New Statistical Investigations of the Ornstein-Uhlenbeck Process

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Abstract—An asymptotic analysis is presented for estimation in the three-parameter Ornstein-Uhlenbeck process, where the parameters are the local mean, the drift, and the variance. We are interested in the case when the damping parameter (λ, or λT = κ) is nearly zero. The asymptotic sufficient statistics can be related to noncentral \chi^2 distribution. The maximum likelihood estimate of the parameter vector is a solution of a rather complicated system of equations. We describe the methods for solving maximum-likelihood equations. Classical and robust estimators are determined for parameters. It is shown that the lower confidence limit of the drift (or damping) parameter is equal to zero with positive probability when it is near to zero. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let us consider the stochastic differential equation

\[ d\xi(t) = -\lambda\xi(t) dt + \sigma w dw(t), \quad E(\xi(t)) = E(w(t)) = 0, \quad \lambda > 0, \]

where \( w \) is the standard Wiener process. \( \xi(t) \) is called an Ornstein-Uhlenbeck or AR(1) process. We are interested in the estimation of \( \lambda, \sigma_w \) and in the case when the observed process is

\[ \eta(t) = \xi(t) + m, \]

in the estimation of \( m \), as well. We examine only the stationary case, i.e., \( \lambda > 0 \). Now, let us suppose that one can observe a realization of \( \xi(t) \) (or \( \eta(t) \)), \( 0 \leq t \leq T \). By a well-known theorem of Baxter [1], \( \sigma_w^2 \) can be estimated by the use of the relation

\[ \lim_{\max(t_k - t_{k-1}) \to 0} \sum \left[ \xi(t_k) - \xi(t_{k-1}) \right]^2 = \sigma_w^2 T, \quad \text{with probability 1}, \]

with probability 1.
where $0 = t_0 < t_1 < \cdots < t_n = T$. Indeed, $\sigma_w^2$ is the only parameter which can be estimated with probability 1 from a realization. Our main aim is to investigate the behaviour of estimators when $\lambda \to 0$ (nearly nonstationarity) and the parameters $m$ and $\lambda$ are unknown. In order to understand this phenomenon, it is necessary to investigate the likelihood (i.e., Radon-Nikodym derivative with respect to some dominating measure on the space of realizations) for the Ornstein-Uhlenbeck model. There exists a huge body of literature of the asymptotic behaviour of the maximum likelihood and other estimators (see [2-5]).

The rest of the paper is organized as follows. In Section 2, we formalize our problem and determine the maximum likelihood estimators for the parameters $m$ and $\lambda$. Furthermore, we give expectations and variances of sufficient statistics and derive the generalized expansions for them. In Section 3, we describe some distributions which are connected to the sufficient statistics when $\lambda \to 0$, and derive the maximum likelihood and robust estimators for distributions used in simulations. Some conclusions and conjectures can be drawn from the simulation results presented in Section 4.

2. MAXIMUM LIKELIHOOD ESTIMATORS

In this section, we present the likelihood for a continuous observation, called realization, $(\eta(t) : 0 \leq t \leq T)$ of the Ornstein-Uhlenbeck process. The derivations are standard and hence omitted. When the parameters $m$ and $\lambda$ are unknowns, we can write the Radon-Nikodym derivative in the following form (see [2]):

$$\left. \frac{\lambda}{\sigma_w^2} \right| s_1^2 + \frac{1}{2} \lambda T s_2^2 + (m - m_1)^2 + \frac{\lambda T}{2} (m - m_2)^2 - \frac{1}{2} \sigma_w^2 T \right\} \right|,$$

where

$$m_1 = \frac{\eta(0) + \eta(T)}{2}, \quad (2.2)$$

$$m_2 = \frac{1}{T} \int_0^T \eta(t) \, dt, \quad (2.3)$$

$$s_1^2 = \frac{[\eta(0) - m_1]^2 + [\eta(T) - m_1]^2}{2} = \frac{[\eta(T) - \eta(0)]^2}{4}, \quad (2.4)$$

$$s_2^2 = \frac{1}{T} \int_0^T [\eta(t) - m_2]^2 \, dt. \quad (2.5)$$

From (2.1) we can conclude that $m_1, m_2, s_1^2,$ and $s_2^2$ form a system of sufficient statistics. One can get simple estimators by the method of moments

$$E(m_1) = m, \quad E(m_2) = m, \quad E\left( \frac{\xi^2(0) + \xi^2(T)}{2} \right) = \frac{1}{2\lambda}, \quad E\left( \frac{1}{T} \int_0^T \xi^2(t) \, dt \right) = \frac{1}{2\lambda}.$$

The maximum likelihood equations of $m$ and $\lambda$ are as follows:

$$\frac{\sigma_w^2}{2\lambda} (1 + \lambda T) - s_1^2 - \lambda Ts_2^2 - (m - m_1)^2 - \lambda T (m - m_2)^2 = 0, \quad (2.6)$$

$$2(m - m_1) + \lambda T (m - m_2) = 0. \quad (2.7)$$

Let $\kappa = \lambda T$; then

$$2(m - m_1) + \kappa (m - m_2) = 0, \quad (2.8)$$

$$[2s_2^2 + 2(m - m_2)^2] \kappa^2 + [2s_1^2 + 2(m - m_1)^2 - \sigma_w^2 T] \kappa - \sigma_w^2 T = 0. \quad (2.9)$$
Moreover, by transformation
\[
t' = \frac{t}{T}, \quad \tilde{\eta}(t') = \frac{\eta(t/T)}{\sigma_w T},
\]
we may assume that \( \sigma_w = 1 \), \( T = 1 \) and \( \kappa = \lambda T \), and \( m \) are the parameters. The results do not depend on \( T \), i.e., on the length of observation time.

From (2.8), we see that the maximum likelihood estimators are related by
\[
m = \frac{2m_1 + \kappa m_2}{2 + \kappa},
\]
and the solution for \( \kappa \) is a root of an equation of the fourth degree
\[
2s_2^2 \kappa^4 + \left[ 8s_2^2 + 2s_1^2 + 2(m_1 - m_2)^2 - 5\sigma_w^2 T \right] \kappa^3
+ \left[ 8s_2^2 + 8s_1^2 + 8(m_1 - m_2)^2 - 5\sigma_w^2 T \right] \kappa^2
+ \left[ 8s_1^2 - 8\sigma_w^2 T \right] \kappa - 4\sigma_w^2 T = 0.
\]

We introduce the following notations of the coefficients:
\[
A = 2s_2^2,
B = 8s_2^2 + 2s_1^2 + 2(m_1 - m_2)^2 - 5\sigma_w^2 T,
C = 8s_2^2 + 8s_1^2 + 8(m_1 - m_2)^2 - 5\sigma_w^2 T,
D = 8s_1^2 - 8\sigma_w^2 T,
E = -4\sigma_w^2 T.
\]

One can observe that
\[
A > 0,
4B = 32s_2^2 + 8s_1^2 + 8(m_1 - m_2)^2 - 4\sigma_w^2 T,
C = 4B - 24s_2^2 - 4\sigma_w^2 T,
D = C - 8s_1^2 - 3\sigma_w^2 T - 8(m_1 - m_2)^2,
E < 0.
\]

Furthermore, if \( B < 0 \), then \( C < 0 \), and if \( C < 0 \), then \( D < 0 \); that is, the number of changes of sign of the sequence of the coefficients equals 1. Therefore, the number of positive zeros of equation (2.11) equals 1 by Descartes’ rule of signs (see [6, Volume II, Part Five, Chapter 1, Problems 36,37]).

We got the following statement. The polynomial (2.11) has only one positive root; that is, the stationary solution is unique.

The roots of a polynomial of fourth degree can be determined by the well-known algebraic method. But, as we want to find the positive root, the numerical approximation is more simple.

The greatest root of equation (2.11) is determined by Newton’s method. Let \( \alpha_n \) be the greatest root of the polynomial
\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0,
\]
and \( x_0 > \alpha_n \). We construct a sequence
\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},
\]
with initial value \( x_0 \). Let \( \alpha_n^{(k)} = x_k - \alpha_n \). Then,
\[
\alpha_n^{(k)} > 0, \quad \forall k \in \mathbb{N},
\]
\[
\alpha_n^{(k)} \leq \left( \frac{n-1}{n} \right)^k (x_0 - \alpha_n), \quad \text{that is,} \quad x_k \to \alpha_n, \quad k \to \infty,
\]
\[
\alpha_n^{(k)} \leq (n-1)(x_{k-1} - x_k);
\]
the last estimation cannot decrease in the general case.
To find the upper bound $x_0$ of the positive roots of an algebraic equation, we can make use of Lagrange’s method. If the coefficients of the polynomial $f(x)$, given above, satisfy the conditions $a_n > 0$, $a_{n-1} \geq 0, \ldots, a_{k+1} \geq 0$, $a_k < 0$, then the upper bound of the positive roots of the equation $f(x) = 0$ can be found from the formula

$$x_0 = 1 + \sqrt[n]{-\frac{B}{a_n}},$$

where $B$ is the greatest of the absolute values of the negative coefficients of $f(x)$. We shall now describe another recursive method, which is called “iteration with roots”, to determine the positive root,

$$m^{(0)} = m_2,$$

$$\kappa^{(k)} = \frac{\sigma_w^2 \lambda - 2s_1^2 - 2(m^{(k)} - m_1)^2}{2[s_2^2 + 2(m^{(k)} - m_2)^2]}$$

$$+ \frac{\sqrt{[2s_1^2 + 2(m^{(k)} - m_1)^2] - \sigma_w^2 \lambda + 8\sigma_w^2 [s_2^2 + (m^{(k)} - m_2)^2]}}{2[s_2^2 + 2(m^{(k)} - m_2)^2]},$$

$$m^{(k+1)} = \frac{m_1 + \kappa^{(k)} m_2}{2 + \kappa^{(k)}}.$$

We compare the two methods by simulations. It turns out that the “iteration with roots” method is faster.

In the rest of this section, we summarize some results for the expectations and the variances of the statistics which take place in the maximum likelihood equations (2.6) and (2.7). To calculate the moments of statistics $m_1$, $m_2$, $s_1^2$, $s_2^2$ directly, one has to use the well-known relation for Gaussian random variables

$$E(\eta_1 \eta_2 \eta_3 \eta_4) = E(\eta_1 \eta_2)E(\eta_3 \eta_4) + E(\eta_1 \eta_3)E(\eta_2 \eta_4) + E(\eta_1 \eta_4)E(\eta_2 \eta_3).$$

We have (see [2], where the characteristic function is given in Section 3.4)

$$D^2(m_1) = \sigma_w^2 \frac{1 + e^{-\lambda T}}{4\lambda},$$

$$D^2(m_2) = \sigma_w^2 \frac{(e^{-\lambda T} + \lambda T - 1)}{2T\lambda^2},$$

$$\text{cov}(m_1, m_2) = \sigma_w^2 \frac{1 - e^{-\lambda T}}{4T\lambda},$$

$$E(s_1^2) = \sigma_w^2 \frac{1 - e^{-\lambda T}}{4T\lambda},$$

$$D^2(s_1^2) = \sigma_w^2 \frac{(1 - e^{-\lambda T})^2}{8T^2\lambda^2},$$

$$E(s_2^2) = \sigma_w^2 \frac{1}{2\lambda} - \frac{\sigma_w^2}{\lambda \kappa} \left[ 1 + \frac{1}{\kappa} (e^{-\kappa} - 1) \right],$$

$$D^2(s_2^2) = \frac{\sigma_w^4}{4\lambda^2 \kappa} \left[ 2 + \frac{e^{-2\kappa} - 1}{\kappa} + \frac{8(\kappa + e^{-\kappa} - 1)^2}{\kappa^3} \right]$$

$$- \frac{\sigma_w^4}{\lambda^2 \kappa^3} \left( 4\kappa + 2\kappa e^{-\kappa} - 7 + 8e^{-\kappa} - e^{-2\kappa} \right).$$
We assume in the following that $T = 1$ and $\sigma_w$ is known ($\sigma_w = 1$), as $\sigma_w$ can be estimated with probability 1. Furthermore, we derive the extended expansions when $\kappa \to 0$ and $\kappa \to \infty$ ($m = 0$), using the Taylor series of $e^{\pm \kappa}$.

\begin{equation}
E(m_1^2) = \begin{cases}
\frac{1}{2\kappa} - \frac{1}{4} + \frac{1}{8} \kappa - \frac{1}{24} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
\frac{1}{4\kappa^2} + \frac{1}{4\kappa \exp(\kappa)}, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
E(m_2^2) = \begin{cases}
\frac{1}{2\kappa} - \frac{1}{6} + \frac{1}{24} \kappa - \frac{1}{120} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
\frac{1}{\kappa^2} - \frac{1}{\kappa^3} + \frac{1}{\kappa^3 \exp(\kappa)}, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
E(m_1 m_2) = \begin{cases}
\frac{1}{2\kappa} - \frac{1}{4} + \frac{1}{12} \kappa - \frac{1}{48} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
\frac{1}{2\kappa^2} - \frac{1}{2\kappa^2 \exp(\kappa)}, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
E((m_1 - m_2)^2) = \begin{cases}
\frac{1}{12} - \frac{1}{120} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
\frac{1}{4\kappa^3} - \frac{1}{4\kappa^3 \exp(\kappa)} + \left(\frac{1}{4\kappa} + \frac{1}{\kappa^2} + \frac{1}{\kappa^3}\right) \frac{1}{\exp(\kappa)}, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
E(s_1^2) = \begin{cases}
\frac{1}{4} - \frac{1}{8} \kappa + \frac{1}{24} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
\frac{1}{4\kappa} - \frac{1}{4\kappa \exp(\kappa)}, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
D^2(s_1^2) = \begin{cases}
\frac{1}{8} - \frac{1}{8} \kappa + \frac{7}{96} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
2 \left(\frac{1}{4\kappa} - \frac{1}{4\kappa \exp(\kappa)}\right)^2, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
E(s_2^2) = \begin{cases}
\frac{1}{6} - \frac{1}{24} \kappa + \frac{1}{120} \kappa^2 + O(\kappa^3), & \text{if } \kappa \to 0, \\
\frac{1}{2\kappa} - \frac{1}{\kappa^2} + \frac{1}{\kappa^3} - \frac{1}{\kappa^3 \exp(\kappa)}, & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

\begin{equation}
D^2(s_2^2) = \begin{cases}
\frac{1}{45} - \frac{1}{60} \kappa + O(\kappa^2), & \text{if } \kappa \to 0, \\
\frac{1}{2\kappa^3} - \frac{9}{4\kappa^4} + \frac{3}{\kappa^5} + \frac{2}{\kappa^6} + O\left(\frac{1}{\kappa^4 \exp(\kappa)}\right), & \text{if } \kappa \to +\infty.
\end{cases}
\end{equation}

In [7] (see also [2]) the expansions of the expectation and the variance of the maximum likelihood estimation of $\lambda$ were given if the parameter $m$ was known.

3. ESTIMATORS AND DISTRIBUTIONS IN THE NEARLY NONSTATIONARY CASE

It is known [2] that when $\kappa \to 0$, the random variables $m_1$ and $s_1^2$ form asymptotically sufficient statistics. In [4] it is given, e.g., that the random variable $s_2^2$ has noncentral $\chi^2$-distribution (see [2, Section 3.4]). Therefore, in this section we shall examine distributions of $m_1$ and $s_1^2$. We shall give estimators for the location and scale parameter. We say that if $\kappa \approx 0$ ($\kappa > 0$), the process $\eta(t)$ is nearly nonstationary. In [2], one of the authors proved the following qualitative statements.

**STATEMENT 1.** (See [2, Section 3.4, p. 188].) Let $p > 0$, and let $\kappa(\eta)$ be a positive and continuous functional in the $C[0, 1]$ with the property $\kappa(\eta) \to \infty$ if $\sup |\eta(t)| \to \infty$. If for any $m$ and $\kappa$ the
condition $P_{\kappa,m}(\kappa > \kappa(\eta)) \geq p$ is satisfied, then

$$P_{\kappa,m}(\kappa(\eta) = 0) \geq g(\kappa,p),$$

where $g(\kappa,p) > 0$ does not depend on the choice of functional and $g(\kappa,p) \to 1$ as $\kappa \to 0$.

**Statement 2.** (See [2].) Let $p > 1/2$, and let $\mu(\eta)$, $\bar{\mu}(\eta)$ be real valued and continuous functionals on $C[0,1]$ (which may assume values $-\infty$, $+\infty$, respectively) satisfying for any $\kappa$, $m$ the conditions

$$P(m > \mu(\eta)) \geq p, \quad P(m \leq \bar{\mu}(\eta)) \geq p.$$

Then,

$$P(\bar{\mu}(\eta) = +\infty) \geq l(\kappa,p), \quad P(\mu(\eta) = -\infty) \geq l(\kappa,p),$$

where $l(\kappa,p)$ does not depend on the choice of the functionals $\mu$, $\bar{\mu}$ and $l(\kappa,p) \to 1/2$ as $\kappa \to 0$.

In this paper, we want to get some qualitative results on the lower confidence limit of $\kappa$, and the corresponding limits $\mu$ and $\bar{\mu}$ for $m$. The problem of estimation can be formulated in the following way. If $\kappa \to 0$, the correlation of $\eta(0)$ and $\eta(1)$ tends to 1 (this means that instead of two observations we have only one), and it is impossible to estimate two parameters $(m, \kappa)$ at the same time. (If $m$ is known, $\kappa$ can be estimated!)

The joint probability density function of a normally distributed random vector $(\xi_1, \xi_2)$ is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp \left\{ -\frac{(x_1-m)^2 - 2\rho(x_1-m)(x_2-m) + (x_2-m)^2}{2\sigma^2(1-\rho^2)} \right\}. \quad (3.1)$$

Let

$$\eta_1 = \frac{\xi_1 + \xi_2}{2}, \quad \eta_2 = \frac{(\xi_1 - \xi_2)^2}{2}, \quad (3.2)$$

with the associated transformation

$$y_1 = \frac{x_1 + x_2}{2}, \quad y_2 = \frac{(x_1 - x_2)^2}{2},$$

where the inverse functions are given by

$$x_1 = y_1 - \sqrt{\frac{y_2}{2}}, \quad x_2 = y_1 + \sqrt{\frac{y_2}{2}}, \quad \text{or}$$

$$x_1 = y_1 + \sqrt{\frac{y_2}{2}}, \quad x_2 = y_1 - \sqrt{\frac{y_2}{2}}. \quad (3.3)$$

Let $A = \mathbb{R}^2 \setminus \{(x_1, x_2) \mid x_1 = x_2\}$. This sample space consists of the union of the two disjoint sets $A_1 = \{(x_1, x_2) \mid x_2 < x_1\}$ and $A_2 = \{(x_1, x_2) \mid x_2 > x_1\}$. Our transformation defines a one-to-one correspondence of each $A_i \ (i = 1, 2)$, onto the set

$$B = \{(y_1, y_2) \mid y_1 \in \mathbb{R}, \ y_2 > 0\};$$

simple algebra shows that the Jacobian has the form

$$|J_1| = |J_2| = \frac{1}{\sqrt{2y_2}}.$$

Thus, the probability density function of the random vector $(\eta_1, \eta_2)$ is

$$g(y_1, y_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \frac{2}{\sqrt{2y_2}} \exp \left\{ -\frac{(2-2\rho)(y_1-m)^2 + (1+\rho)y_2}{2\sigma^2(1-\rho^2)} \right\}$$

$$= \sqrt{\frac{2}{\sqrt{2\pi\sigma^2\sqrt{1-\rho}}}} \exp \left\{ -\frac{(y_1-m)^2}{\sigma^2(1+\rho)} \right\} \frac{1}{\sqrt{2\pi\sigma^2\sqrt{1-\rho}}} \frac{1}{\sqrt{y_2}} \exp \left\{ -\frac{y_2}{2\sigma^2(1-\rho)} \right\}. \quad (3.4)$$
so \( \eta_1 \) and \( \eta_2 \) are stochastically independent. Furthermore, \( \eta_1 \) is normally distributed

\[
E(\eta_1) = m, \quad \text{and} \quad D^2(\eta_1) = \frac{\sigma^2(1 + \rho)}{2},
\]

and

\[
\frac{\eta_2}{\sigma^2(1 - \rho)}
\]

has chi-square distribution with one degree of freedom \( \left( \chi^2_1 \right) \).

\[
\varphi(t_1, t_2) = E \left( \exp \left\{ it_1 \eta_1 + it_2 \frac{\eta_2}{2} \right\} \right) = \varphi_{\eta_1}(t_1) \varphi_{\eta_2/2}(t_2)
\]

\[
= \exp \left\{ it_1 m - \frac{t_1^2 \sigma^2(1 + \rho)}{4} \right\} \frac{1}{\sqrt{1 - it_2 \sigma^2(1 - \rho)}},
\]

and

\[
\varphi_{\eta_1^2 + \eta_2/2}(t) = \varphi_{\eta_1^2}(t) \varphi_{\eta_2/2}(t)
\]

\[
= \frac{1}{\sqrt{1 - it^2 \sigma^2(1 + \rho)}} \exp \left\{ - \frac{itm^2 \sigma^2(1 + \rho)}{2 - it^2 \sigma^2(1 + \rho)} \right\} \frac{1}{\sqrt{1 - it^2 \sigma^2(1 - \rho)}}.
\]

\( \eta_1^2 \) has noncentral \( \chi^2 \) distribution with noncentrality \( m^2 + (\sigma^2(1 + \rho))/2 \). From the above formulae, we can easily determine the characteristic function of

\[
\left( \frac{\xi_1 + \xi_2}{2} = \eta_1, \quad \frac{\xi_1^2 + \xi_2^2}{2} = \eta_1^2 + \frac{\eta_2}{2} \right),
\]

as the sum of squares of random variables with joint probability density function (3.1) equals the sum of two independent random variables with distribution \( \chi^2_1 \), and the expectations

\[
E(\eta_1^2) = \frac{\sigma^2(1 + \rho)}{2} + m^2 \quad \text{and} \quad E(\eta_2) = \sigma^2(1 - \rho),
\]

respectively. We have, if \( m = 0 \),

\[
\varphi(t_1, t_2) = E \left( \exp \left\{ it_1 \eta_1 + it_2 \left( \eta_1^2 + \frac{\eta_2}{2} \right) \right\} \right)
\]

\[
= \frac{1}{\sqrt{1 - it_2 \sigma^2(1 - \rho)}} \frac{1}{\sqrt{1 - it_2 \sigma^2(1 + \rho)}} \exp \left\{ - \frac{t_1^2 \sigma^2(1 + \rho)}{2 - 4it_2 \sigma^2(1 + \rho)} \right\}.
\]

Let \( \xi \) be central \( \chi^2 \)-distributed and \( \eta = \sigma / \xi \); then its probability distribution function

\[
F_\eta(x) = 2 - 2\Phi \left( \sqrt{\frac{x}{\eta}} \right), \quad \text{if} \ x > 0,
\]

where \( \Phi \) denotes the standard normal distribution function. It is well known, or one can show by easy computation that the expectation of the random variable \( \eta \) does not exist. The maximum likelihood estimator of the scale parameter \( \sigma \) is given by

\[
\hat{\sigma} = \frac{n}{\sum_{i=1}^{n} 1/\eta_i},
\]

where the random variables \( \eta_1, \eta_2, \ldots, \eta_n \) are independent, identically distributed with distribution function \( F_\eta \).

The noncentral \( \chi^2 \) distribution depends on two parameters, the degree of freedom and the noncentrality. The system of the maximum likelihood equations cannot be solved explicitly (see [8]). When the degree of freedom is equal to 1, results for estimating parameters of folded
normal distributions are applicable. The general folded normal distribution is the distribution of \(|\sigma \xi + \mu|\). The maximum likelihood equations for estimators \(\hat{\mu}, \hat{\sigma}\) of \(\mu\) and \(\sigma\) can be given in the form

\[
\hat{\mu}^2 + \hat{\sigma}^2 = \frac{1}{m} \sum_{i=1}^{m} \eta_i^2, \\
\hat{\mu} = \frac{1}{m} \sum_{i=1}^{m} \eta_i \left( \frac{\hat{\mu} \eta_i}{\hat{\sigma}^2} \right). 
\]

For comparison with other estimators we shall give a robust method to determine the location (\(\mu\)) and scale (\(\sigma\)) parameters. This method also can be applied to describe the nearly nonstationary case (\(\lambda \to 0\)), if we use for distribution of asymptotic model the reciprocal of \(\chi_1^2\) distribution. And, it may be useful for non-Gaussian cases, too. Our location and scale problem is the following. Let us assume that \(\xi = \sigma \eta + \mu\), where the distribution of the random variable \(\eta\) is \(G_0(x)\). Given the sample \(\xi_1, \xi_2, \ldots, \xi_n\) and the type of distribution \(G_0\), the distribution of the random variable \(\xi_t\) is \(G_0((x - \mu)/\sigma)\) and estimates the location (\(\mu \in \mathbb{R}\)) and scale (\(\sigma > 0\)) parameters from the sample.

The system of equations for the parameters \(\mu\) and \(\sigma\), using Huber’s [9] notations, is

\[
\sum_{i=1}^{n} \psi \left( \frac{\xi_i - \mu}{\sigma} \right) = 0, \\
\sum_{i=1}^{n} \chi \left( \frac{\xi_i - \mu}{\sigma} \right) = 0,
\]

where \(\psi(x) = G_0(x) - 0.5\), \(\chi(x) = \psi^2(x) - 1/12\). Therefore,

\[
\sum_{i=1}^{n} \left( G_0 \left( \frac{\xi_i - T_n}{s_n} \right) - \frac{1}{2} \right) = 0, \\
\sum_{i=1}^{n} \left( \left( G_0 \left( \frac{\xi_i - T_n}{s_n} \right) - \frac{1}{2} \right)^2 - \frac{1}{12} \right) = 0.
\]

If the solutions \(T_n\) and \(s_n\) of this system of equations exist, \(T_n\) and \(s_n\) are called the probability integral transformation (PT)-estimators of the location and the scale parameters, respectively. Assuming that \(G_0\) is differentiable, strictly monotone increasing, and \(G_0(0) = 0.5\), then \(T_n\) and \(s_n\) are well defined; that is, (3.14) has a unique solution with \(s_n > 0\). The joint distribution of \((T_n, s_n)\) converges to the normal one

\[
\sqrt{n}((T_n, s_n) - (\mu, \sigma)) \xrightarrow{d} N(0, \Sigma),
\]

where the covariance matrix \(\Sigma\) is given by \(\Sigma = C^{-1} S (C^{-1})^T\). The matrix

\[
C = \begin{bmatrix}
E \left( \frac{\partial}{\partial \mu} \psi \left( \frac{\xi - \mu}{\sigma} \right) \right) & E \left( \frac{\partial}{\partial \sigma} \psi \left( \frac{\xi - \mu}{\sigma} \right) \right) \\
E \left( \frac{\partial}{\partial \mu} \chi \left( \frac{\xi - \mu}{\sigma} \right) \right) & E \left( \frac{\partial}{\partial \sigma} \chi \left( \frac{\xi - \mu}{\sigma} \right) \right)
\end{bmatrix},
\]

and

\[
S = \begin{bmatrix}
E (\psi^2(\eta)) & E(\psi(\eta)\chi(\eta)) \\
E(\psi(\eta)\chi(\eta)) & E (\chi^2(\eta))
\end{bmatrix} = \begin{bmatrix}
1/12 & 0 \\
0 & 1/180
\end{bmatrix}.
\]

The (PT)-estimators are B-robust, V-robust, qualitatively robust, and their breakdown points (for definitions, see [9,10])

\[
\varepsilon^*(T_n) = \frac{\delta}{1 + \delta} = 0.5, \quad \text{where} \quad \delta = \min \left\{ -\psi(-\infty)/\psi(+\infty), -\psi(+\infty)/\psi(-\infty) \right\},
\]

and

\[
\varepsilon^*(s_n) = \frac{-\chi(0)}{\chi(-\infty) - \chi(0)} = \frac{1}{3}
\]

(see [11]). We propose an algorithm to estimate the location and the scale simultaneously.
STEP 1. Preestimation of location and scale by median (med) and median absolute deviation (MAD), i.e.,

\[ T_n^{(0)} = \text{med}\{\xi_i\} \quad \text{and} \quad s_n^{(0)} = \text{MAD}\{\xi_i\}. \]

STEP 2. Estimation of location by

\[ T_n^{(m+1)} = T_n^{(m)} + \frac{s_n^{(m)}}{n} \sum_{i=1}^{n} \psi \left( \frac{\xi_i - T_n^{(m)}}{s_n^{(m)}} \right) \]

STEP 3. Estimation of scale by

\[ \left( s_n^{(m+1)} \right)^2 = \frac{12}{n-1} \sum_{i=1}^{n} \psi^2 \left( \frac{\xi_i - T_n^{(m+1)}}{s_n^{(m)}} \right) \left( s_n^{(m)} \right)^2. \]

STEP 4. Stop or go to Step 2.

This method can be applied for the system of equations (3.12),(3.13).

4. SIMULATION RESULTS

We have to keep in mind the following ‘surprising’ facts. First, if \( \kappa \) is known, estimator \( m_1 \) is better than estimator \( m_2 \) for \( 0 < \kappa < 2 \) in the sense \( D^2(m_1) < D^2(m_2) \); see (2.19),(2.20). Moreover, if \( m \) is known, then

\[ E(s_2^2) = \frac{1}{2\lambda}, \quad E(s_1^2) = \frac{1}{2\lambda}, \]

\[ D^2(s_2^2) = \frac{1 + e^{-2\lambda}}{4\lambda^2}, \quad D^2(s_1^2) = \frac{2\lambda + e^{-2\lambda} - 1}{4\lambda^4}; \]

that is, \( D^2(s_1^2) < D^2(s_2^2) \) for \( 0 < \lambda < 1 \). Let \( T = 1, \sigma^2_0 = 1 \), and \( n \) be a positive integer. Introduce the following notations:

\[ \xi \left( \frac{i}{n} \right) = \xi_i, \quad i = 0, 1, \ldots, n, \]

\[ \rho = e^{-\lambda/n}, \quad \sigma^2_\xi = \frac{1 - \rho^2}{2\lambda} \approx \frac{1}{n} - \frac{2\lambda}{n^2}. \]

Then, the algorithm of the simulation is the following:

\[ u = \Phi^{-1} (\text{random}), \quad \xi_0 = \frac{u}{\sqrt{2\lambda}}, \]

\[ \xi_{i+1} = \rho \xi_i + \sigma_\xi \Phi^{-1} (\text{random}), \quad i = 0, 1, \ldots, n - 1, \]

where \( \Phi^{-1} \) is the inverse of the standard normal distribution function and the function \( \text{random} \) generates uniformly distributed pseudorandom numbers between 0 and 1.

Furthermore, we introduce the following notations to the description of the investigations of the simulations:

\[ m_1 = \frac{\xi_0 + \xi_n}{2}, \quad m_2 = \frac{\sum_{k=0}^{n} \xi_k}{n + 1}, \]

\[ s_1^2 = \frac{(\xi_n - \xi_0)^2}{4}, \quad s_2^2 = \frac{\sum_{k=0}^{n} (\xi_k - m_2)^2}{n + 1}, \quad s_0^2 = \frac{\xi_0^2 + \xi_n^2}{2}, \]

\[ \lambda_1 = \frac{0.5}{s_1^2}, \quad \lambda_2 = \frac{0.5}{s_2^2}, \quad \lambda_0 = \frac{0.5}{s_0^2}. \]
Figure 1. Quantiles for $\lambda_1$.

- $\overline{\vartheta}$: the mean of statistics $\vartheta$,
- $\sigma^*_\vartheta$: the unbiased empirical standard deviation of $\vartheta$,
- $\hat{\mu}_\vartheta(F)$: the maximum likelihood estimator of the location parameter $\mu$ with respect to $\vartheta$, if the distribution is $F$,
- $\hat{\sigma}_\vartheta(F)$: the maximum likelihood estimator of the scale parameter $\sigma$ with respect to $\vartheta$, if the distribution is $F$,
- $T_{\vartheta}(F)$: the (PT)-estimator of the location parameter $\mu$ with respect to $\vartheta$, and distribution $F$,
- $s_{\vartheta}(F)$: the (PT)-estimator of the scale parameter $\sigma$ with respect to $\vartheta$, and distribution $F$.

The first step in the direction of the nearly nonstationary case was done by Kormos [12,13]; see also [14]. Simulations were used by Arató and Benczúr [15].
If there is $\lambda_01$ in a formula or notation, then we suppose that the parameter $m$ is known.

In Figures 2–4, the empirical quantiles are given by the estimators $\hat{\lambda}$ (the maximum likelihood estimator), $\lambda_1$, and $\lambda_2$. We can easily find the lower confidence limit for a given $\kappa$, but near to 0 we see that they are all biased and the lower confidence limit is greater than the value of the parameter $\kappa$. For the estimator $\lambda_1$ we can calculate the lower confidence limit for a given probability $p$. From (3.10), we know that the statistics $\lambda_1$ has the probability distribution function

$$F(x) = 2 - 2\Phi \left( \sqrt{\frac{\sigma}{x}} \right), \quad \text{if } x > 0,$$

(4.5)

where

$$\sigma = \frac{2\lambda}{1 - e^{-\lambda}}.$$
Furthermore, 
\[ \lim_{\lambda \to 0} \sigma = 2, \]
and thus, we can give the quantiles of \( \lambda_1 \) when \( \lambda \to 0 \). For \( \lambda = 0 \) and \( \sigma = 2 \), we have (4.6).

<table>
<thead>
<tr>
<th>( p )</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F^{-1}(p) )</td>
<td>0.3014</td>
<td>0.5206</td>
<td>0.739</td>
<td>4.396</td>
<td>126.6</td>
<td>508.6</td>
<td>12731.7</td>
</tr>
</tbody>
</table>

(4.6)

For the proof, let \( \xi \) be \( \chi^2 \)-distributed and \( \eta = \sigma / \xi \) and denote \( x_p \) the quantiles of \( \eta \) for a given \( p \); that is, 
\[ P(\eta < x_p) = p. \]  
(4.7)
By the formula (3.10),

\[ P(\eta < x_p) = 2 - 2\Phi\left(\frac{\sigma}{x_p}\right), \quad \sigma = \frac{2\lambda}{1 - e^{-\lambda}}; \] (4.8)

that is,

\[ p = 2 \left[ 1 - \Phi\left(\frac{\sigma}{x_p}\right) \right] \quad \text{and} \quad \frac{1}{\Phi^{-1}(1 - p/2)^2} = \frac{x_p}{\sigma}. \] (4.9)

We know that

\[ \lambda_1 = \frac{1}{2s_1^2} = \frac{2}{(\xi_n - \xi_0)^2}; \]

then

\[ P(\lambda_1 < z_{\lambda_1,p}) = p, \quad \text{i.e.,} \quad z_{\lambda_1,p} = \frac{x_p}{\sigma}. \] (4.10)
We have to solve the inequality
\[ \lambda < z_{\lambda_1,p}; \] (4.11)
that is,
\[ \lambda < \frac{z_p}{\sigma} = \frac{2\lambda}{1 - e^{-\lambda}} \frac{1}{x_p} = \frac{2\lambda}{1 - e^{-\lambda}} \frac{1}{[\Phi^{-1}(1 - p/2)]^2}. \] (4.12)
Denote by \( u_p \) the solution of
\[ u_p = -\ln \left( 1 - \frac{1}{[\Phi^{-1}(1 - p/2)]^2} \right). \] (4.13)
If \( x_p < 0.5 \), a solution exists, which means \( p < 0.157299 \). We give some values of \( u_p \) (see Figure 1).

<table>
<thead>
<tr>
<th>( p )</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.157299</th>
<th>0.16</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_p )</td>
<td>0.1507</td>
<td>0.2603</td>
<td>0.3696</td>
<td>0.4825</td>
<td>0.4999995</td>
<td>0.5065</td>
<td>0.6088</td>
</tr>
<tr>
<td>( u_p )</td>
<td>0.3587</td>
<td>0.7353</td>
<td>1.344</td>
<td>3.356</td>
<td>13.8155</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The following table contains some empirical values of \( u_p \) using the statistics \( \hat{\lambda} \), \( \lambda_1 \), and \( \lambda_2 \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_{p,\hat{\lambda}} )</td>
<td>0.4483</td>
<td>1.0941</td>
<td>2.0571</td>
</tr>
<tr>
<td>( u_{p,\lambda_1} )</td>
<td>0.3527</td>
<td>0.7325</td>
<td>1.3521</td>
</tr>
<tr>
<td>( u_{p,\lambda_2} )</td>
<td>0.9013</td>
<td>1.8399</td>
<td>3.0750</td>
</tr>
</tbody>
</table>

The following tables show the simulation results for the estimators and the empirical quantiles. In the simulations, \( n = 1000 \) and the sample size is equal to 1000. The empirical results support the theoretical statements (generalized expansions); e.g., if \( \lambda \to 0 \), then the variance of the mean \( \bar{m} \) tends to infinite. From Tables 6 and 7, we see that if \( m \) is known then \( \lambda_0^1 \) is a good estimator for \( \lambda \), when \( \lambda \to 0 \). But, Tables 1–5 show that if \( \lambda < 0.1 \), then we cannot distinguish between the values of \( \lambda \).

**Table 1.** The means and the standard deviations for \( \bar{m} \), \( m_1 \), and \( m_2 \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \bar{m} )</th>
<th>( \bar{m}_1 )</th>
<th>( \bar{m}_2 )</th>
<th>( \sigma_{\bar{m}}^2 )</th>
<th>( \sigma_{m_1}^2 )</th>
<th>( \sigma_{m_2}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-9}</td>
<td>-328.184908</td>
<td>-328.193052</td>
<td>-328.179580</td>
<td>22567.43198</td>
<td>22567.43349</td>
<td>22567.43164</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.008145</td>
<td>-0.003466</td>
<td>-0.010227</td>
<td>2.17374</td>
<td>2.17675</td>
<td>2.18135</td>
</tr>
<tr>
<td>1</td>
<td>-0.011508</td>
<td>-0.003422</td>
<td>-0.018721</td>
<td>0.58614</td>
<td>0.58266</td>
<td>0.61386</td>
</tr>
<tr>
<td>2</td>
<td>0.022027</td>
<td>0.026912</td>
<td>0.017645</td>
<td>0.36130</td>
<td>0.38266</td>
<td>0.37737</td>
</tr>
<tr>
<td>10</td>
<td>0.004124</td>
<td>0.000694</td>
<td>0.004402</td>
<td>0.09211</td>
<td>0.15534</td>
<td>0.09604</td>
</tr>
<tr>
<td>100</td>
<td>-0.000198</td>
<td>0.000907</td>
<td>-0.000213</td>
<td>0.01002</td>
<td>0.04931</td>
<td>0.01007</td>
</tr>
<tr>
<td>1000</td>
<td>0.000030</td>
<td>0.000203</td>
<td>0.000030</td>
<td>0.00107</td>
<td>0.01612</td>
<td>0.00107</td>
</tr>
</tbody>
</table>

**Table 2.** \( (PT) \)-estimators if \( G_0 = \Phi \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( T_{\lambda}^3(\Phi) )</th>
<th>( T_{\lambda_1}(\Phi) )</th>
<th>( T_{\lambda_2}(\Phi) )</th>
<th>( s_\lambda(\Phi) )</th>
<th>( s_{\lambda_1}(\Phi) )</th>
<th>( s_{\lambda_2}(\Phi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-9}</td>
<td>4.259</td>
<td>11.174</td>
<td>4.865</td>
<td>3.429</td>
<td>15.987</td>
<td>3.545</td>
</tr>
<tr>
<td>0.1</td>
<td>4.243</td>
<td>10.281</td>
<td>4.867</td>
<td>3.333</td>
<td>14.144</td>
<td>3.412</td>
</tr>
<tr>
<td>1</td>
<td>4.871</td>
<td>15.532</td>
<td>5.415</td>
<td>3.455</td>
<td>21.386</td>
<td>3.534</td>
</tr>
<tr>
<td>2</td>
<td>5.742</td>
<td>20.859</td>
<td>6.257</td>
<td>3.689</td>
<td>28.462</td>
<td>3.739</td>
</tr>
<tr>
<td>10</td>
<td>13.714</td>
<td>97.204</td>
<td>13.952</td>
<td>5.227</td>
<td>138.000</td>
<td>5.284</td>
</tr>
<tr>
<td>100</td>
<td>103.536</td>
<td>996.798</td>
<td>103.569</td>
<td>14.951</td>
<td>1445.055</td>
<td>15.057</td>
</tr>
<tr>
<td>1000</td>
<td>1004.917</td>
<td>9652.539</td>
<td>1004.914</td>
<td>51.744</td>
<td>12940.382</td>
<td>51.791</td>
</tr>
</tbody>
</table>
### Table 3. The means and the standard deviations for $\lambda$, $\lambda_1$, and $\lambda_2$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\lambda$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\sigma^*_\lambda$</th>
<th>$\sigma^*_1$</th>
<th>$\sigma^*_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-9}$</td>
<td>4.829</td>
<td>1655.893</td>
<td>5.406</td>
<td>4.063</td>
<td>20082.158</td>
<td>4.110</td>
</tr>
<tr>
<td>0.1</td>
<td>4.820</td>
<td>262466.628</td>
<td>5.441</td>
<td>4.058</td>
<td>7926294.253</td>
<td>4.147</td>
</tr>
<tr>
<td>1</td>
<td>5.405</td>
<td>1739.684</td>
<td>5.951</td>
<td>4.103</td>
<td>25694.612</td>
<td>4.175</td>
</tr>
<tr>
<td>100</td>
<td>104.018</td>
<td>2476187.904</td>
<td>14.904</td>
<td>5.951</td>
<td>45390009.139</td>
<td>14.956</td>
</tr>
<tr>
<td>1000</td>
<td>1005.018</td>
<td>217378846.779</td>
<td>15.063</td>
<td>5.951</td>
<td>6866120869.699</td>
<td>51.719</td>
</tr>
</tbody>
</table>

### Table 4. (PT)-estimators if $G_0$ is equal to distribution of $\chi^2_1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$T_\lambda(x_1)$</th>
<th>$T_{\lambda_1}(x_1)$</th>
<th>$T_{\lambda_2}(x_1)$</th>
<th>$s_\lambda(x_1)$</th>
<th>$s_{\lambda_1}(x_1)$</th>
<th>$s_{\lambda_2}(x_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-9}$</td>
<td>1.971</td>
<td>-0.233</td>
<td>1.671</td>
<td>4.366</td>
<td>17.154</td>
<td>4.553</td>
</tr>
<tr>
<td>0.1</td>
<td>1.217</td>
<td>-0.072</td>
<td>1.773</td>
<td>4.330</td>
<td>15.853</td>
<td>4.439</td>
</tr>
<tr>
<td>1</td>
<td>1.700</td>
<td>-0.105</td>
<td>2.183</td>
<td>4.543</td>
<td>23.639</td>
<td>4.637</td>
</tr>
<tr>
<td>2</td>
<td>2.389</td>
<td>-0.063</td>
<td>2.866</td>
<td>4.845</td>
<td>31.804</td>
<td>4.910</td>
</tr>
<tr>
<td>100</td>
<td>90.321</td>
<td>-23.148</td>
<td>90.246</td>
<td>19.066</td>
<td>1519.424</td>
<td>19.196</td>
</tr>
<tr>
<td>1000</td>
<td>958.745</td>
<td>34.938</td>
<td>958.801</td>
<td>67.063</td>
<td>14877.598</td>
<td>66.981</td>
</tr>
</tbody>
</table>

### Table 5. (PT)- and maximum likelihood estimators if $G_0$ is equal to distribution of noncentral $\chi^2_1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$T_\lambda(x'_1)$</th>
<th>$T_{\lambda_1}(x'_1)$</th>
<th>$T_{\lambda_2}(x'_1)$</th>
<th>$s_\lambda(x'_1)$</th>
<th>$s_{\lambda_1}(x'_1)$</th>
<th>$s_{\lambda_2}(x'_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-9}$</td>
<td>1.969</td>
<td>2.125</td>
<td>2.016</td>
<td>2.172</td>
<td>0.877</td>
<td>0.837</td>
</tr>
<tr>
<td>0.1</td>
<td>1.971</td>
<td>2.133</td>
<td>2.018</td>
<td>2.183</td>
<td>0.858</td>
<td>0.808</td>
</tr>
<tr>
<td>1</td>
<td>2.130</td>
<td>2.261</td>
<td>2.173</td>
<td>2.306</td>
<td>0.823</td>
<td>0.788</td>
</tr>
<tr>
<td>2</td>
<td>2.330</td>
<td>2.443</td>
<td>2.378</td>
<td>2.493</td>
<td>0.804</td>
<td>0.772</td>
</tr>
<tr>
<td>10</td>
<td>3.669</td>
<td>3.701</td>
<td>3.701</td>
<td>3.732</td>
<td>0.713</td>
<td>0.716</td>
</tr>
<tr>
<td>100</td>
<td>10.161</td>
<td>10.163</td>
<td>10.173</td>
<td>10.174</td>
<td>0.735</td>
<td>0.740</td>
</tr>
<tr>
<td>1000</td>
<td>31.695</td>
<td>31.695</td>
<td>31.691</td>
<td>31.691</td>
<td>0.816</td>
<td>0.817</td>
</tr>
</tbody>
</table>

### Table 6. Estimators of $\lambda$ if $m$ is known.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\lambda_01$</th>
<th>$T_{\lambda_01}(\Phi)$</th>
<th>$T_{\lambda_01}(x_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-9}$</td>
<td>0.000000004520</td>
<td>0.000000000054</td>
<td>-0.00000000001</td>
</tr>
<tr>
<td>0.1</td>
<td>2.9584553512</td>
<td>0.3849289055</td>
<td>0.0169395202</td>
</tr>
<tr>
<td>1</td>
<td>36.2521922522</td>
<td>2.1221856267</td>
<td>0.3786405268</td>
</tr>
<tr>
<td>2</td>
<td>14.366714050</td>
<td>4.0161238720</td>
<td>0.698441780</td>
</tr>
<tr>
<td>10</td>
<td>99.6942998767</td>
<td>19.2742707771</td>
<td>4.5206277313</td>
</tr>
<tr>
<td>100</td>
<td>681.3721028164</td>
<td>201.7760313855</td>
<td>36.6146373116</td>
</tr>
<tr>
<td>1000</td>
<td>6485.1994744147</td>
<td>2121.9602044310</td>
<td>362.5585998607</td>
</tr>
</tbody>
</table>

<table>
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<th>$\lambda$</th>
<th>$\sigma^*_{\lambda_01}$</th>
<th>$s_{\lambda_01}(\Phi)$</th>
<th>$s_{\lambda_01}(x_1)$</th>
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Table 7. The maximum likelihood estimators for $\lambda$ if the supposed distribution is central $\chi^2$ (see (3.10)).

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<th>$\lambda$</th>
<th>$\hat{\lambda}_1(x_1)$</th>
<th>$\hat{\lambda}_2(x_1)$</th>
<th>$\hat{\lambda}_3(x_1)$</th>
<th>$\hat{\lambda}_4(x_1)$</th>
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REFERENCES