Fuzzy Boolean Algebras

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The purpose of this paper is to generalize the following situation: from the concrete structure $\mathcal{P}(E)$, we define the notion of Boolean algebra; the Stone representation theorem allows us to replace the algebraic study of Boolean algebras by a topological one. Let $E$ be a non-empty set, and $J$ a non-empty ordered set. Note $\mathcal{P}(E)$ the set of all fuzzy subsets of $(E,J)$. We shall introduce the concept of fuzzy Boolean algebra and find a representation theorem. But it will be difficult to speak of the dual fuzzy topological space of a fuzzy Boolean algebra as we shall see further, except in certain particular cases.

1. FUZZY BOOLEAN ALGEBRAS

1.1. The Concept of Fuzzy Boolean Algebra

Let $(E,J)$ be a fuzzy structure. A fuzzy subset $\tilde{A}$ is a mapping from $E$ to $J$. To such an $\tilde{A}$, it is possible to associate a family $(N_\alpha(\tilde{A}))_{\alpha \in J}$ of crisp subsets of $E$ by this way: $\forall \alpha \in J, N_\alpha(\tilde{A}) = \{ a \in E / \tilde{A}(a) \geq \alpha \}$ (called the $\alpha$-cut of $\tilde{A}$).

We have

$$\forall \alpha \in J, \forall \beta \in J: \alpha < \beta \Rightarrow N_\alpha(\tilde{A}) \supset N_\beta(\tilde{A})$$

$$\forall \alpha \in J: N_\alpha(\tilde{A}) = \bigcup_{\gamma \geq \alpha} N_\gamma(\tilde{A}).$$

Let us call, for every $\alpha$, $L_\alpha(\tilde{A}) = \{ a \in E / \tilde{A}(a) = \alpha \}$. Then, $(L_\alpha(\tilde{A}))_{\alpha \in \tilde{J}(E)}$ is a partition of $E$. and

$$\forall \alpha \in J: N_\alpha(\tilde{A}) = \bigcup_{\gamma \geq \alpha} L_\gamma(\tilde{A}).$$

Conversely, is it possible to build a fuzzy set the cuts of which are given? We have the following answer:
PROPOSITION. Let \((A_\alpha)_{\alpha \in J}\) be a family of crisp subsets of \(E\). The following assertions are equivalent:

1. Note, for \(\alpha \in J\), \(L_{\alpha} = A_\alpha \cap \bigcap \left(\bigcup_{\gamma > \alpha} A_\gamma\right)\). Then:
   \[
   \forall \alpha \in J: A_\alpha = \bigcup_{\gamma > \alpha} L_\gamma.
   \]
   \((L_{\alpha})_{\alpha \in J}\) is a pseudo-partition of \(E\).

2. There is a non-empty subset \(K\) of \(J\), and a family \((M_\alpha)_{\alpha \in K}\) of crisp subsets of \(E\) such that:
   \[
   \forall \alpha \in J: A_\alpha = \bigcup_{\gamma \in K, \gamma > \alpha} M_\gamma.
   \]

3. There is a non-empty subset \(K\) of \(J\), such that if we note, for \(\alpha \in K\),
   \[
   M^*_\alpha = A_\alpha \cap \bigcap_{\gamma \in K, \gamma > \alpha} \left(\bigcup_{\gamma \in K} A_\gamma\right) \cap \bigcap_{\gamma, \alpha \text{ incomparable}} \left(\bigcup_{\gamma \in K} A_\gamma\right)
   \]
   \[
   \forall \alpha \in J: A_\alpha = \bigcup_{\gamma \in K, \gamma > \alpha} A_\gamma.
   \]
   \((M^*_\alpha)_{\alpha \in K}\) is a partition of \(E\).

If these conditions are realized, there exists a unique fuzzy subset of \(E\), \(\tilde{A}\), such that:

\[
\begin{align*}
K &= \tilde{A}(E) \\
\forall \alpha \in K: N_\alpha(\tilde{A}) &= A_\alpha.
\end{align*}
\]

Moreover

\[
\begin{align*}
\forall \alpha \in J: N_\alpha(\tilde{A}) &= A_\alpha, & L_\alpha(\tilde{A}) &= L_\alpha \\
\forall \alpha \in K: L_\alpha &= M_\alpha = M^*_\alpha.
\end{align*}
\]

We can generalize this proposition in the following manner:

THEOREM 1. Let \(B\) be a complete Boolean algebra, and \(J\) a non-empty ordered set. Let \(x = (x_\alpha)_{\alpha \in J}\) be an element of \(B^J\). The following assertions are equivalent:
(1) Note, for $a \in J$, $l_a = x_a \cdot \bigvee_{\gamma \supset a} x_\gamma$. Then:

$$
\left\{
\begin{array}{l}
\forall a \in J: x_a = \bigvee_{\gamma \supset a} l_\gamma \\
(l_a)_{a \in J}
\end{array}
\right.
$$

is a pseudo-partition of the unit.

(2) There is a non-empty subset $K$ of $J$, and a family $(m_a)_{a \in K}$ of elements of $B$ such that:

$$
\left\{
\begin{array}{l}
(m_a)_{a \in K}
\forall a \in J: x_a = \bigvee_{\gamma \in K} m_\gamma.
\end{array}
\right.
$$

is a partition of the unit

(3) There is a non-empty subset $K$ of $J$ such that if we note, for $a \in K$.

$$
m^*_a = x_a \cdot \bigvee_{\gamma \in K} x_\gamma \cdot \bigvee_{\delta \in K} x_\delta
d$$

is a partition of the unit.

Definition. The subset $\tilde{B}$ of $\mathcal{B}^J$ of all the elements $(x_a)_{a \in J}$ for which we have $|\{1\} - \{2\} - \{3\}|$ is called a fuzzy Boolean algebra (or a $J$-fuzzy Boolean algebra, when necessary).

If $x = (x_a)_{a \in J} \in \tilde{B}$, there is a unique subset $K$ of $J$ such that $|\{2\} - \{3\}|$, and we shall note it $x$.

1.2. The Structure of $\tilde{B}$

**Theorem 2.** (a) If $J$ has a lowest element $\square$, $\tilde{0} (\tilde{0}_\square = 1, \tilde{0}_a = 0$ for $a \neq \square$) is the lowest element of $\tilde{B}$. It is not the lowest element of $\mathcal{B}^J$.

We can remark that, if $x = (x_a)_{a \in J} \in \tilde{B}$, $x_\square = 1$.

(b) If $J$ has a greatest element $\uparrow$, $\tilde{1} (\tilde{1}_a = 1)$ is the greatest element of $\tilde{B}$. (If $J$ has no greatest element, $\tilde{1}$ is the greatest element of $\mathcal{B}^J$ but does not belong to $\tilde{B}$.)

(c) If $J$ is an $\wedge$-lattice, $\tilde{B}$ is a sub-$\wedge$-lattice of $\mathcal{B}^J$, and:

$$
\left\{
\begin{array}{l}
(x_a)_{a \in J} \wedge (y_a)_{a \in J} = (x_a y_a)_{a \in J}.
\end{array}
\right.
$$

Moreover, if $(l_a)_{a \in J}$ and $(l'_a)_{a \in J}$ are the pseudo-partitions of the unit which are, respectively, associated to $(x_a)_{a \in J}$ and $(y_a)_{a \in J}$, $x_a y_a = \bigvee_{r \wedge a} l_r l'_r$. 

(d) If \( J \) is a \( \vee \)-lattice, \( \bar{B} \) is a \( \vee \)-lattice but not necessarily a sub-\( \vee \)-lattice of \( B^J \). We have only:

\[
(x_a)_{a \in J} \vee (y_a)_{a \in J} = (z_a)_{a \in J}, \quad \text{where} \quad z_a = \bigvee_{r \leq a} l_r.
\]

(e) If \( J \) is a chain, \( B \) is a sub-lattice of \( B^J \). In this case:

\[
(x_a)_{a \in J} \vee (y_a)_{a \in J} = (x_a \vee y_a)_{a \in J}.
\]

If we denote by \( x'_a \) the element \( \bigvee_{y \geq a} x_y = \bigvee_{y \geq a} l_y \), we have

\[
(x_a)_{a \in J} \wedge (y_a)_{a \in J} = (t_a)_{a \in J}
\]

\[
(x_a)_{a \in J} \vee (y_a)_{a \in J} = (z_a)_{a \in J}
\]

where

\[
\begin{align*}
t_a &= x_a \cdot y_a \\
t'_a &= x'_a \cdot y'_a \\
z_a &= x_a \vee y_a \\
z'_a &= x'_a \vee y'_a.
\end{align*}
\]

Henceforth in this paragraph, \( J \) is a closed lattice (a lattice with a lowest element \( \boxempty \) and a greatest element \( \boxcheck \)).

* First, we remark that the mapping \( p \mapsto \tilde{p} (\tilde{p}_\boxempty = 1, \tilde{p}_a = p \text{ for } a \neq \boxempty) \) is a closed lattice monomorphism from \( B \) to \( \bar{B} \).

* Another remark: if \( \tilde{A} \) is a fuzzy subset of \( E \), \( \tilde{A} = \bigcup_{a \in \tilde{A}(E)} |N_a(\tilde{A}) \cap E_a| \) (\( E_a \) is the constant function the value of which is \( a \)).

* Let \( \bar{B} \) be a fuzzy Boolean algebra.

For \( a \in J \), the element \( e^a = (e^a_\beta)_{\beta \in J} \) (\( e^a_\emptyset = 1 \iff \emptyset \supseteq a \), \( e^a_\beta = 0 \iff \beta \nsubseteq a \)) belongs to \( \bar{B} \).

**Theorem 3.** For any \( x = (x_a)_{a \in J} \) belonging to \( \bar{B} \), the family \( (\tilde{x}_a \wedge e_a)_{a \in J} \) has a lowest upper bound in \( \bar{B} \) which is \( x \), so \( x = \bigvee_{a \in J} (\tilde{x}_a \wedge e_a) \).

1.3. Complemented Elements of \( \bar{B} \)

An element \( a \) of a closed lattice \( L \) is said to be complemented if there exists an element \( b \) of \( L \) such that \( a \wedge b = 0 \), \( a \vee b = 1 \). Let us denote \( C(L) \) the set of all these elements.

We know that, if \( J \) is a distributive closed lattice, \( C(\tilde{p}(E)) = C(J)^\varnothing \). (For
instance, if \( J \) is a chain, the crisp subsets of \( E \) are the only complemented fuzzy subsets of \( E \).

We shall characterize the complemented elements of a fuzzy Boolean algebra.

**Lemma.** Let \( x = (x_a)_{a \in J} \) and \( y = (y_a)_{a \in J} \) be two elements of \( \tilde{B} \), the pseudo-partitions of which are denoted, respectively, \( (l_a)_{a \in J} \) and \( (L_a)_{a \in J} \). Then,

\[
(x \land y = \emptyset \text{ and } x \lor y = \top) \iff \left( \bigvee_{a \land \beta = \emptyset} l_a L_\beta = 1 \right).
\]

**Theorem 4.** \( J \) is a closed lattice.

1. Any element of \( B \) has a complement in \( \tilde{B} \) (\( \widetilde{\neg} \) is a complement for \( \tilde{B} \)).
2. If \( J \) is a chain \( C(\tilde{B}) = B \).
3. \( x \in C(\tilde{B}) \Rightarrow x \subset C(J) \).
4. In the following situations:
   a. \( J \) is a distributive lattice (then \( \tilde{B} \) is also a distributive lattice).
   b. There is a bijective mapping \( \varphi : C(J) \rightarrow C(J) \) such that
      \[
      \forall r \in C(J), \quad r \land \varphi(r) = \emptyset, \quad r \lor \varphi(r) = \top,
      \]
   we have the equivalence: \( x \in C(\tilde{B}) \Leftrightarrow x \subset C(J) \).
5. This last result is true, without any hypothesis concerning \( J \), if we admit the axiom of choice.

2. A Representation Theorem for Fuzzy Boolean Algebras

In this section, \( J \) is a non-empty ordered set.

2.1. A Natural Idea

The first idea we had was to use the Stone representation theorem cut by cut: let \( X \) denote the dual space of \( B \), and \( \sigma : B \rightarrow \wp(X) \) denote the Stone function. To \( x = (x_a)_{a \in J} \in \tilde{B} \), we can associate the family \( (\sigma(x_a))_{a \in J} \in \wp(X)' \). But \( (\sigma(x_a))_{a \in J} \) is not necessarily an element of \( \wp(X) \).

**Example**

\( J \) is a complete chain \( \alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots < \alpha_\omega \) (type: \( \omega + 1 \)).
FUZZY BOOLEAN ALGEBRAS

$X_1$ is the discrete space $N$.

$X$ is the Alexandroff compactification of $X_1$ ($X = X_1 \cup \{\omega\}$). It is a Boolean space.

$C$ is the dual Boolean algebra of $X$.

$X'$ is the dual Boolean space of $C$, and $\lambda: X \to X'$ the natural homeomorphism. Let $\omega' = \lambda(\omega)$.

$B$ is the complete Boolean algebra of the regular open sets of $X$.

$Y$ is the dual Boolean space of $B$, and $\sigma: B \to \mathcal{B}(Y)$ the Stone function.

To the natural embedding $i$ from $C$ into $B$, we associate—by duality—a continued surjection $\theta$ from $Y$ onto $X'$. Let $\omega_1$ be an element of $Y$ such that $\theta(\omega_1) = \omega'$.

We consider the fuzzy Boolean algebra $\tilde{B}$. [For any $n$, $l_{\omega_1} = |n|$; $l_{\omega_2} = \emptyset$] is a pseudo-partition of the unit in $B$: let us call $x$ the associated element of $\tilde{B}$.

We remark that $\sigma(x_{\omega_1}) = \sigma(x_{\omega_2}) \neq \cap_n \sigma(x_{\omega_n})$. A result of Ralescu [4] proves then that $(\sigma(x_n))_{a \in J} \in \mathcal{B}(Y)$.

However, this method gives a representation theorem in the following cases:

(a) $J$ is a complete chain, in which each element of $J - \{\square\}$ has a predecessor.

(b) $J$ is a finite lattice (moreover, if $B$ is a finite Boolean algebra, we obtain a bijective representation).

2.2. A Representation Theorem

If $U \in X$, and $S = \{a \in J/x_a \in U\}$, we have

\[ S \text{ is an initial cut of } J \]
\[ U \in \bigcap_{a \in S} \sigma(x_a) \]
\[ \text{If } T \text{ is an initial cut of } J, \text{ and if } T \not\supseteq S, U \notin \bigcap_{a \in T} \sigma(x_a) \]
\[ U \in \left( \bigcap_{a \in S} \sigma(x_a) \right) \cap \bigcup_{T \not\supseteq S} \left( \bigcap_{a \in T} \sigma(x_a) \right). \]

So, we shall try to get a representation theorem of $\tilde{B}$ in the algebra of fuzzy subsets of $(X, \mathcal{F})$, where $\mathcal{F}$ is the set of initial cuts of $J$.

In fact, for technical reasons, we must give a slightly different definition of $\mathcal{F}$ when $J$ has a lowest element. We obtain the following theorem:
THEOREM 5.

(1) If $J$ has no $\cap$:

Let $\mathcal{J}$ be the set of initial cuts of $J$ (with inclusion, it is a complete lattice). The mapping

$$\delta: \mathcal{B} \to \mathcal{B}(X) \ (\mathcal{J} - \text{fuzzy Boolean algebra})$$

$$x \mapsto \left( \bigcap_{a \in S} \sigma(x_a) \right)_{s \in \mathcal{J}}$$

is increasing and one-to-one.

(2) If $J$ has $\emptyset$:

Let $\mathcal{J}$ be the set of non-empty initial cuts of $J$ (with inclusion, it is a complete lattice). $\delta$ denotes the mapping

$$\delta: \mathcal{B} \to \mathcal{B}(X) \ (\mathcal{J} - \text{fuzzy Boolean algebra})$$

$$x \mapsto \left( \bigcap_{a \in S} \sigma(x_a) \right)_{s \in \mathcal{J}}$$

$\delta$ is increasing and one-to-one

$$\delta(\emptyset) = \emptyset$$

If $J$ has $\forall$, $\delta(\top) = \bar{X}$

If $J$ has $\forall$, and if $\emptyset \neq \forall$, $\delta(p) = \sigma(p)$ for any $p \in B$

If $J$ is an $\wedge$-lattice: $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$

If $J$ is a chain: $\delta(x \lor y) = \delta(x) \lor \delta(y)$

(this last result is wrong if $J$ is not a chain).

2.3. Involutive Fuzzy Boolean Algebras

Now, we suppose that there is a decreasing and involutive mapping:

$$n: J \to J.$$

THEOREM 6. (a) Let $x = (x_a)_{a \in J}$ be an element of $B$. If for any $a \in J$

$$y_a = \bigvee_{y < y_a} l_y,$$

then $(y_a)_{a \in J} \in \bar{B}$. This last element is denoted $n \cdot x$. Its pseudo-partition is $(l_{a \cdot x})_{a \in J}$.

(b) $x \mapsto n \cdot x$ is decreasing and involutive in $\bar{B}$.

(c) If $J$ is a chain: $(n \cdot x)_a = \neg x^a_{na}$.

Is Theorem 5 a good representation for involutive Boolean algebras? A first question: is there an extension $N: \mathcal{J} \to \mathcal{J}$ of $n$ which is decreasing and involutive?
The answer is no except if $J$ is a regular chain (every element of $J - \{\square\}$ has a successor then every element of $J - \{\square\}$ has a predecessor because of $n$). Then the only extension $N$ is given by

$$NS = \bigcap_{a \in S} \{\square, na\}.$$  

In this case: $\bar{s}(n \cdot x) = N \cdot \bar{s}(x)$.

3. Fuzzy Boolean Algebras and Multivalent Logics

3.1. J-Lukasiewicz Algebras

Let $J$ be a closed chain.

A J-Lukasiewicz algebra is a \([L, \land, \lor, 1, 0, (\varphi_a)_{a \in J - \{\square\}}]\), where:

- $[L, \land, \lor, 1, 0]$ is a closed distributive lattice
- $\varphi_a$ is a closed lattice endomorphism of $L$
- $\varphi_a \varphi_b = \varphi_b$
- $a \leq \beta \Rightarrow \varphi_b \leq \varphi_a$
- $(\forall a \in J \{\square\} : \varphi_a x = \varphi_a y) \Rightarrow x = y$
- $\varphi_a(L) \subset C(L)$.

**Example.** If $J$ is a closed chain $\bar{B}$ is a J-Lukasiewicz algebra $(\varphi_a x = \bar{x}_a)$.

More generally the following theorem has been proved in [5]:

Let \([L, \land, \lor, 1, 0, (\varphi_a)_{a \in J - \{\square\}}]\) be a J-Lukasiewicz algebra.

Let $X$ be the Boolean space of the Boolean algebra $C(L)$ and $\sigma : C(L) \to \mathfrak{B}(X)$ the Stone function.

Let $\mathcal{J}$ be the chain of non-empty initial cuts of $J$. We define $f : L \to \mathfrak{B}(X)$ ($\mathcal{J}$-fuzzy Boolean algebra) by $f(x)_S = \bigcap_{a \in S} \sigma(\varphi_a x)$. It is a good representation for $L$.

3.2. Duality for Fuzzy Boolean Algebras

Let $J$ be a closed chain.

$B, X, \sigma, \bar{B}, \mathcal{J}, \bar{\sigma}$ are those of Theorem 5. The usual topology on $X$ induces an $H$-fuzzy topology on $(X, \mathcal{J})$ [1].

We can remark that, for any $x \in B$, $\bar{\sigma}(x)$ is closed fuzzy subset of $(X, \mathcal{J})$. 
REFERENCES