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NOTE

ON 'k-SETS' IN THE PLANE

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Given an arrangement of n points in the plane, a k-set in the plane is a k element subset of these that can be separated from the others by a straight line. The question of how many j-sets there are with j less than k was considered by Pach and by Goodman and Pollack [1], who obtained upper bounds of $2kn$, and $2kn-2k^2-k$, for this number, with $k \leq n/2$. Recently Alon and Gyori [2] have obtained proofs of the best possible upper bound for it, namely nk.

It is the purpose of this note to provide another different but very simple proof of this last result.

Theorem. *The number of j-sets for $j \leq k$, given n points in the plane is at most kn for all k.*

The argument is based on two observations, the first of which is a standard result in this subject. We may assume without loss of generality that no three of our points are on a line, since if we ever have a collection of lines that define our j-sets they will remain so if we move points slightly to avoid three on a line.

Observation 1. The number of j-sets is, with no three points on a line, exactly the number of edges joining points whose lines separate $j-1$ other points from the rest.

Proof. If we choose a direction, take a line normal to it, and sweep it across our system of points always maintaining its normal direction, at some stage it will separate j points from the rest, unless the line is parallel to such an edge. If we rotate the normal direction by 360 degrees, the j-set obtained changes each time we reach such an edge. Since we end with the j-set we began with, the number of changes or edges is the same as the number of j-sets we see, which is all there are. □

Observation 2. In the graph consisting of the edges as above separating $k-1$ or fewer points from the rest, no vertex can have degree above $2k$.

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Proof. Consider any point and replace the edges linking it to other points in graph by the lines containing these edges. Associate to each such line the nor directions that points to the side of it which has k or fewer points. Any two s lines determine four quadrants and there is exactly one quadrant that the norn both point away from, which we will call their 'good quadrant'.

Now choose two such lines whose good quadrant contains no other line. (Suc quadrant exists obviously if there are only two lines; and of the two regio obtained when a new line splits a good empty quadrant one will still be good a empty, so that one such must exist by induction.) There are at most $2(k-1)$ poin in the interior of the three quadrants other than the good one by the defin property of these lines. There can be at most $2(k-1)$ other edges or lin emanating from our point, since each of these must contain a point in the quadrants. □

The theorem follows from these observations, since a graph on n vertices, all of whose degrees are at most $2k$ will have at most nk edges. □

Alon and Gyori have two arguments for this same result, both based on a different notion. In the terms used here they get the upper bound of $2k$ on the degree of our vertices by obtaining a lower bound on the degree of vertices in the complementary graph. As one rotates about any particular vertex, the j values associated with edges in the complete graph change by one each time a point is crossed and must, if ever less than k, become $n-k$ upon a 180 degree rotation, so that all intermediate values must be traversed, which fact leads to the desired answer. Their other argument makes use of somewhat more detailed concepts, but has the same general idea.

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References

[1] J.E. Goodman and R. Pollack, On the number of k-subsets of a set of n points in the plane, J. Combin. Theory Ser. A 36 (1984) 101–104.

[2] N. Alon and E. Gyori, The number of small semispaces of a finite set of points in the plane, J. Combin. Theory Ser. A (to appear).