \( DT = \sum_{s \in S} s D_s \) with \( D_s \) reduced. Hence it suffices to determine the local intersection numbers \((D_s \cdot \Delta_C)_{(x,x)} \) at \( (x,x) \) for a ramification point \( x \in C \) of \( p : C \to \mathbb{P}^1 \). Let \( b \in \mathbb{P}^1 \) be the corresponding branch point and assume that the local monodromy at \( b \) is given by \( \sigma_b \in S_{md} \). Further, let \( \tau \in S_{md} \) be the cycle factor of \( \sigma_b \) which describes the ramification at \( x \) and assume that it is of order \( l \). For each \( s \in S \) we define a set \( T_{\tau,s} \) of elements \( t \in \{1, \ldots, l-1\} \) for which there exists a \( j \in \{1, \ldots, md\} \) such that \( s_j \tau^i(j) = s \). Then:

**Lemma 4.5.** For each \( s \in S \) we have \((D_s \cdot \Delta_C)_{(x,x)} = \#(T_{\tau,s}).\)

**Proof.** After a suitable choice of coordinates on a small open neighborhood of \( x \) the covering \( p \) is given by \( z \mapsto z^l \). Then near the point \((x,x)\) the reduced divisor \( D_s \) can be described as the union of graphs of the multiplications \( z \mapsto \zeta_t^i z \) (for \( t \in T_{\tau,s} \)) with \( \zeta_t = \exp(\frac{2\pi}{l} \sqrt{-1}) \). Obviously these graphs intersect \( \Delta_C \) transversally in \((x,x)\), thus implying \((D_s \cdot \Delta_C)_{(x,x)} = \#(T_{\tau,s}). \)

Hence, given the branch locus \( B \) of \( p \) and, for each \( b \in B \), the set \( R_b \) of cycle factors in the cycle decomposition of \( \sigma_b \), we can calculate the intersection number as a sum
\[
(D_T \cdot \Delta_C) = \sum_{b \in B} \sum_{\tau \in R_b} \sum_{s \in S} s \#(T_{\tau, s}).
\]

For an application of the lemma we refer to Example 5.3.

5. Examples

We briefly recall the definition of a Nielsen class, as coined by M.D. Fried. For a finite group \(G\) and an \(n\)-tuple \(C = (\Sigma_1, \ldots, \Sigma_n)\) of conjugacy classes from \(G\), the Nielsen class \(Ni(G, C)\) is the set of all \((g_1, \ldots, g_n) \in G^n\) such that \(g_n \cdots g_1 = 1\), the elements \(g_1, \ldots, g_n\) generate \(G\), and \((g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \in \Sigma_1 \times \cdots \times \Sigma_n\) for some \(\sigma \in S_n\), where \(n \geq 2\).

Now assume that \(G \subset S_d\) is a transitive subgroup and \(Ni(G, C) \neq \emptyset\). Let \(p : C \to \mathbb{P}^1\) be a covering of degree \(d\) with branch locus \(\{b_1, \ldots, b_n\}\) and monodromy group \(G\) with respect to a labeling class \([v]\). Then the pair \((p, [v])\) is said to have Nielsen class \(Ni(G, C)\) if there exists an \(n\)-tuple \((g_1, \ldots, g_n) \in Ni(G, C)\) with the property that the monodromy of \(p\) at \(b_i\) is induced by \(g_i\), for all \(i\).

Our first example has been covered by Kanev in [7,8]. We will look at it from a different angle.

Example 5.1 (Schläfli graph). Let \(\mathcal{L}\) be the intersection graph of the 27 lines on a non-singular cubic surface in \(\mathbb{P}^3\). In the notation of Example 2.1 we take \(\tau_i (i = 1, \ldots, 5)\) (respectively \(\tau_6\)) to be the transformation that interchanges the rows of the double-six \(M_{i, i+1}\) (respectively \(M_{1, 2, 3}\)). Denoting the adjacency matrix of \(\mathcal{L}\) by \(A\), we recall that \(\text{Stab}(A) = \langle \tau_1, \ldots, \tau_6 \rangle\) is a transitive subgroup of \(S_{27}\). Note that each \(\tau_j\) is an involution with exactly 15 fixed points.

For an integer \(n \geq 7\), take a \(2n\)-tuple \(\mathcal{C} = (\Sigma_1, \ldots, \Sigma_{2n})\) with the property that each \(\Sigma_i\) is the \(\text{Stab}(A)\)-conjugacy class of a \(\tau_j\), and \(\text{Ni}(\text{Stab}(A), \mathcal{C}) \neq \emptyset\). Given a subset \(B \subset \mathbb{P}^1\) of cardinality \(2n\), let \(T = (A, p, [v])\) be a Prym triple, where \(p : C \to \mathbb{P}^1\) is a covering of degree 27 with branch locus \(B\), such that \((p, [v])\) has Nielsen class \(\text{Ni}(\text{Stab}(A), \mathcal{C})\). According to Hurwitz’ formula the curve \(C\) is of genus \(6n - 26\). Since no vertex \(v\) of \(\mathcal{L}\) is adjacent to \(\tau_j(v)\), it follows that \(D_T\) is fixed point free. As \(A\) has eigenvalues \(k = 10, r_+ = 1\) and \(r_- = -5\), Proposition 4.4 implies that \(P_+(T)\) is an \((n - 6)\)-dimensional Prym–Tjurin variety of exponent 6 for the curve \(C\). With regard to moduli, note that for \(n = 12\) we have \(g(C) = 46\), \(\dim P_+(T) = 6\) and \(\#(B) = \dim A_6 + \dim \text{Aut}(\mathbb{P}^1) = 24\), where \(A_6\) is the moduli space of 6-dimensional principally polarized abelian varieties.

Example 5.2 (Lattice graphs). For \(n \geq 3\) let \(A\) be the adjacency matrix of the lattice graph \(L_2(n)\) with vertex set \(\{1, \ldots, n\}^2\). We define a group \(G = \langle \varphi_0, \varphi_1, \varphi_2, \varphi_3 \rangle\) generated by the involutions \(\varphi_h := (\tau_h, \tau_h^{-1}) \circ t\) in \(\text{Stab}(A)\), where \(t\) acts on \([1, \ldots, n]^2\) by exchange of coordinates and \(\tau_0, \tau_1, \tau_2, \tau_3 \in S_n\) are given by \(\tau_0 = (1)\), \(\tau_1 = (1 n)\), \(\tau_2 = (2 n)\) and \(\tau_3 = (1 2 \cdots n)\). Then \(G\) is a transitive subgroup of \(\text{Stab}(A)\); indeed, identifying \([1, \ldots, n]\) and \(\mathbb{Z}/n\mathbb{Z}\), we have

(a) \((\varphi_0 \circ \varphi_3)^m(1, 1) = (1, h + 1)\) for \(m = 1, \ldots, n - 2\);
(b) \((\varphi_3 \circ \varphi_0)(i, j) = (i + 1, j - 1)\) for \(i, j = 1, \ldots, n\) with \(j \neq n - i\);
(c) \((\varphi_3 \circ \varphi_0)^{m-1} \circ \varphi_2)(2, 1) = (m, n - m + 1)\) for \(m = 1, \ldots, n\).

Given an integer \(l \geq 0\), let \(\mathcal{C} = (\Sigma_1, \ldots, \Sigma_{2l+8})\), where each \(\Sigma_i\) is the \(G\)-conjugacy class of a \(\varphi_j\), and \(\text{Ni}(G, \mathcal{C}) \neq \emptyset\). Given a subset \(B \subset \mathbb{P}^1\) of cardinality \(2l + 8\), we may assume that
$T = (A, p, [v])$ is a Prym triple for a covering $p : C \rightarrow \mathbb{P}^1$ of degree $n^2$ branched over $B$, such that $(p, [v])$ has Nielsen class $\text{Ni}(G, C)$. Then the curve $C$ is of genus $(n-1)^2 + \frac{1}{2}ln(n-1)$. As $D_T$ is fixed point free and $A$ has eigenvalues $k = 2(n-1)$, $r_+ = n - 2$ and $r_- = -2$, it follows that

$$\dim P_+(T) = (n-1)(n-3) + \frac{1}{2}l(n-1)(n-2).$$

Hence, for $n = 3$ and $l \geq 1$ we obtain a finite number of finite-dimensional families of $l$-dimensional Prym–Tyurin varieties of exponent 3 for curves of genus $3l + 4$. In anticipation of Section 7 we shall say that $T$ is of type $l$ whenever $n = 3$ and $l \geq 1$.

**Example 5.3.** For an example involving symmetric correspondences with fixed points, let $n \geq 3$ and assume that $A$ is the adjacency matrix of the graph $L_2(n)$ (the complement of $L_2(n)$) with vertex set $\{1, \ldots, n\}^2$. Assume that $t \in \text{Stab}(A)$ acts on $\{1, \ldots, n\}^2$ by exchange of coordinates and for $h = 1, \ldots, n-1$ define the involution $\sigma_h := ((1, h+1), (1))$ in $S_n \times S_n$. We observe that $\text{Stab}(A)$ is generated by the elements $t$ and $\sigma_1, \ldots, \sigma_{n-1}$. Clearly, no vertex $(i, j) \in \{1, \ldots, n\}^2$ is adjacent to $\sigma_h(i, j)$, and $(i, j)$ is adjacent to $t(i, j)$ if and only if $i \neq j$. Note that $\sigma_2, \ldots, \sigma_{n-1}$ are $\text{Stab}(A)$-conjugates of $\sigma_1$.

For integers $l_1, l_2 \geq 0$, let $C = (\Sigma_{1,1}, \ldots, \Sigma_{1,2(l_1+1)}; \Sigma_{2,1}, \ldots, \Sigma_{2,2(l_2+n-1)})$, where each $\Sigma_{i,j}$ (respectively $\Sigma_{2,j}$) is the $\text{Stab}(A)$-conjugacy class of $t$ (respectively $\sigma_1$). Given a subset $B \subset \mathbb{P}^1$ of cardinality $2(l_1 + l_2 + n)$, we may choose a Prym triple $T = (A, p, [v])$ for a covering $p : C \rightarrow \mathbb{P}^1$ of degree $n^2$ branched over $B$, with the property that $(p, [v])$ has Nielsen class $\text{Ni}(\text{Stab}(A), C)$. Then $C$ is of genus $\frac{1}{2}(n-1)(n-2) + \frac{1}{2}ln(n-1) + l_2n$ and Lemma 4.5 implies $(D_T \cdot \Delta_C) = (l_1 + 1)(n-1)n$. It follows that $P_+(T)$ is of dimension $l_1(n-1) + l_2$.

In view of moduli, note that for $l_1 = 0$ and $n \geq \frac{1}{3}(l_2^2 - 3l_2 + 6)$ we have $\dim P_+(T) = l_2$ and $\#(B) \geq \dim \mathcal{A}_{l_2} + \dim \text{Aut}(\mathbb{P}^1)$. In particular, if $l_1 = 0$ and $n = l_2 = 6$, then $g(C) = 46$. Moreover, there exist $n \geq 3$ such that $S_2 \times (S_n \times S_n)$ has no subgroup of index $n$, in which case there is no factorization $p : C \xrightarrow{n:1} C' \xrightarrow{n:1} \mathbb{P}^1$.

**Example 5.4** *(Latin square graphs).* Given an integer $n \geq 3$, we assume that $A$ is the adjacency matrix of the Latin square graph $L_3(n)$. We recall from Example 2.5 that $(\mathbb{Z}/n\mathbb{Z})^2$ induces a transitive subgroup of $\text{Stab}(A)$ via translation; as such it coincides with $G := \langle (1, 1), (1, 2) \rangle$. Viewed as permutations of the vertex set $(\mathbb{Z}/n\mathbb{Z})^2$, the translations $(1, 1)$ and $(1, 2)$ split into $n$ mutually disjoint $n$-cycles. For $n \geq 4$ the vertices $(i + 1, j + 1)$ and $(i + 1, j + 2)$ of $L_3(n)$ are non-adjacent to $(i, j)$.

Now assume that $n \geq 4$. For an integer $l \geq 2$, choose an $ln$-tuple $C = (\Sigma_1, \ldots, \Sigma_{ln})$, where each $\Sigma_i$ is the $G$-conjugacy class of one the translations $(1, 1), (1, 2)$, with the property that $\text{Ni}(G, C) \neq \emptyset$. Given a subset $B \subset \mathbb{P}^1$ of cardinality $ln$, let $T = (A, p, [v])$ for a covering $p : C \rightarrow \mathbb{P}^1$ of degree $n^2$ with branch locus $B$, such that $(p, [v])$ has Nielsen class $\text{Ni}(G, C)$. We find that $C$ is of genus $1 - n^2 + \frac{1}{2}ln^2(n-1)$. Moreover, since $\deg(p) = \#(\mathbb{Z}/n\mathbb{Z})^2$, it is immediately seen that $p$ is a Galois covering. Using the fact that $D_T$ is fixed point free and $A$ has eigenvalues $k = 3(n-1)$, $r_+ = n - 3$ and $r_- = -3$, we compute

$$\dim P_+(T) = -(n-1)(n-2) + \frac{1}{2}ln(n-1)(n-3).$$
Hence, for \( n = 4 \) we get finitely many finite-dimensional families of \( 6(l - 1) \)-dimensional Prym–Tyurin varieties of exponent 4 for curves of genus \( 24l - 15 \).

6. A splitting

We show that for certain Prym triples \( T = (A^\oplus m, p, [v]) \) the covering \( p : C \to \mathbb{P}^1 \) splits into a covering \( f : C \to C' \) of degree \( d \) and a covering \( h : C' \to \mathbb{P}^1 \) of degree \( m \) such that \( f \) depends essentially on \( D_T \). Recall that \( k \) is the eigenvalue of the eigenvector \((1, \ldots, 1)\) of \( A \).

**Theorem 6.1.** Assume that \( A \in \{0, 1\}^{d \times d} \) is a Prym matrix with zero diagonal and eigenvalue \( k \) of multiplicity 1. Given \( m \geq 2 \), let \( T = (A^\oplus m, p, [v]) \) be a Prym triple for a covering \( p : C \to \mathbb{P}^1 \) with branch locus \( B \). Denote \( C_0 := p^{-1}(\mathbb{P}^1 \setminus B) \). Then there exists a unique splitting

\[
p : C \xrightarrow{f} C' \xrightarrow{h} \mathbb{P}^1
\]

such that, for any \((x, x') \in (C_0 \times C_0) \setminus \Delta_{C_0} \), the points \( x, x' \) are in the same fiber of \( f \) if and only if there is a finite sequence of points \( x = x_0, \ldots, x_l = x' \) on \( C_0 \) with \((x_j, x_{j+1}) \in D_T \) for all \( j = 0, \ldots, l - 1 \).

**Proof.** Fix a point \( q_0 \in \mathbb{P}^1 \setminus B \) and assume that \( v \) is a labeling of the fiber \( p^{-1}(q_0) \). We denote \( S = \{1, \ldots, m\}, T = \{1, \ldots, d\} \) and identify \( S \times T \) with \( \{1, \ldots, md\} \) via the bijection \((s, t) \leftrightarrow (s - 1)m + t \). Then \( v \) turns into a bijection \((v_1, v_2) \) of \( p^{-1}(q_0) \) with \( S \times T \). Let \( \Sigma \) be the monodromy group of \( p \) with respect to \((v_1, v_2) \) and split its elements accordingly into \( \sigma = (\sigma_1, \sigma_2) \). Denoting \( A := (a_{i,j})_{i,j=1}^d \) we may view \( A^\oplus m \) as the matrix of entries \( c_{u,u'} \), where \( u = (s, t) \) and \( u' = (s', t') \) run through the set \( S \times T \), such that \( c_{u,u'} = \sigma_{s,t} a_{s',t'} \) if \( s = s' \) and \( c_{u,u'} = 0 \) else. According to Proposition 2.3 the matrix \( A \) is the adjacency matrix of a connected strongly regular graph \( G \) on \( d \) vertices. Thus, for \( u = (s, t) \) and \( u' = (s', t') \) there exists a finite sequence \( u = u_0, \ldots, u_l = u' \) in \( S \times T \) such that \( c_{u_j,u_{j+1}} = 1 \) for all \( j = 0, \ldots, l - 1 \) and if only if \( s = s' \). Hence \( \sigma_{1, t} = \sigma_{1, t} \) for all \( \sigma \in \Sigma \), i.e., there is a unique \( \tau_\sigma \in S_m \) such that \( \sigma_{1, t} = \tau_\sigma(\cdot) \) for all \( t \in T \).

Let \( \pi : X \to \mathbb{P}^1 \) be the Galois closure of \( p \) and denote \( G = \text{Gal}(\pi) \). As we have seen in Section 4, there exists an isomorphism \( \phi : G \to \Sigma \) such that the Galois group \( H \) of \( X \to C \) is the stabilizer of \((1, 1) \in S \times T \) w.r.t. \( \phi \) and any Galois labeling of a fiber of \( \pi \) induces a labeling in the class \([v]\) via the identification \( H g \leftrightarrow g^{-1}(1, 1) \). We let \( H' \) be the stabilizer of \( 1 \in S \) with respect to \( \psi \circ \phi \), where \( \psi : \Sigma \to S_m \) is the transitive representation induced by \( \sigma \mapsto \tau_\sigma \). Write \( C' = X/H'; \) since \( H \subset H' \) (respectively \( H' \subset G \)) is a subgroup of index \( d \) (respectively \( m \)), there are canonical coverings \( f : C \to C' \) of degree \( d \) and \( h : C' \to \mathbb{P}^1 \) of degree \( m \) such that \( p = h \circ f \). Take a point \( q \in \mathbb{P}^1 \setminus B \) and a Galois labeling \( \pi^{-1}(q) \leftrightarrow G \). For any element \( g \in G \), if \((s, t) = g^{-1}(1, 1) \), then \( H' g \leftrightarrow g^{-1}(1, 1) = s \), i.e., on the fiber \( p^{-1}(q) \) the covering \( f \) is given by \((s, t) \mapsto s \). With reference to Lemma 4.2 we conclude that \( f \) has the desired properties. Using the monodromy of \( p \), the reader will easily check that the splitting is unique. \( \square \)

With \( T, f \) and \( h \) as above, we say that the pair of coverings \((f, h)\) represents the **canonical splitting** for \( T \).

**Corollary 6.2.** For integers \( d, m \geq 2 \), assume that \( T = ((J_d - I_d)^\oplus m, p, [v]) \) is a Prym triple associated to a covering \( p : C \to \mathbb{P}^1 \) and let \((f, h)\) be its canonical splitting. Then \( P_+(T) \) is...
the usual Prym variety associated to the covering \( f \), i.e., \( P_+(T) \) and \( \text{im} f^* \) are complementary subvarieties of \( J(C) \).

**Proof.** According to Theorem 6.1 we have \( D_T(x) = -x + f^* f(x) \) for all \( x \in C \) in an unramified fiber of \( p \). Hence \( \gamma_T + \text{id}_{J(C)} = f^* N_f \) and thus \( P_-(T) = \text{im} f^* \). As \( (J_d - I_d)^{\oplus m} \) is a Prym matrix, Proposition 3.5 implies that \( P_+(T) \) and \( \text{im} f^* \) are complementary in \( J(C) \). □

Corollary 6.2 has a natural converse. Before addressing this, we recall that a smooth projective curve of genus \( g \) is \( m \)-gonal for all \( m \geq \lceil \frac{g}{2} \rceil + 1 \) (cf. [1, Existence Theorem, p. 206]).

**Corollary 6.3.** Assume that \( f : C \to C' \) is a covering of degree \( d \geq 2 \) of a curve \( C' \) of genus \( g \geq 1 \) and let \( h : C' \to \mathbb{P}^1 \) be a covering of degree \( m \geq \lceil \frac{g}{2} \rceil + 1 \). Then there exists a labeling class \([v]\) for the covering \( h \circ f \) such that \( T = ((J_d - I_d)^{\oplus m}, h \circ f, [v]) \) is a Prym triple and \( P_+(T) \) is the usual Prym variety associated to \( f \).

**Proof.** Take a point \( q \in \mathbb{P}^1 \) outside the branch locus of \( h \circ f \). Then we can define a bijection \( v = (v_1, v_2) : (h \circ f)^{-1}(q) \to \{1, \ldots, m\} \times \{1, \ldots, d\} \) such that \( v_1(x) = v_1(x') \iff f(x) = f(x') \), for all \( x, x' \in (h \circ f)^{-1}(q) \). It is immediately seen that \( ((J_d - I_d)^{\oplus m}, h \circ f, [v]) \) represents a Prym triple with canonical splitting \((f, h)\). Now apply Corollary 6.2. □

Given integers \( d, m \geq 2 \), assume that \( T = ((J_d - I_d)^{\oplus m}, p, [v]) \) is a Prym triple. We shall call \( T \) simple if its canonical splitting \((f, h)\) is simple, i.e., if \( f \) and \( h \) are simply branched coverings such that no ramified fiber of \( h \) contains a branch point of \( f \) and no unramified fiber of \( h \) contains more than one branch point of \( f \). It should be noted that simplicity can also be described in terms of the monodromy of \( p \) alone, without reference to \( f \) and \( h \).

To conclude this section, we use simple Prym triples to characterize (at least up to isogeny) abelian varieties corresponding to the general points of \( A_4 \) and \( A_5 \).

**Lemma 6.4.**

(1) The general 4-dimensional principally polarized abelian variety is isogenous to a Prym variety \( P_+(T) \) for a simple Prym triple \( T = ((J_2 - I_2)^{\oplus 3}, p, [v]) \) such that the covering \( p \) has exactly 4 simple and 10 double branch points.

(2) The general 5-dimensional principally polarized abelian variety is of the form \( P_+(T) \), where \( T = ((J_2 - I_2)^{\oplus 4}, p, [v]) \) is a simple Prym triple such that the covering \( p \) has exactly 18 double branch points.

**Proof.** For integers \( g \geq 1 \) and \( n \geq 0 \), let \( \mathcal{R}(g, n) \) be the moduli space of equivalence classes of double coverings \( f : C \to C' \) with \( C' \) of genus \( g \) and \( f \) branched at 2n distinct points of \( C' \). We shall need the following fact: Let \( m \) be an integer. If \( m \geq \lceil \frac{g}{2} \rceil + 1 \), then for a double covering \( f : C \to C' \) corresponding to a general point of \( \mathcal{R}(g, n) \) there exists an \( m \)-fold covering \( h : C' \to \mathbb{P}^1 \) such that the covering pair \((f, h)\) is simple. The proof is left to the reader. As in [2, p. 122], we let \( p(g, n) : \mathcal{R}(g, n) \to A_{g+n-1}(\delta) \) be the usual Prym morphism, where \( A_{g+n-1}(\delta) \) is the moduli space of abelian \( g \)-folds with polarization type \( \delta \). According to [2, Theorem 2.2], the morphism \( p(3, 2) : \mathcal{R}(3, 2) \to A_4(1, 2, 2, 2) \) is dominant. Moreover, for the general double covering \( f : C \to C' \) with 4 branch points and \( g(C') = 3 \) there exists a 3-fold covering \( h : C' \to \mathbb{P}^1 \) such that the pair \((f, h)\) is simple and the covering \( h \circ f \) has exactly 4 simple and
10 double branch points. Together with Corollary 6.3 this shows (1). To prove (2) we recall that \( p_{(6,0)} : \mathcal{R}(6,0) \to A_5 \) is dominant (cf. [3]). Hence it suffices to note that for the general étale double covering \( f : C \to C' \) with \( g(C') = 6 \) there exists a 4-fold covering \( h : C' \to \mathbb{P}^1 \) branched at 18 points such that the pair \((f, h)\) is simple. \( \Box \)

7. Prym–Tyurin varieties of arbitrary exponent \( \geqslant 3 \)

We show how the graph \( \overline{L_2(n)} \subseteq \text{SRG}(n^2, (n-1)^2, (n-2)^2, (n-1)(n-2)) \) for \( n \geqslant 3 \) can be employed to construct families of Prym–Tyurin varieties of exponent \( n \). These varieties turn out to be the product of the Jacobians of two \( n \)-gonal curves.

**Example 7.1.** Given an integer \( n \geqslant 3 \), we shall construct Prym–Tyurin varieties of exponent \( n \). Assume that \( A \) is the adjacency matrix of the graph \( \overline{L_2(n)} \) with vertex set \( \{1, \ldots, n\}^2 \). Recall that \( S_n \times S_n \) is a transitive subgroup of \( \text{Stab}(A) \). For \( i = 1, \ldots, n-1 \) we define involutions \( \sigma_{1,i} := ((1 + i + 1), 1) \) and \( \sigma_{2,i} := (1, 1 + i) \) in \( S_n \times S_n \). Clearly, \( S_n \times S_n \) is generated by these elements, and each \( \sigma_{1,i} \) (respectively \( \sigma_{2,j} \)) is an \( S_n \times S_n \)-conjugate of \( \sigma_{1,1} \) (respectively \( \sigma_{2,1} \)).

For integers \( l_1, l_2 \geqslant 0 \) with \( l_1 + l_2 \geqslant 1 \), take a \( 2(l_1 + l_2 + 2n - 2) \)-tuple \( C = (\Sigma_{1,1}, \ldots, \Sigma_{1,2(l_1+n-1)}, \Sigma_{2,1}, \ldots, \Sigma_{2,2(l_2+n-1)}) \), where each \( \Sigma_{1,i} \) (respectively \( \Sigma_{2,j} \)) is the \( S_n \times S_n \)-conjugacy class of \( \sigma_{1,1} \) (respectively \( \sigma_{2,1} \)). Given a subset \( B \subseteq \mathbb{P}^1 \) of cardinality \( 2(l_1 + l_2 + 2n - 2) \), we may assume that \( T = (A, p, [v]) \) is a Prym triple for a covering \( p : C \to \mathbb{P}^1 \) of degree \( n^2 \) with branch locus \( B \), such that \( (p, [v]) \) has Nielsen class \( \text{Ni}(S_n \times S_n, C) \). We then say that \( T \) is of type \((l_1, l_2)\). Since no vertex \((i, j)\) of \( \overline{L_2(n)} \) is adjacent to \( \sigma_{m,h}(i, j) \), the correspondence \( D_T \) is fixed point free. Moreover, it is easily seen that the \( \sigma_{m,h} \) decompose into \( n \) mutually disjoint transpositions on the set \( \{1, \ldots, n\}^2 \). Hence, as \( A \) has eigenvalues \( k = (n-1)^2, r_+ = 1 \) and \( r_- = -n + 1 \), it follows that \( P_{\pm}(T) \) is an \((l_1 + l_2)\)-dimensional Prym–Tyurin variety of exponent \( n \) for the curve \( C \) of genus \((n-1)^2 + (l_1 + l_2)n\).

Recall from Example 5.2 that a Prym triple \( T \) of type \( l \) yields an \( l \)-dimensional Prym–Tyurin variety \( P_{\pm}(T) \) of exponent \( 3 \). We will show that for \( n = 3 \) the Prym–Tyurin varieties of the preceding example are the same as those of Example 5.2. More precisely, let \( A \) be the adjacency matrix of the lattice graph \( L_2(3) \) and assume that \([v]\) is a labeling class for a covering \( p : C \to \mathbb{P}^1 \) of degree 9. Given the isomorphism of graphs \( \xi : L_2(3) \cong L_2(3) \) induced by the matrix \( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) as in Example 2.4, we have:

**Lemma 7.2.** Let \( l \geqslant 1 \) be an integer. Then \( T = (A, p, [v]) \) is a Prym triple of type \( l \) if and only if there exist integers \( l_1, l_2 \geqslant 0 \) such that \( l_1 + l_2 = l \) and \( T' = (J_9 - I_9 - A, p, [\xi^{-1} \circ v]) \) is a Prym triple of type \((l_1, l_2)\). In particular, if \( T \) is of type \( l \), then \( P_{\pm}(T) = P_{\pm}(T') \).

**Proof.** It suffices to note that the elements \( \varphi_0, \varphi_1, \varphi_2, \varphi_3 \) in \( \text{Aut}(L_2(3)) \) of Example 5.2 satisfy the identities \( \xi^{-1} \sigma_{1,1} \xi = \varphi_0, \xi^{-1} \sigma_{1,2} \xi = \varphi_0 \varphi_3 \varphi_0^{-1}, \xi^{-1} \sigma_{2,1} \xi = \varphi_1 \varphi_2 \varphi_1^{-1} \) and \( \xi^{-1} \sigma_{2,2} \xi = \varphi_2 \). \( \Box \)

For Prym–Tyurin varieties \( P_{\pm}(T) \) with \( T \) of type \((l_1, l_2)\) we have the following characterization.

**Theorem 7.3.** Assume that \( A \) is the adjacency matrix of \( \overline{L_2(n)} \) with \( n \geqslant 3 \). For non-negative integers \( l_1, l_2 \) such that \( l_1 + l_2 \geqslant 1 \), let \( T \) be a Prym triple of type \((l_1, l_2)\) associated to \( A \) and a
covering $C \to \mathbb{P}^1$. Then there exist $n$-gonal curves $C_1$ of genus $l_1$ and $C_2$ of genus $l_2$ such that $C = C_1 \times_{\mathbb{P}^1} C_2$ and $P_+(T) \simeq J(C_1) \times J(C_2)$.

**Proof.** Let $p : C \to \mathbb{P}^1$ and $[v]$ be the covering and labeling class such that $T = (A, p, [v])$. Then $S_n \times S_n$ is the monodromy group of $p$. We take the inclusion $i : S_n \times S_n \to \text{Perm}(N \times N)$ and write $N = \{1, \ldots, n\}$. Let $\pi : X \to \mathbb{P}^1$ be the Galois closure of $p$ and denote $G = \text{Gal}(\pi)$. Then there exists an isomorphism $\phi : G \to \Sigma$ such that the Galois group $H$ of $X \to C$ is the stabilizer of $(1, 1) \in N \times N$ w.r.t. $i \circ \phi$ and any Galois labeling of a fiber of $\pi$ induces a labeling in the class $[v]$ via the identification $Hg \leftrightarrow g^{-1}(1, 1)$. Take the projection mappings $\text{pr}_1, \text{pr}_2 : S_n \times S_n \to S_n$ onto the first and second factor and let $H_1$ (respectively $H_2$) be the stabilizer of the letter $1 \in N$ w.r.t. $\phi_1 := \text{pr}_1 \circ \phi$ (respectively $\phi_2 := \text{pr}_2 \circ \phi$). Observing that $H = H_1 \cap H_2$, we take the quotient curves $C_m = X/H_m$ for $m = 1, 2$ and let $f_m : C \to C_m$ and $h_m : C_m \to \mathbb{P}^1$ be the canonical coverings. The transitivity of $\phi_m$ implies that $f_m$ and $h_m$ are of degree $m$. Moreover, $h_m$ is a simple covering with branch locus $B_m$, where $B_m$ is the set of points $b \in B$ such that the monodromy of $p$ at $b$ is given by an involution $\sigma_m$, i.e., defined in Example 7.1. Since $\#(B_m) = 2(l_m + n - 1)$, we obtain $g(C_m) = l_m$. In addition to $H = H_1 \cap H_2$ we have $G = \{H_1, H_2\}$, hence by elementary Galois theory

$$C(C) = C(C_1) \otimes_{C(\mathbb{P}^1)} C(C_2).$$

As $B_1 \cap B_2 = \emptyset$, it follows that $C_1 \times_{\mathbb{P}^1} C_2$ is smooth and therefore identical to $C$.

It remains to show that $P_+(T) \simeq J(C_1) \times J(C_2)$. Choose a point $q \in \mathbb{P}^1 \setminus B$ and a Galois labeling $\pi^{-1}(q) \leftrightarrow G$. Then take a labeling $\{y_1, \ldots, y_n, \gamma\}$ for $p^{-1}(q)$ and a labeling $\{z_m, 1, \ldots, z_m, n\}$ for $h_m^{-1}(q)$, $m = 1, 2$ such that $y_1 g^{-1}(1, 1) \leftrightarrow Hg$ and $z_m, (\phi_m(g))^{-1}(1) \leftrightarrow H_m g$, for all $g \in G$. We observe that $f_1^{-1}(z_1, s) = \{y_s, j \mid j \in N\}$ and $f_2^{-1}(z_2, t) = \{y_i, t \mid i \in N\}$, for all $s, t \in N$. According to Lemma 4.2 we have $T(y_s, t) = \sum_{i \neq j, j \neq t} y_{i,j}$ and therefore

$$f_1^* z_1 + f_2^* z_2 = \sum_{j \in N} y_{s,j} + \sum_{i \in N} y_{i,t} = p^* p(y_{s,t}) + y_{s,t} - D_T(y_{s,t}).$$

Hence, for $y, y' \in p^{-1}(\mathbb{P}^1 \setminus B)$ and $z_m = f_m(y)$, $z_m' = f_m(y')$ with $m = 1, 2$ we obtain, using divisor class notation,

$$f_1^* [z_1 - z_1'] + f_2^* [z_2 - z_2'] = -(\gamma T - \text{id}_{J(C)})([y - y']).$$

Consequently, defining $\psi = f_1^* \psi_1 + f_2^* \psi_2 : J(C_1) \times J(C_2) \to J(C)$, where $\psi_m : J(C_1) \times J(C_m) \to J(C_m)$ is the projection on the $m$th factor, we get $P_+(T) = \text{im}(\gamma T - \text{id}_{J(C)}(\gamma T - \text{id}_{J(C)})([y - y']))$. Because $\dim P_+(T) = l_1 + l_2 = \dim J(C_1) + \dim J(C_2)$, it thus follows that $\psi : J(C_1) \times J(C_2) \to P_+(T)$ is an isogeny. As $P_+(T)$ is a Prym–Tyurin variety of exponent $n$ for $C$, the restriction of the canonical polarization $\Theta_C$ to $P_+(T)$ is of type $(n, \ldots, n)$. Lemma 12.3.1 of [4] implies that the polarization $\varphi^* \Theta_C$ of $J(C_1) \times J(C_2)$ is of type $(n, \ldots, n)$, as well. Hence $\varphi : J(C_1) \times J(C_2) \to P_+(T)$ is an isogeny of degree 1, i.e., an isomorphism. □

**Remark.** Despite the similarities between Examples 5.4 and 7.1, the preceding theorem does not fully extend to Prym triples $T$ such as in Example 5.4. In fact, defining $n$-gonal curves $C_1$ and $C_2$ analogously to those in the proof, we get $C = C_1 \times_{\mathbb{P}^1} C_2$. A simple computation shows, however, that the dimensions of $P_+(T)$ and $J(C_1) \times J(C_2)$ do not match.
A different construction. In [9], Lange, Recillas and Rojas define non-trivial families of Prym–Tyurin varieties of exponent 3. Here is a recap of their construction: Given a hyperelliptic curve $X$ of genus $g \geq 3$ and an étale covering $f : \tilde{X} \to X$ of degree 3, let $h : X \to \mathbb{P}^1$ be the covering given by the $g_2^1$ and define the curve $C = (f(2))^{-1}(g_2^1)$, where $f(2) : \tilde{X}(2) \to X(2)$ is the second symmetric product of $f$. Assume for the moment that $C$ is smooth and irreducible. Denote $\tilde{C} = \mu^{-1}(C)$, where $\mu : \tilde{X}(2) \to \tilde{X}(2)$ is the canonical 2 : 1 map and let $\tau : \tilde{C} \to \tilde{X}$ be the projection on the first factor, where $\tilde{C}$ is considered as a curve in $\tilde{X}^2$. Now define the covering $p : \tilde{C} \to \mathbb{P}^1$ induced by $h \circ f \circ \tau : \tilde{C} \to \mathbb{P}^1$ and let $f : X \to X$ be the hyperelliptic involution. Then $p$, $h \circ f$ and $h$ have the same branch locus $B$, which may be assumed to be of cardinality $2l + 8$ for $l \geq 0$. To obtain a divisorial correspondence on $C \times C$, we choose a point $q \in \mathbb{P}^1 \setminus B$ and denote the fiber $h^{-1}(q)$ by $\{x, t(x)\}$. Write $f^{-1}(x) = \{y_1, y_2, y_3\}$ and $f^{-1}(t(x)) = \{z_1, z_2, z_3\}$; then $p^{-1}(q) = \{y_i + z_j | i, j = 1, 2, 3\}$ and the identity

$$D(y_s + z_t) = \sum_{j \neq i} (y_s + z_j) + \sum_{i \neq s} (y_i + z_i)$$

defines a fixed point free symmetric correspondence $D$ of bidegree $(2, 2)$ on $C \times C$. Note that the matrix of entries $a(s,j),(i,t)$ (for $(s, j), (i, t) \in \{1, 2, 3\}^2$) given by

$$a(s,j),(i,t) = \begin{cases} 1 & \text{if } (y_s + z_j, y_i + z_t) \in D, \\ 0 & \text{else} \end{cases}$$

is the adjacency matrix of $L_2(3)$. Hence the canonical endomorphism $\gamma_D$ of $J(C)$ satisfies the equation

$$(\gamma_D - \text{id}_{J(X)})\gamma_D + 2\text{id}_{J(X)} = 0.$$ 

For $l \geq 1$ it follows that $P := \text{im}(\gamma_D - \text{id}_{J(C)})$ is an $l$-dimensional Prym–Tyurin variety of exponent 3 for the curve $C$ of genus $3l + 4$. We shall call $P$ (respectively $p$) the Prym variety (respectively covering) associated to $f$ and $h$.

The preceding construction is a special case of Example 7.1. This is a direct consequence of Lemma 7.2 and the following result, for integers $l \geq 1$.

Proposition 7.4. The Lange–Recillas–Rojas family of $l$-dimensional Prym–Tyurin varieties of exponent 3 coincides with the family of Prym–Tyurin varieties $P_+\mathcal{T}$ for Prym triples $\mathcal{T}$ of type $l$.

Proof. Let $P$ (respectively $p : C \to \mathbb{P}^1$) be the Prym variety (respectively covering) associated to an étale threefold covering $f : \tilde{X} \to X$ and a double covering $h : X \to \mathbb{P}^1$ with branch locus $B$ of cardinality $2l + 8$. We fix a point $q \in \mathbb{P}^1 \setminus B$ and write $N = \{1, 2, 3\}$. Using the notation of the preceding construction, we define the bijections $\nu : p^{-1}(q) \to N \times N$, $y_i + z_j \mapsto (i, j)$ and $\mu : (h \circ f)^{-1}(q) \to \{1, 2\} \times N$, sending $y_i \mapsto (1, i)$ and $z_j \mapsto (2, j)$. We let $\rho$ (respectively $\varrho$) be the monodromy representations for $p$ (respectively $h \circ f$) induced by $\nu$ (respectively $\mu$). Choose a small $q$-based loop $\lambda \subset \mathbb{P}^1 \setminus B$ around a point $b \in B$. Then $\varrho(\lambda) = \nu_1 \nu_2 \nu_3$ is the product of mutually disjoint transpositions $\nu_j = ((1, s_j), (2, t_j))$, $s_j, t_j \in N$. Employing the fact that $p$ comes from $h \circ f \circ \pi$ with $\pi$ as in the construction, one easily checks that $\varrho(\xi \rho(\lambda) \xi^{-1})$ is a conjugate of an involution $\sigma_{m, 1} \in S_3 \times S_3$ (as defined in Example 7.1). By transitivity of $\text{im} \rho$ it
thus follows that $\xi(\text{im} \rho)\xi^{-1} = S_3 \times S_3$. Hence, if $A$ denotes the adjacency matrix of the graph $L_2(3)$, then $T = (A, p, [v])$ is a Prym triple of type $l$ and Lemma 4.2 implies that $P = P_+(T)$.

Conversely, let $T$ be a Prym triple of type $l$ associated to a covering $p : C \rightarrow \mathbb{P}^1$ with branch locus $B$ and labeling class $[v]$, where $v : p^{-1}(q) \rightarrow \{1, 2, 3\}^2$ is a labeling for the fiber of $p$ over a point $q \in \mathbb{P}^1 \setminus B$. Given the monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus B, q) \rightarrow \text{Perm}(\{1, 2, 3\}^2)$ for $p$ induced by $v$, we take coordinate mappings $\rho_1, \rho_2$ such that $\rho(\beta)$ splits as $(\rho_1(\beta), \rho_2(\beta))$ for $\beta \in \pi_1(\mathbb{P}^1 \setminus B, q)$. We then have a transitive representation $\varrho : \pi_1(\mathbb{P}^1 \setminus B, q) \rightarrow \text{Perm}(\{1, 2\} \times \{1, 2, 3\})$, defined by the relations $\varrho(\beta)(1, i) = (2, \rho_2(\beta)(i, 1))$ and $\varrho(\beta)(2, j) = (1, \rho_1(\beta)(1, j))$. Using the local monodromy of $p$, one shows by analogy with the proof of Theorem 6.1 that $\varrho$ is a monodromy representation for a covering

$$h \circ f : \tilde{X} \xrightarrow{3:1} X \xrightarrow{2:1} \mathbb{P}^1,$$

where $f$ is étale and $g(X) \geq 4$. Then $\rho$ is easily seen to act as a monodromy representation for the covering that is associated to $f$ and $h$. Hence $P_+(T)$ (respectively $p$) is the Prym variety (respectively covering) associated to $f$ and $h$. $\square$

References