JOURNAL OF DIFFERENTIAL EQUATIONS 116, 419–430 (1995)

Exceptional Solutions of Hill Equations*

Á. ELBERT

Mathematical Institute of Hungary, Academy of Sciences, Budapest, Realtanoda u. 13–15, 1053, Hungary

AND

F. NEUMAN

Mathematical Institute of the Academy of Sciences, Žižkova 22, CZ-61662 Brno, The Czech Republic

Received November 2, 1992

A unified global classification of Hill equations summarizing classical and other results is presented. Connection between Lyapunov exponent and the rotation number of Hill equation is explained. It is shown that the notion of exceptional solutions introduced in [Elbert *et al.*, *Differential Integral Equations* 5 (1992), 945–960] is applicable to every Hill equation. A complete list of exceptional solutions for all Hill equations is given.

1. INTRODUCTION

The notion of the principal solution as an exceptional solution of a non-oscillatory second order linear differential equation is connected with the names W. Leighton, M. Morse, and Ph. Hartman (see, e.g., [4]).

In the oscillatory case the situation is rather complicated. For certain oscillatory equations it is possible to extend the notion of a principal solution. For another class of oscillatory equations (including Bessel, Airy, and other equations) in contrast to *single* exceptional solutions, exceptional *pairs* of solutions were introduced in [3], namely a *principal pair* and an *extremal pair of solutions*. It is shown that each Hill equation admits either a principal solution or an extremal pair of solutions.

In Section 2 of this paper a unified global classification of Hill equations is presented summarizing classical and other results on these equations. It is shown how the Lyapunov exponent and the rotation number of the dynamical system induced by the Hill equation occur naturally in this classification. In Section 3 the notion of the principal solution is extended

* The research was supported by the Hungarian National Foundation for Scientific Research Grant 6032/6319 and by Czechoslovak Academy of Sciences Grant 11902.

0022-0396/95 $6.00

to some oscillatory equations and some definitions concerning extremal pairs of solutions, introduced in [3], are given. Then the complete description of principal solutions and extremal pairs of solutions for all Hill equations is given in our Theorem. Section 4 contains some auxiliary results needed for the proof of Theorem in Section 5.

2. Classification of Hill Equations

First we summarize the known facts from the theory of Hill equations, see, e.g., [1], [5] or [7].

Let $u_1(t)$, $u_2(t)$ be two solutions of the differential equation

$$y''=q(t)\,y, \qquad q\in C^0(\mathbb{R}), \qquad q(t+\pi)=q(t), \tag{1}$$

subject to the initial conditions

$$u_1(0)=0,\ u_1'(0)=1,\ u_2(0)=1,\ u_2'(0)=0.$$

The matrix

$$H=\begin{pmatrix} u_1'(\pi) & u_1(\pi) \\ u_2'(\pi) & u_2(\pi) \end{pmatrix}$$

plays an important role in global behavior of solutions of (1). Its characteristic equation is

$$\lambda^2-d\lambda+1=0,$$

where $d=u_1'(\pi)+u_2(\pi)$. The roots λ_1 and λ_2 can be written in the form

$$\lambda_1=e^{(\varrho+i\nu)\pi}, \qquad \lambda_2=e^{-(\varrho+i\nu)\pi}, \qquad \varrho\geqslant 0,\ \nu\geqslant 0. \tag{2}$$

The relation $d=\lambda_1+\lambda_2$ implies

$$\begin{aligned} d&=2\cosh\varrho\pi\cdot\cos\nu\pi \\ 0&=\varrho\sin\nu\pi. \end{aligned} \tag{3}$$

We distinguish the following cases (see also [2] or [8]).

Case $d>2$. The roots λ_1 and λ_2 are real, positive, and different, $\lambda_1>1>\lambda_2>0$, $\lambda_1=1/\lambda_2$. There exist two linearly independent solutions y_1, y_2 of (1) which can be written in the form

$$\begin{aligned} y_1(t) &= e^{\varrho t} p_1(t), & p_1(t+\pi) &= p_1(t), & p_1 &\in C^2(\mathbb{R}) \\ y_2(t) &= e^{-\varrho t} p_2(t), & p_2(t+\pi) &= p_2(t), & p_2 &\in C^2(\mathbb{R}). \end{aligned} \tag{4}$$

The number of zeros of y_1 (or y_2) on $[0, \pi)$ is even, say, $2n$ ($n = 0$ if Eq. (1) is nonoscillatory).

Case $d < -2$. The roots λ_1 and λ_2 are negative, $\lambda_1 < -1 < \lambda_2 < 0$. There exist two linearly independent solutions of (1) of the form

$$\begin{aligned} y_1(t) &= e^{\varrho t} p_1(t), & p_1(t+\pi) &= -p_1(t), & p_1 &\in C^2(\mathbb{R}) \\ y_2(t) &= e^{-\varrho t} p_2(t), & p_2(t+\pi) &= -p_2(t), & p_2 &\in C^2(\mathbb{R}). \end{aligned} \tag{5}$$

Equation (1) is oscillatory, the number of zeros of y_1 on $[0, \pi)$ is odd, say, $2n + 1$.

Case $d = 2\varepsilon$, $\varepsilon = \pm 1$. Then $\lambda_1 = \lambda_2 = \varepsilon$ and this case is divided into two subcases.

Subcase 1°, when the Jordan form of the matrix H is $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)$. Then there exist linearly independent solutions of the form

$$\begin{aligned} y_1(t) &= p_1(t), & p_1(t+\pi) &= \varepsilon p_1(t), & p_1 &\in C^2(\mathbb{R}) \\ y_2(t) &= p_2(t), & p_2(t+\pi) &= \varepsilon p_2(t), & p_2 &\in C^2(\mathbb{R}). \end{aligned} \tag{6}$$

Subcase 2°, when the Jordan form of H is $\left(\begin{smallmatrix} \varepsilon & 0 \\ 1 & \varepsilon \end{smallmatrix}\right)$. Then for an arbitrary constant $c \neq 0$ there are two linearly independent solutions of (1) of the form

$$\begin{aligned} y_1(t) &= p_1(t), & p_1(t+\pi) &= \varepsilon p_1(t), & p_1 &\in C^2(\mathbb{R}) \\ y_2(t) &= p_2(t) + ctp_1(t), & p_2(t+\pi) &= \varepsilon p_2(t), & p_2 &\in C^2(\mathbb{R}). \end{aligned} \tag{7}$$

Denote the number of zeros of y_1 on $[0, \pi)$ by $2n$ if $\varepsilon = 1$ and $2n + 1$ if $\varepsilon = -1$.

Case $|d| < 2$. There are two linearly independent solutions of (1) of the form

$$\begin{aligned} y_1(t) &= \frac{\cos(P(t) + (2n+\mu)t)}{\sqrt{P'(t) + 2n + \mu}} \\ y_2(t) &= \frac{\sin(P(t) + (2n+\mu)t)}{\sqrt{P'(t) + 2n + \mu}}, \end{aligned} \tag{8}$$

where $P \in C^3(\mathbb{R})$ is a π-periodic function, $n \geqslant 0$ is an integer, $\mu \in (0, 1) \cup (1, 2)$, $P'(t) + 2n + \mu > 0$, and the roots λ_1 and λ_2 satisfy

$$\lambda_1 = e^{i(2n+\mu)} \qquad \text{and} \qquad \lambda_2 = e^{-i(2n+\mu)}.$$

Remark 1. This result was established by the second author in [8].

Let $N(T)$ denote the number of the zeros of a solution of (1) on the interval $[0, T)$ (see also [4]). We find that for each Hill equation (1) there exists the limit

$$\lim_{T \to \infty} \frac{\pi N(T)}{T} = \nu^*$$

independent of the chosen solution. In particular, $\nu^* = 0$ if (1) is nonoscillatory. The number ν^* can be viewed as an average number of the zeros of any solution of (1) on an interval of the length π, the period of the coefficient $q(t)$ of (1). The general theory of dynamical systems shows that ν^* is the rotation number of the dynamical system induced by Eq. (1) (see, e.g., in [6]). According to this theory the rotation number depends continuously on the coefficient $q(t)$.

The connection between ν^* and the qualitative properties of the solutions is given by the following observation.

PROPOSITION 1. *We have*

$$\nu^* = \begin{cases} 2n & \text{for } d \geqslant 2, \\ 2n+1 & \text{for } d \leqslant -2, \\ 2n+\mu & \text{in the case } |d| < 2. \end{cases}$$

Moreover, we may put $\nu = \nu^*$ in (2).

Proof. For the cases $d > 2$, $d = 2$, $d = -2$, and $d < -2$ Proposition 1 is an immediate consequence of the expressions (4)–(7) and the specification of the integer n there. Case $|d| < 2$ is the same as the Corollary in [8].

Remark 2. The Lyapunov exponent of the solutions of Eq. (1) (see, e.g., [4]) is given by the formula

$$\limsup_{t \to \infty} \frac{\log \sqrt{y_1^2(t) + y_2^2(t)}}{t}.$$

It can easily be checked in the all above cases that the Lyapunov exponent equals ϱ in (2). Its connection with the rotation number ν^* is given in (3) with $\nu = \nu^*$.

3. Exceptional Solutions

Let us recall some facts about exceptional solutions. For a nonoscillatory equation (1) there always exists a solution $y(t)$ satisfying

$$\int^{\infty} \frac{dt}{y(t)^2} = \infty,$$

called principal (at ∞), while for any other solution $\tilde{y}(t)$, independent of $y(t)$, we have (see [4]):

$$\int^{\infty} \frac{dt}{\tilde{y}(t)^2} < \infty.$$

For an oscillatory equation (1) consider a pair (y_1, y_2) of solutions with Wronskian 1 and consider—in accordance with [3]—the limits

$$A = \lim_{T \to \infty} \frac{\int_0^T y_1'^2\, dt}{\int_0^T (y_1'^2 + y_2'^2)\, dt},$$

$$B = \lim_{T \to \infty} \frac{\int_0^T y_1' y_2'\, dt}{\int_0^T (y_1'^2 + y_2'^2)\, dt},$$

$$C = \lim_{T \to \infty} \frac{\int_0^T y_2'^2\, dt}{\int_0^T (y_1'^2 + y_2'^2)\, dt}.$$

Let us note here that the condition on Wronskian to be 1 is not essential in defining the quantities A, B, and C.

If these limits exist, Eq. (1) is called *regular*. It will turn out that Hill equations are regular. Clearly, we have $A \geqslant 0$, $C \geqslant 0$, $A + C = 1$, and for $D = AC - B^2$ the relation $0 \leqslant D \leqslant 1/4$ holds. A regular equation (1) is *degenerate* if $D = 0$. Theorem 4.1 in [3] guarantees that for regular equation (1) which is *not degenerate*, i.e., for which $D > 0$, there always exists a pair $(\bar{y}_1, \bar{y}_2)$ of solutions with Wronskian 1 so that

$$\bar{y}_1^2 + \bar{y}_2^2 = \bar{a} y_1^2 + \bar{b} y_1 y_2 + \bar{c} y_2^2,$$

where

$$(\bar{a}, \bar{b}, \bar{c}) = \left(\frac{C}{\sqrt{D}}, -\frac{2B}{\sqrt{D}}, \frac{A}{\sqrt{D}} \right).$$

In fact, this triple $(\bar{a}, \bar{b}, \bar{c})$ gives the minimum of the expression

$$L = aA + bB + cC \qquad \text{on} \quad \{(a, b, c) \in \mathbb{R}^3;\ a > 0,\ c > 0,\ 4ac - b^2 = 4\}.$$

This is the reason why the pair $(\bar{y}_1, \bar{y}_2)$ is called *extremal*. An extremal pair is unique up to orthogonal transformations, see again [3].

In the degenerate case, $D=0$, there exits a solution $\bar{y}_1$ unique up to a constant factor for which $A=0$. We call this solution *principal*. In this way we *extend* the notion of *principal solutions* to some *oscillatory equations*.

Our main result concerning the exceptional solutions refers to the cases and notations given in Section 2 and reads as follows.

THEOREM. *The Hill equation has the following exceptional solutions*:

in case $|d|>2$ *solution* y_2 *in* (4) *or in* (5) *is the only solution of* (1) *which tends to zero as* $t\to\infty$. *It is a principal solution at* $+\infty$ *both in the non-oscillatory and in the oscillatory case, as well. In the latter case Eq.* (1) *is degenerate. Similar statement holds for* y_1 *at* $-\infty$;

in case $|d|=2$, *subcase* 1° *Eq.* (1) *is oscillatory, regular, and not degenerated. There exists a unique* (*up to an orthogonal transformation*) *extremal pair of solutions*;

in case $|d|=2$, *subcase* 2° *if Eq.* (1) *is nonoscillatory then the periodic solution* y_1 *in* (7) *is principal. If* (1) *is oscillatory then it is degenerated and the periodic solution* y_1 *in* (7) *is a principal solution* (*both at* ∞ *and* $-\infty$);

in case $|d|<2$ *Eq.* (1) *is oscillatory, not degenerated. The pair*

$$y_1(t)=\frac{\cos(P(t)+\nu^*t)}{\sqrt{P'(t)+\nu^*}},\qquad y_2(t)=\frac{\sin(P(t)+\nu^*t)}{\sqrt{P'(t)+\nu^*}}$$

in (8) *is an extremal pair.*

4. LEMMAS

We derive some estimates on integrals connected with periodic functions. These estimates may have an independent interest in their own right.

LEMMA 1. *Let* $P(t)$ *be a continuous and* π*-periodic function. For* $a>0$ *define*

$$\varphi(T)=\int_0^T e^{at}P(t)\,dt \qquad and \qquad Q(T)=\int_0^\pi e^{at}P(T+t)\,dt.$$

Then

$$\varphi(T)=\frac{e^{aT}Q(T)-\varphi(\pi)}{e^{a\pi}-1}.$$

Proof of Lemma 1. Since

$$\varphi(T+\pi)=\varphi(T)+\int_T^{T+\pi} e^{at}P(t)\,dt=\varphi(\pi)+\int_\pi^{\pi+T} e^{at}P(t)\,dt,$$

we obtain

$$\varphi(T)+e^{aT}\int_0^\pi e^{at}P(T+t)\,dt=\varphi(\pi)+e^{a\pi}\varphi(T),$$

whence the relation follows.

Now, for $n=0, 1, 2, \ldots$ and for a function $P(t)$ as in Lemma 1 we define the functions

$$\varphi_n(T)=\int_0^T t^n P(t)\,dt. \tag{9}$$

LEMMA 2. *For the functions $\varphi_n(T)$ defined by (9) we have the representation*

$$\varphi_n(T)=\frac{1}{n+1}T^{n+1}\frac{\int_0^\pi P(t)\,dt}{\pi}+T^nQ_n(T)+\cdots+TQ_1(T)+Q_0(T),$$

where $Q_i(T)$ are π-periodic functions for $i=0, 1, \ldots, n$, n fixed.

Proof of Lemma 2. The proof is given by induction. For $n=0$ consider the function

$$Q_0(T)=\varphi(T)-T\frac{\int_0^\pi P(t)\,dt}{\pi},$$

which is evidently π-periodic. Suppose Lemma 2 holds for $n=0, 1, \ldots, k-1$ ($k\geqslant 1$). Then the case $n=0$ implies

$$\int_0^T P(t)\,dt=T\frac{\int_0^\pi P(t)\,dt}{\pi}+Q_0(T)$$

and an integration by parts gives

$$\begin{aligned}\phi_k(T)&=\int_0^T t^kP(t)\,dt\\&=\left[t^k\left(t\frac{\int_0^\pi P(t)\,dt}{\pi}+Q_0(t)\right)\right]_0^T-\int_0^T kt^{k-1}\left(t\frac{\int_0^\pi P(t)\,dt}{\pi}+Q_0(t)\right)dt\\&=\frac{T^{k+1}}{k+1}\frac{\int_0^\pi P(t)\,dt}{\pi}+T^kQ_0(T)-\int_0^T kt^{k-1}Q_0(t)\,dt,\end{aligned}$$

where the last integral can be evaluated by the induction assumption, which completes the proof of Lemma 2.

LEMMA 3. *Let*

$$F(T)=\int_0^T \cos 2\alpha \cdot h(t)\,dt, \qquad G(T)=\int_0^T \sin 2\alpha \cdot h(t)\,dt,$$

where $\alpha=\alpha(t)=P(t)+\nu t$, ν *is positive and not an integer, and* P, h *are continuous* π*-periodic functions. Then* $F(T)$ *and* $G(T)$ *are bounded on* $\mathbb{R}$.

Proof of Lemma 3. Put $\beta=\alpha(\pi)-\alpha(0)$. Then $\alpha(t+\pi)=\beta+\alpha(t)$ and $\beta=\nu\pi$ with $\sin\beta\neq 0$. Hence

$$F(T+\pi)=F(t)+\int_T^{T+\pi}\cos 2\alpha\cdot h\,dt=F(\pi)+\int_\pi^{T+\pi}\cos 2\alpha\cdot h\,dt$$
$$=F(\pi)+\cos 2\beta\,F(T)-\sin 2\beta\,G(T);$$

similarly

$$G(T+\pi)=G(T)+\int_T^{T+\pi}\sin 2\alpha\cdot h\,dt=G(\pi)+\cos 2\beta\,G(T)+\sin 2\beta\,F(T).$$

The system of linear equations for the functions $F(T)$ and $G(T)$,

$$(1-\cos 2\beta)\,F(T)+\sin 2\beta\,G(T)=F(\pi)-\int_T^{T+\pi}\cos 2\alpha\cdot h\,dt$$

$$-\sin 2\beta\,F(T)+(1-\cos 2\beta)\,G(T)=G(\pi)-\int_T^{T+\pi}\sin 2\alpha\cdot h\,dt,$$

has been found, where the determinant

$$\begin{vmatrix} 1-\cos 2\beta & \sin 2\beta \\ -\sin 2\beta & 1-\cos 2\beta \end{vmatrix}=4\sin^2\beta>0.$$

Since the integrals on the right hand side of the system are periodic, consequently they are uniformly bounded on $\mathbb{R}$, Lemma 3 is proved.

5. PROOF OF THEOREM

The proof follows the cases as they occur in Theorem.

Case $d>2$. If Eq. (1) is nonoscillatory then consider the solution y_2 in (4), where $p_2(t)$ is a π-periodic and nowhere vanishing function. Then for any integer $k>0$

$$\int_0^{k\pi}\frac{dt}{y_2(t)^2}=\int_0^{k\pi}\frac{e^{2\rho t}\,dt}{p_2^2(t)}>k\int_0^{\pi}\frac{dt}{p_2^2(t)}.$$

Hence the solution y_2 is principal at $+\infty$.

Now we may assume that Eq. (1) is oscillatory. By using formulas (4) and Lemma 1 we obtain

$$\int_0^T y_1'^2(t)\,dt + \int_0^T e^{2\rho t}(p_1' + \rho p_1)^2\,dt = \frac{e^{2\rho T}Q(T) - \int_0^\pi y_1'^2(t)\,dt}{e^{2\rho\pi} - 1},$$

where $Q(T) > 0$ and π-periodic, and similarly

$$\int_0^T y_2'^2(t)\,dt < \int_0^\infty y_2'^2(t)\,dt < \infty.$$

Hence the limit A, defined in Section 3, exists; $A = 1$, hence $C = 0$. Moreover the relation $D = AC - B^2 \geqslant 0$, also mentioned in Section 3, implies $B = 0$ and $D = 0$. Therefore, in the case $d > 2$ the oscillatory equation (1) is degenerate. Up to a nonzero constant factor, solution y_2 in (4) is the principal solution of Eq. (1). This solution tends to zero as $t \to \infty$, while any other linearly independent solution of (1) is unbounded. A similar role is played by y_1 as $t \to -\infty$.

Case $d < -2$. Now Eq. (1) is oscillatory. Analogously to the case $d > 2$, we have $A = 1$, $B = 0$, $C = 0$, $D = 0$ by using expressions (5). This means that Eq. (1) is degenerate and y_2 is a principal solution at ∞.

Case $|d| = 2$, *subcase* 1°. First we show that (1) is oscillatory. This is evident when $d = -2$. If for $d = 2$, i.e., for $\varepsilon = 1$, Eq. (1) would be non-oscillatory then the periodic functions $p_1(t)$ and $p_2(t)$ in (6) have no zeros. Consequently we would have

$$\int^\infty \frac{dt}{y_i^2(t)} = \infty \qquad \text{for} \quad i = 1, 2,$$

contradicting the fact that this relation can hold only for one, for the principal solution. So (1) is oscillatory.

By using formulas (6) and Lemma 2 for $n = 0$, we get for $i = 1, 2$

$$\int_0^T p_i'^2(t)\,dt = T\,\frac{\int_0^\pi p_i'^2(t)\,dt}{\pi} + Q_{0i}(T),$$

where Q_{0i} is a π-periodic function. Hence

$$A = \frac{\int_0^\pi p_1'^2\,dt}{\int_0^\pi (p_1'^2 + p_2'^2)\,dt}, \qquad B = \frac{\int_0^\pi p_1' p_2'\,dt}{\int_0^\pi (p_1'^2 + p_2'^2)\,dt}, \qquad C = \frac{\int_0^\pi p_2'^2\,dt}{\int_0^\pi (p_1'^2 + p_2'^2)\,dt},$$

$$D = AC - B^2 = \frac{\int_0^\pi p_1'^2\,dt \int_0^\pi p_2'^2\,dt - \left(\int_0^\pi p_1' p_2'\,dt\right)^2}{\left(\int_0^\pi (p_1'^2 + p_2'^2)\,dt\right)^2}$$

are well-defined. Moreover we claim that $D>0$, i.e., Eq. (1) is not degenerated. Indeed, consider the quadratic polynomial $R(x)$ of x

$$R(x)=\int_0^\pi [p_1'(t)-xp_2'(t)]^2\,dt.$$

Evidently, $R(x)>0$ for all x, because $R(x_0)=0$ would imply $p_1'(t)=x_0p_2'(t)$, which gives for the Wronskian W of $y_1=p_1$ and $y_2=p_2$

$$W=y_1'y_2-y_1y_2'=p_2'[x_0p_2-p_1].$$

However, the Wronskian W is a nonzero constant, so the function p_2' should never vanish, contradicting the fact that p_2 is a periodic function. Hence we have

$$R(x)=\int_0^\pi p_1'^2\,dt-2x\int_0^\pi p_1'p_2'\,dt+x^2\int_0^\pi p_2'^2\,dt>0$$

for all $x\in\mathbb{R}$, which yields the negative discriminant

$$\left(\int_0^\pi p_1'p_2'\,dt\right)^2-\int_0^\pi p_1'^2\,dt\int_0^\pi p_2'^2\,dt<0,$$

as we needed for $D>0$. Hence Eq. (1) is regular, not degenerate and it admits an extremal pair of solutions due to Theorem 4.1 in [3].

Case $|d|=2$, *subcase* 2°. For $\varepsilon=1$ Eq. (1) may be nonoscillatory. The solution $y_1(t)$ is periodic, hence $\int^\infty dt/y_1^2=\infty$, i.e., $y_1(t)$ is the principal solution.

For oscillatory equation (1) we have to calculate the quantities A, B, C. Making use of formula (7) and Lemma 2 with $n=0$ we get

$$\int_0^T y_1'^2\,dt=T\,\frac{\int_0^\pi p_1'^2\,dt}{\pi}+Q_0(T)$$

$$\begin{aligned}\int_0^T y_2'^2\,dt&=\int_0^T [ctp_1'+cp_1+p_2']^2\,dt\\&=c^2\int_0^T t^2p_1'^2\,dt+2c\int_0^T tp_1'[cp_1+p_2']\,dt+\int_0^T [2cp_1p_2'+p_2'^2]\,dt\\&=c^2\,\frac{T^3}{3}\,\frac{\int_0^\pi p_1'^2\,dt}{\pi}+\mathcal{O}(T^2),\end{aligned}$$

where $\mathcal{O}(T^2)/T^2$ is bounded as $T \to \infty$. Hence we have $C = 1$, and consequently $A = B = D = 0$. We conclude that Eq. (1) is degenerate and its principal solution is y_2 which is periodic at the same time.

Case $|d| < 2$. Consider the pair (y_1, y_2) in (8). Equation (1) is evidently oscillatory and the Wronskian of this pair is 1. Now we shall evaluate the quantities A, B, C. Due to Proposition 1 we have $2n + \mu = v^*$. Put $\alpha(t) = P(t) + v^* t$. Then $y_1' = -\sin\alpha \cdot \sqrt{\alpha'} - 1/2 \cos\alpha \cdot (\alpha')^{-3/2} P''$, hence

$$\int_0^T y_1'^2\, dt = \int_0^T \sin^2\alpha \cdot \alpha'\, dt + \int_0^T \sin\alpha \cdot \cos\alpha \cdot \frac{P''}{\alpha'}\, dt + \frac{1}{4}\int_0^T \cos^2\alpha \cdot \frac{P''^2}{\alpha'^3}\, dt.$$

Here α' and P'' are π-periodic functions. Since $\cos^2\alpha = (1 + \cos 2\alpha)/2$, $\sin^2\alpha = (1 - \cos 2\alpha)/2$, we can apply Lemmas 2 and 3 to get

$$\int_0^T y_1'^2\, dt = \frac{1}{2}\alpha(T) + \frac{T}{8\pi}\int_0^\pi \frac{P''^2}{(\alpha')^3}\, dt + \mathcal{O}(1),$$

where $\alpha(T) = v^* T + \mathcal{O}(1)$. Similarly

$$\int_0^T y_2'^2\, dt = \frac{1}{2}\alpha(T) + \frac{T}{8\pi}\int_0^\pi \frac{P''^2}{(\alpha')^3}\, dt + \mathcal{O}(1),$$

$$\int_0^T y_1' y_2'\, dt = \mathcal{O}(1).$$

Hence $A = 1/2$, $B = 0$, $C = 1/2$, and $D = 1/4$; consequently Eq. (1) is regular, non degenerated, and

$$\left(\frac{C}{\sqrt{D}}, -\frac{2B}{\sqrt{D}}, \frac{A}{\sqrt{D}}\right) = (1, 0, 1),$$

showing that y_1, y_2 in (10) is an extremal pair of (1). This completes the proof of Theorem.

References

1. L. Cesari, "Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations," Nauka, Moscow, 1964. [Russian translation]
2. M. S. P. Eastham, "The Spectral Theory of Periodic Differential Equations," Scottish Academic Press, Edinburgh/London, 1973.
3. Á. Elbert, F. Neuman, and J. Vosmanský, Principal pairs of solutions of linear second order oscillatory differential equations, *Differential Integral Equations* **5** (1992), 945–960.

4. PH. HARTMAN, "Ordinary Differential Equations," Wiley, New York/London, 1964.
5. J. HORN, "Gewöhnliche Differentialgleichungen," W. de Gruyter, Berlin/Leipzig, 1937.
6. I. P. KORNFELD, YA. G. SINAI, AND S. V. FOMIN, "Ergodic Theory," Nauka, Moscow, 1980. [in Russian]
7. W. MAGNUS AND S. WINKLER, "Hill's Equations," Wiley, New York/London, 1966.
8. F. NEUMAN, Note on bounded non-periodic solutions of second order linear differential equations with periodic coefficients, *Math. Nachr.* **39** (1969), 217–222.