On the Connectedness of Attractors for Dynamical Systems

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For a dynamical system on a connected metric space $X$, the global attractor (when it exists) is connected provided that either the semigroup is time-continuous or $X$ is locally connected. Moreover, there exists an example of a dynamical system on a connected metric space which admits a disconnected global attractor.

1. INTRODUCTION

Dynamical systems (or semigroups) are a fundamental tool in the description and in the study of many important problems of natural sciences.

In Banach spaces, time-continuous semigroups arise in a natural way, e.g., from the Cauchy problem for the autonomous differential equation $u'(t) = f(u(t))$, provided that, for any initial data $u(0) = u_0$, there exists a unique global solution for all positive times, which depends continuously on $u_0$.

In metric spaces discrete semigroups arise simply by considering the successive powers of any continuous map $f: X \rightarrow X$, i.e., by setting $S_n(x) := f^n(x)$. In this case "$f$ generates $S_n$".

One of the most fascinating problems of the theory of semigroups is the so-called asymptotical dynamic, i.e., the long-term behaviour of the system $(t \rightarrow \infty)$. In many applications this study is very hard, due to several difficulties related to chaos, bifurcation, or sensitivity to initial data.

Some aspects of the complex asymptotic flow can be explained by the existence of the global attractor to which trajectories converge as $t \rightarrow \infty$. This global attractor may sometimes be a strange fractal, but a good understanding of this mathematical object is often necessary for a better understanding of the dynamic that it describes. Mathematical literature provides results of existence of the global attractor for dynamical systems, if the semigroup satisfies some reasonable hypotheses [Ar, Ha88, Lady88, Te87].
The aim of this paper is to examine the question whether the global attractor is connected. An affirmative answer to this problem was given by Hale [Ha88], in the case when the phase space is a Banach space (cf. [BV89]). In [Lady88] it is stated that if the phase space is a connected complete metric space, and the global attractor \( A \) exists, then \( A \) is connected, but actually the proof works only if \( A \) is contained in some bounded connected set (cf. [Har91]).

In this paper we establish that:

1. If the phase space is connected and the semigroup is time-continuous then the global attractor, if it exists, is connected (Theorem 3.1);
2. If the phase space is connected, and the global attractor \( A \) exists, the number of its connected components is either one or infinity (Proposition 4.3); if moreover the phase space is locally connected then \( A \) turns out to be connected (Theorem 4.5); this generalizes Hale's result;
3. There exists a discrete semigroup on a connected metric space which admits a disconnected global attractor (Theorem 5.1).

In this last case, the phase space may be taken complete and path-connected, while the semigroup may be extended to a group (Remark 5.2).

This paper is organized as follows: in Section 2 we provide the basic notations and definitions; in Section 3 we deal with time-continuous semigroups and we prove (1); in Section 4 we prove Proposition 4.3 and we deduce (2); and, in Section 5 we prove (3).

2. PRELIMINARIES

In this section we give notations and we recall basic definitions from the theory of semigroups of continuous operators. Throughout this paper, unless otherwise stated, \( X \) will denote a generic (not necessarily complete) metric space, sometimes called phase space, with distance function \( d \). For any \( A \subseteq X \), we denote by \( \overline{A} \) the closure of \( A \) in \( X \), and for any \( \varepsilon > 0 \) we denote by \( A_\varepsilon \) the open \( \varepsilon \)-neighbourhood of \( A \) in \( X \), i.e.,

\[
A_\varepsilon := \{ x \in X : \inf_{a \in A} d(x, a) < \varepsilon \}.
\]

We denote by \( \mathbb{R}_{>0} \) the set of nonnegative real numbers.

In order to give a unified treatment of continuous and discrete semigroups, we give here a rather general definition of semigroup, very similar to the definition given in [Lady88, Lady90].
Definition 2.1. A subset $P \subseteq \mathbb{R}$ is said to be a parameter space if and only if:

- there exists $\alpha > 0$ such that $\{0, \alpha\} \subseteq P$;
- $P$ is additively closed, i.e., for all $t \in P$, $s \in P$ we have that $t + s \in P$.

Remark 2.2. We will hereafter assume, without loss of generality, that $1 \in P$, hence $\mathbb{N} \subseteq P$.

Definition 2.3. Let $X$ be a metric space and let $P \subseteq \mathbb{R}$ be a parameter space. A semigroup of continuous operators on $X$, parametrized by $P$, is a family of maps $\{S_t\}_{t \in P}$, satisfying:

- $S_t : X \to X$ is continuous, for every $t \in P$;
- $S_0$ is the Identity on $X$;
- $S_{t+s} = S_t \circ S_s$, for every $t, s \in P$.

When in addition $P$ is an additive subgroup of $\mathbb{R}$, we call $\{S_t\}_{t \in P}$ a group of continuous operators.

Definition 2.4. Let $\{S_t\}_{t \in P}$ be a semigroup of continuous operators on a metric space $X$.

- We call $\{S_t\}_{t \in P}$ a discrete semigroup (resp. a discrete group) provided that $P \cap \mathbb{R}_{\geq 0} = \mathbb{N}$ (resp. $P = \mathbb{Z}$).
- We call $\{S_t\}_{t \in P}$ a time-continuous semigroup (resp. a time-continuous group) provided that $P \supseteq \mathbb{R}_{\geq 0}$ (resp. $P = \mathbb{R}$) and, for every $x \in X$, the function $t \mapsto S_t(x)$ is continuous on $P$.

Sometimes we will use the expression “arbitrary semigroup” to emphasize that we are dealing with a semigroup in the sense of Definition 2.3, i.e., without any further assumption on $P$.

Definition 2.5. Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$, and let $A \subseteq X$.

- The $\omega$-limit of $A$ is defined as

$$\omega(A) := \bigcap_{s \geq 0} \bigcup_{t \geq s} S_t(A);$$

- $A$ is positively invariant if and only if $S_t(A) \subseteq A$, for every $t \geq 0$, $t \in P$;
- $A$ is invariant if and only if $S_t(A) = A$, for all $t \in P$. 

3 CONNECTEDNESS OF ATTRACTORS
For a detailed discussion of the properties of the \( \omega \)-limit operator, the reader is referred to the wide literature on this subject [AG95, Ar, Ha88, Lady88, Te87].

**Definition 2.6.** Let \( \{S_t\}_{t \in \mathbb{R}} \) be an arbitrary semigroup of continuous operators on a metric space \( X \), and let \( A \subseteq X \), \( B \subseteq X \). We say that \( A \) attracts \( B \) if and only if, for any \( \varepsilon > 0 \), there exists \( t_\varepsilon \geq 0 \) such that

\[
S_t(B) \subseteq A, \quad \forall t \geq t_\varepsilon, \quad t \in \mathbb{R}.
\]

**Definition 2.7.** Let \( \{S_t\}_{t \in \mathbb{R}} \) be an arbitrary semigroup of continuous operators on a metric space \( X \). A subset \( A \subseteq X \) is a **global attractor** if and only if:

- \( A \) is compact;
- \( A \) is invariant;
- \( A \) attracts any bounded subset of \( X \).

The global attractor, when it exists, is necessarily unique. The reader interested in existence results for the global attractor under suitable assumptions on \( X \) and \( S_t \) is referred to [Ar, Ha88, Lady88, Te87]; in this paper we need only the following characterization (the trivial proof is omitted).

**Proposition 2.8.** Let \( \{S_t\}_{t \in \mathbb{R}} \) be an arbitrary semigroup of continuous operators on a metric space \( X \), which admits a global attractor \( A \). Then \( A = \omega(U) \), for any bounded set \( U \supseteq A \).

In the proof of Lemma 4.2 we will need the following lemma.

**Lemma 2.9.** [Ha88, Lemma 3.1.1]. Let \( \{S_t\}_{t \in \mathbb{R}} \) be an arbitrary semigroup of continuous operators on a metric space \( X \), and let \( B \subseteq X \). Let us assume that \( \omega(B) \) is compact and attracts \( B \). Then \( \omega(B) \) is invariant.

### 3. The Case of Time-Continuous Semigroups

**Theorem 3.1.** Let \( \{S_t\} \) be a time-continuous semigroup of continuous operators on a connected metric space \( X \). If there exists the global attractor \( A \), then \( A \) is connected.

**Proof.** Step 1. We argue by contradiction. Let us assume that \( A \) is not connected and let \( A = A_1 \cup A_2 \), where \( A_1 \) and \( A_2 \) are nonempty, disjoint, compact sets. Fix \( \varepsilon > 0 \) such that \( (A_i)_\varepsilon \cap (A_j)_\varepsilon = \emptyset \).
Let us set:

\[ X_1 := \{ x \in X : S_t(x) \in (A_1)_\text{e} \text{ for } t \text{ large enough} \}, \]
\[ X_2 := \{ x \in X : S_t(x) \in (A_2)_\text{e} \text{ for } t \text{ large enough} \}. \]

It is clear that \( X_1 \cap X_2 = \emptyset \). If we show that \( X = X_1 \cup X_2 \) and that \( X_1 \) and \( X_2 \) are nonempty open sets, then we have found a contradiction, since \( X \) was assumed to be connected.

**Step 2.** We show that \( X = X_1 \cup X_2 \).

Let \( x \in X \). Since \( \{ x \} \) is attracted by \( A \), there exists \( t_* > 0 \) such that
\[
S_{(t_* \to +\infty)}(x) := \{ S_t(x) : t > t_* \} \subseteq A_\text{e}.
\]

By the time-continuity of the semigroup the set \( S_{(t_* \to +\infty)}(x) \) is connected and therefore it is either totally contained in \( (A_1)_\text{e} \) or totally contained in \( (A_2)_\text{e} \).

**Step 3.** We show that \( X_1 \supseteq A_1 \neq \emptyset, \) for \( i = 1, 2 \).

Let \( x \in A_i \). Since \( A \) is invariant, by the same argument of Step 2 with \( t_* = 0 \) we conclude that \( S_{[0, +\infty)}(x) \subseteq A_i \) and therefore \( x \in X_i \).

**Step 4.** We show that \( X_1 \) and \( X_2 \) are open sets.

Let \( x \in X_i \) and let \( B \) be a bounded neighbourhood of \( x \). Since \( A \) attracts \( B \) and \( x \in X_i \), there exists \( t_* > 0 \) such that
\[
S_t(B) \subseteq A_\text{e}, \quad \forall t \geq t_*;
\]
\[
S_{t_*}(x) \in (A_i)_\text{e}.
\]

By the continuity of \( S_{t_*} \), there exists a neighbourhood \( U \subseteq B \) of \( x \) such that
\[
S_{t_*}(U) \subseteq (A_i)_\text{e}.
\]

Applying the same argument of Step 2 to every point of \( U \), it follows that \( S_t(U) \subseteq (A_i)_\text{e} \), for any \( t \geq t_* \), hence \( U \subseteq X_i \).

For the interested reader, we note that the above theorem admits the following generalization:

Let \( \{ S_t \} \) be a time-continuous semigroup of continuous operators on a metric space \( X \), which admits a global attractor \( A \). Then every connected component of \( X \) contains exactly one connected component of \( A \).
Let indeed \( Y \) be any connected component of \( X \), and let \( \bar{A} := A \cap Y \). It is easy to see that \( Y \) is a closed positively invariant set and that \( \bar{A} := A \cap Y \) is the global attractor for the restriction of \( \{ S_t \} \) to \( Y \). From Theorem 3.1, it follows that \( \bar{A} \) is connected.

4. THE CASE OF ARBITRARY SEMIGROUPS

In this section we prove that attractors for arbitrary semigroups defined on connected and locally connected phase spaces are connected. The proof of this result is based on Proposition 4.3, which can thus be considered as the technical core of this section.

**Lemma 4.1.** Let \( X \) be a metric space. Let \( M \subseteq X \) be a compact set and let \( m \) be a positive integer. Let us assume that, for any \( \varepsilon > 0 \), there exists \( C \subseteq X \) such that:

(i) \( M \subseteq C \subseteq M_\varepsilon = \{ x \in X : d(x, M) < \varepsilon \} \);

(ii) \( C \) has at most \( m \) connected components.

Then \( M \) has at most \( m \) connected components.

**Proof.** We argue by contradiction. Let us suppose that \( M \) has at least \( m + 1 \) connected components. Then we can write

\[ M = M_1 \cup M_2 \cup \cdots \cup M_{m+1}, \]

where the \( M_i \)'s are compact, nonempty, pairwise disjoint sets. Fix \( \varepsilon > 0 \) such that the \( (M_\varepsilon)_i \)'s, \( 1 \leq i \leq m + 1 \), are pairwise disjoint sets. By hypothesis, there exists \( C \subseteq X \), with at most \( m \) connected components, such that \( M \subseteq C \subseteq M_\varepsilon \).

Let us set

\[ C_i := C \cap (M_\varepsilon)_i, \quad 1 \leq i \leq m + 1. \]

It is easy to see that \( C_i \supseteq M_i \neq \emptyset \) and that the \( C_i \)'s are open pairwise disjoint subsets of \( C \).

This is a contradiction, since \( C \) has at most \( m \) connected components.

**Lemma 4.2.** Let \( \{ S_t \}_{t \in \mathbb{R}} \) be an arbitrary semigroup of continuous operators on a metric space \( X \). Let \( B \subseteq X \) and let \( m \) be a positive integer such that:
(i) $B$ has $m$ connected components;
(ii) $\omega(B)$ is compact and attracts $B$;
(iii) $\omega(B) \subseteq B$.

Then $\omega(B)$ has at most $m$ connected components.

**Proof.** We consider the compact set $M := \omega(B)$ and we show that, for any $\varepsilon > 0$, $M$ fulfills conditions (i) and (ii) of Lemma 4.1 with $C = S_t(B)$ for a large enough $t \in P$.

Since the operators of the semigroup are continuous, it follows that, for any $t \in P$, the set $S_t(B)$ has at most $m$ connected components. Furthermore, by (ii) and Lemma 2.9, $M$ is invariant and therefore, by (iii):

$$M = S_t(M) \subseteq S_t(B), \quad \forall t \in P.$$  

Finally, since $M$ attracts $B$, it follows that $S_t(B) \subseteq M$, for a large enough $t \in P$.

Applying Lemma 4.1 we deduce that $M$ has at most $m$ connected components. 

**Proposition 4.3.** Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a connected metric space $X$. If there exists the global attractor $A$, then either $A$ is connected, or $A$ has infinitely many connected components.

**Proof.** Step 1. We argue by contradiction. Let us assume that $A$ has a finite number $m > 1$ of connected components $A_1, \ldots, A_m$. Since $A$ is invariant and each $A_i$ is connected, there exists a permutation $\sigma$ of $\{1, 2, \ldots, m\}$ such that

$$S_t(A_i) = A_{\sigma(i)}, \quad 1 \leq i \leq m. \quad (1)$$

Fix $\varepsilon > 0$ such that the $(A_i)$’s are pairwise disjoint sets, and then fix $0 < \delta < \varepsilon$ such that

$$S_t((A_i)_\delta) \subseteq (A_{\sigma(i)})_\delta, \quad 1 \leq i \leq m. \quad (2)$$

This choice is possible since the $A_i$’s are compact.

Now let us set, for $1 \leq i \leq m$:

$$X_i := \{ x \in X : S_{n,m}(x) \in (A_i)_\delta \text{ for } n \text{ large enough} \}.$$ 

The $X_i$’s are clearly pairwise disjoint sets. If we show that the $X_i$’s are nonempty open sets such that $X = X_1 \cup X_2 \cup \cdots \cup X_m$, then we have found a contradiction, since $X$ was assumed to be connected.
Step 2. We show that \( X_i \supseteq A \), and therefore \( X_i \) is nonempty.

Since the \((n \cdot m!)\)-th power of any permutation of \([1, 2, \ldots, m]\) is the identity for any \( n \), then by applying \( n \cdot m! \) times equality (1) it follows that, for any \( n \in \mathbb{N} \):

\[
S_{n \cdot m!}(A_i) = A_{\sigma^{n \cdot m!}(i)} = A_i,
\]
i.e., \( A_i \subseteq X_i \).

Step 3. We show that \( X = X_1 \cup X_2 \cup \cdots \cup X_m \).

Let \( x \in X \). Since \( A \) attracts \( \{x\} \), there exists \( n_0 \in \mathbb{N} \) such that

\[
S_k(x) \in (A)_d = (A_1)_d \cup \cdots \cup (A_m)_d, \quad \forall k \geq n_0 \cdot m!.
\] (3)

Let us assume that \( S_{n_0 \cdot m!}(x) \in (A_i)_d \). By (2) and (3) we have that

\[
S_{n_0 \cdot m! + 1}(x) \in (A_{\sigma^{n_0 \cdot m!}}(i))_d \cap (A_d)_d = (A_d)_d, \\
S_{n_0 \cdot m! + 2}(x) \in (A_{\sigma^{n_0 \cdot m!}}(i))_d \cap (A_{\sigma^2(i)})_d, \\
\vdots \\
S_{n_0 \cdot m! + m}(x) \in (A_{\sigma^{n_0 \cdot m!}}(i))_d \cap (A_{\sigma^m(i)})_d = (A_i)_d.
\]

By an easy induction we conclude that \( S_{n_0 \cdot m!}(x) \in (A_i)_d \) for any \( n \geq n_0 \), hence \( x \in X_i \).

Step 4. We show that the \( X_i \)'s are open sets.

Let \( x \in X_i \) and let \( B \) be a bounded neighbourhood of \( x \). Since \( A \) is the global attractor and \( x \in X_i \), there exists \( n_0 \in \mathbb{N} \) such that

\[
S_k(B) \subseteq A_d, \quad \forall k \geq n_0 \cdot m!; \\
S_{n_0 \cdot m!}(x) \in (A_i)_d.
\]

By the continuity of \( S_{n_0 \cdot m!} \), there exists a neighbourhood \( U \subseteq B \) of \( x \) such that

\[
S_{n_0 \cdot m!}(U) \subseteq (A_i)_d.
\]

Applying the same argument of Step 3 (with \( U \) instead of \( x \)) we conclude that

\[
S_n(U) \subseteq (A_i)_d, \quad \forall n \geq n_0.
\]

This proves that \( U \subseteq X_i \).

Remark 4.4. In an analogous way, we can prove that if \( X \) has a finite number \( m \) of connected components and there exists the global attractor \( A \)
for \( \{ S_t \}_{t \in P} \), then either \( A \) has a finite number \( m' \leq m \) of connected components, or \( A \) has infinitely many connected components.

In [Har91, Lady88] it is proved that the global attractor \( A \) is connected provided that it is contained in a connected bounded subset of the (connected) phase space \( X \). As a matter of fact, we can prove that \( A \) is connected if it is contained in a bounded subset of \( X \) with a finite number of connected components. In particular, we have the following result.

**Theorem 4.5.** Let \( \{ S_t \}_{t \in P} \) be an arbitrary semigroup of continuous operators on a metric space \( X \). If \( X \) is connected and locally connected, and there exists the global attractor \( A \), then \( A \) is connected.

**Proof.** Since \( A \) is compact and \( X \) is locally connected, there exists a bounded neighbourhood \( U \) of \( A \) with a finite number of connected components. By Proposition 2.8 we have that \( A = \omega(U) \), so applying Lemma 4.2 with \( B := U \) it follows that \( A \) has a finite number of connected components. Since \( X \) is connected, it follows from Proposition 4.3 that \( A \) is connected.

**Remark 4.6.** In an analogous way (by applying Remark 4.4), we can prove that if \( X \) is a locally connected space with a finite number \( m \) of connected components and there exists the global attractor \( A \) for \( \{ S_t \}_{t \in P} \), then \( A \) has at most \( m \) connected components.

In order to restrict the search of the counterexample of point (3) of the Introduction, we develop an idea contained in [La76, Theorem I.5.2].

**Definition 4.7.** (cf. [La76]). Let \( \{ S_t \}_{t \in P} \) be an arbitrary semigroup of continuous operators on a metric space \( X \). A subset \( A \subseteq X \) is said to be invariantly connected if and only if it may not be represented as the union of two disjoint, nonempty, closed, positively invariant sets.

**Proposition 4.8.** Let \( \{ S_t \}_{t \in P} \) be an arbitrary semigroup of continuous operators on a connected metric space \( X \). If there exists the global attractor \( A \), then \( A \) is invariantly connected.

**Proof.** We argue by contradiction. Let us assume that \( A \) is not invariantly connected and let \( A = A_1 \cup A_2 \), where \( A_1 \) and \( A_2 \) are disjoint, nonempty, compact, positively invariant sets. Fix \( \varepsilon > 0 \) and \( 0 < \delta < \varepsilon \) such that

\[
(A_i)_\varepsilon \cap (A_j)_\varepsilon = \emptyset; \\
S_i((A_i)_\delta) \subseteq (A_i)_\varepsilon, \quad i = 1, 2.
\]
Let us set, for \( i = 1, 2, \)
\[
X_i := \{ x \in X : S_n(x) \in (A_i)_a \text{ for } n \text{ large enough} \}.
\]

With the same argument used in Steps 2, 3, 4 of the proof of Proposition 4.3 (this time without the complication due to the permutation \( \sigma \)) it is easy to verify that \( X = X_1 \cup X_2 \), and that \( X_1 \) and \( X_2 \) are disjoint, nonempty open subsets of \( X \).

This leads to a contradiction, since \( X \) was assumed to be connected. \( \square \)

Proposition 4.8 above clarifies that any disconnected global attractor in a connected phase space may not simply reduce to the set of fixed points of the semigroup.

5. DISCONNECTED ATTRACTORS IN CONNECTED PHASE SPACES

In this section we show that, if the phase space is not locally connected and the semigroup is not time-continuous, then the global attractor in a connected phase space may fail to be connected. By Proposition 4.3, such an attractor need have infinitely many connected components.

**Theorem 5.1.** There exists a connected metric space \( X \) and a discrete semigroup \( \{ S_n \} \) of continuous operators on \( X \) which admits a disconnected global attractor \( A \).

**Proof.** Step 1. The Metric Space \( X \). Let us set \( s_0 = 0 \) and \( s_n = \sum_{i=1}^{n} 2^{-i} \), for \( n \geq 1 \). Let us consider the following points in \( \mathbb{R}^2 \):

\[
\begin{align*}
P_0 &:= (0, 0); & P_{-\infty} &:= (-1, 0); & P_{+\infty} &:= (2, 0); \\
P_n &:= (1 + s_{n-1}, 0), & n &\geq 1; \\
P_{-n} &:= (-s_n, 0), & n &\geq 1.
\end{align*}
\]

For any \( z \in \mathbb{Z} \), let \( T_z \) be the isosceles triangle contained in the halfplane \( \{(x, y) \in \mathbb{R}^2 : y \geq 0 \} \) with base \( P_z P_{z+1} \) and height \( 2^{-z} \). Let \( X_z \) be the union of the two equal sides of \( T_z \). We remark that, for any \( z \in \mathbb{Z} \), the height of \( T_z \) is one half the height of \( T_{z-1} \).

Finally let us consider the set:

\[
X := \{ P_{-\infty} \} \cup \{ P_{+\infty} \} \cup \bigcup_{z \in \mathbb{Z}} X_z,
\]

with the metric induced by \( \mathbb{R}^2 \). It is easy to see that \( X \) is connected.
Figure 1 gives a pictorial idea of $X$.

**Step 2. The Semigroup $\{S_n\}$.** Let us consider the map $S : X \to X$ defined by:

- $S(P_{-\infty}) = P_{-\infty}$, $S(P_{+\infty}) = P_{+\infty}$;
- if $n \geq 0$, $P \in X_n$ and $P = (1 + s_n - x, y)$, then
  $$S(P) = (1 + s_{n+1} - x/2, y/2);$$
- if $n \geq 1$, $P \in X_{-n}$ and $P = (-s_n + x, y)$, then
  $$S(P) = (-s_{n-1} + 2x, y/2).$$

The following properties of $S$ are an easy consequence of the given definition:

1. the fixed points of $S$ are $P_{-\infty}$ and $P_{+\infty}$;
2. $S(P_z) = P_{z+1}$, for any $z \in \mathbb{Z}$. 
(3) \( S(X_z) = X_{z+1} \), for any \( z \in \mathbb{Z} \);
(4) for any \( P \in X \), the \( y \)-coordinate of \( S(P) \) is one half the \( y \)-coordinate of \( P \);
(5) \( S \) is continuous on \( X \).

Thanks to (5), \( S \) generates a discrete semigroup (actually a group) on \( X \), which we denote by \( \{ S_n \} \).

\section*{Step 3: The Global Attractor \( A \).}

Let us set
\[
A := \{ P_{-\infty} \} \cup \{ P_{+\infty} \} \cup \{ P_z : z \in \mathbb{Z} \}.
\]

We claim that \( A \) is the (disconnected) global attractor for \( \{ S_n \} \).

The set \( A \) is trivially compact, and it is invariant thanks to properties (1) and (2) of \( S \). Thus we have only to prove that \( A \) attracts any bounded subset of \( X \). To this end, it is enough to prove that, for any \( a \geq 0 \), \( A \) attracts \( B_{sa} \), where
\[
B_a := \{ (x, y) \in X : y \leq a \}.
\]

Applying \( n \) times property (4) of \( S \), it follows that \( S_n(B_a) = B_{a+2^n} \), for any \( n \in \mathbb{N} \).

The thesis now follows remarking that the sets \( B_{\varepsilon} \), with \( \varepsilon > 0 \), are a fundamental system of neighbourhoods of \( A \) in \( X \).

\section*{Remark 5.2.}

With some small modification, the same phenomenon of Theorem 5.1 can be obtained with a path-connected complete phase space.

Let indeed \( X \) and \( S \) be like in the proof of Theorem 5.1 and let us set:
\[
\begin{align*}
Y &:= \{ (x, y) \in \mathbb{R}^2 : y \leq 0, (x - 1/2)^2 + y^2 = 9/4 \}, \\
W &:= \{ (0, y) \in \mathbb{R}^2 : y \geq 0 \}, \\
\tilde{X} &:= X \cup Y \cup W.
\end{align*}
\]

and let us define \( \tilde{S} : \tilde{X} \to \tilde{X} \) by
\[
(\tilde{S}(P)) = \begin{cases} 
S(P) & \text{if } P \in X, \\
P & \text{if } P \in Y, \\
(0, y/2) & \text{if } P = (0, y) \in W.
\end{cases}
\]

It is easy to verify that:
\begin{itemize}
  \item \( \tilde{X} \) is a path-connected complete metric space (see Figure 2);
  \item \( \tilde{S} \) generates a discrete group of continuous operators on \( \tilde{X} \);
  \item \( \tilde{A} := A \cup Y \) is the (disconnected) global attractor for the discrete group generated by \( \tilde{S} \).
\end{itemize}
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