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Further properties of the star, left-star, right-star, and minus partial orderings

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Abstract

Certain classes of matrices are indicated for which the star, left-star, right-star, and minus partial orderings, or some of them, are equivalent. Characterizations of the left-star and right-star orderings, similar to those devised by Hartwig and Styan [Linear Algebra Appl. 82 (1986) 145] for the star and minus orderings, are established along with other auxiliary results, which are of independent interest as well. Some inheritance-type properties of matrices are also given. The class of EP matrices appears to be essential in several points of our considerations.

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1. Introduction

Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ complex matrices. The symbols $\mathbf{K}^*$, $\mathscr{R}(\mathbf{K})$, and $\mathrm{r}(\mathbf{K})$ will denote the conjugate transpose, range, and rank, respectively, of $\mathbf{K} \in \mathbb{C}_{m,n}$.

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Further, $\mathbf{K}^+$ will stand for the Moore–Penrose inverse of $\mathbf{K}$, i.e., the unique matrix satisfying the equations

$$\mathbf{KK}^+\mathbf{K} = \mathbf{K}, \quad \mathbf{K}^+\mathbf{KK}^+ = \mathbf{K}^+, \quad \mathbf{KK}^+ = (\mathbf{KK}^+)^*, \quad \mathbf{K}^+\mathbf{K} = (\mathbf{K}^+\mathbf{K})^*, \tag{1.1}$$

and $\mathbf{I}_n$ will be the identity matrix of order n. Moreover, $\mathbb{C}_{m,n}^{\mathsf{PI}}$ will denote the subset of $\mathbb{C}_{m,n}$ comprising partial isometries, i.e., $\mathbb{C}_{m,n}^{\mathsf{PI}} = \{\mathbf{K} \in \mathbb{C}_{m,n} : \mathbf{KK}^*\mathbf{K} = \mathbf{K}\}$, and $\mathbb{C}_n^{\mathsf{U}}$, $\mathbb{C}_n^{\mathsf{EP}}$, $\mathbb{C}_n^{\mathsf{N}}$, $\mathbb{C}_n^{\mathsf{H}}$, and $\mathbb{C}_n^{\mathsf{HI}}$ will denote the subsets of $\mathbb{C}_{n,n}$ consisting of unitary, EP, normal, Hermitian, and Hermitian idempotent matrices (orthogonal projectors), respectively, i.e., $\mathbb{C}_n^{\mathsf{U}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathbf{KK}^* = \mathbf{I}_n\}$, $\mathbb{C}_n^{\mathsf{EP}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathscr{R}(\mathbf{K}) = \mathscr{R}(\mathbf{K}^*)\}$, $\mathbb{C}_n^{\mathsf{N}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathbf{KK}^* = \mathbf{K}^*\mathbf{K}\}$, $\mathbb{C}_n^{\mathsf{H}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathbf{K} = \mathbf{K}^*\}$, and $\mathbb{C}_n^{\mathsf{HI}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathbf{K} = \mathbf{K}^*, \mathbf{K} = \mathbf{K}^2\}$. It is clear that $\mathbb{C}_n^{\mathsf{HI}} \subseteq \mathbb{C}_n^{\mathsf{H}} \subseteq \mathbb{C}_n^{\mathsf{N}} \subseteq \mathbb{C}_n^{\mathsf{EP}}$.

Four matrix partial orderings defined in $\mathbb{C}_{m,n}$ are considered in this paper. The first of them is the star ordering introduced by Drazin [8], which is determined by

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \Leftrightarrow \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \text{ and } \mathbf{AA}^* = \mathbf{BA}^*. \tag{1.2}$$

Modifying (1.2), Baksalary and Mitra [4, p. 76] proposed the left-star and right-star orderings characterized as

$$\mathbf{A} * \!\leqslant \mathbf{B} \Leftrightarrow \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \text{ and } \mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}), \tag{1.3}$$

$$\mathbf{A} \leqslant\! * \mathbf{B} \Leftrightarrow \mathbf{AA}^* = \mathbf{BA}^* \text{ and } \mathscr{R}(\mathbf{A}^*) \subseteq \mathscr{R}(\mathbf{B}^*). \tag{1.4}$$

The fourth partial ordering of interest is the minus (rank subtractivity) ordering devised by Hartwig [10] and independently by Nambooripad [14]. It can be specified as

$$\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \Leftrightarrow \mathrm{r}(\mathbf{B} - \mathbf{A}) = \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A}). \tag{1.5}$$

From (1.3) and (1.4) it is seen that

$$\mathbf{A} \leqslant\! * \mathbf{B} \Leftrightarrow \mathbf{A}^* * \!\leqslant \mathbf{B}^*. \tag{1.6}$$

Moreover, Theorem 2.1 in [4] asserts that the left-star and right-star orderings are located between the star and minus orderings in the sense that

$$\mathbf{A} \overset{*}{\leqslant} \mathbf{B} \Leftrightarrow \mathbf{A} * \!\leqslant \mathbf{B} \text{ and } \mathbf{A} \leqslant\! * \mathbf{B}, \tag{1.7}$$

$$\mathbf{A} * \!\leqslant \mathbf{B} \Rightarrow \mathbf{A} \overset{-}{\leqslant} \mathbf{B} \quad \text{and} \quad \mathbf{A} \leqslant\! * \mathbf{B} \Rightarrow \mathbf{A} \overset{-}{\leqslant} \mathbf{B}. \tag{1.8}$$

The purpose of the present paper is to consider relationships between orderings defined in (1.2)–(1.5) with the emphasis laid on indicating classes of matrices for which all or some of them are equivalent. Characterizations of the left-star and right-star orderings, similar to those devised by Hartwig and Styan [12] for the star and minus orderings, are established along with other such general properties, which seems to be of interest exceeding the frames of the present paper. Some inheritance-type properties of matrices are also given.

2. Preliminary results

The two lemmas below contain modified versions of the characterizations of the orders $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ and $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ developed by Hartwig and Styan [12] (see their Theorems 1 and 2, Corollary 1(b), and comments on page 154). The modifications consist in replacing unitary matrices by semiunitary ones, obtained by removing the columns corresponding to singular values equal to zero in cases where $\mathbf{B}$ is singular. In addition notice that the assumption $\mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B})$ adopted in several results of this paper is natural, for otherwise each of the orders considered holds merely when $\mathbf{A} = \mathbf{B}$.

Lemma 2.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ and let $a = \mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B}) = b$. Then $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ if and only if*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^* \text{ and } \mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{D}_2\end{pmatrix}\mathbf{V}^*, \tag{2.1}$$

for some $\mathbf{U} \in \mathbb{C}_{m,b}$, $\mathbf{V} \in \mathbb{C}_{n,b}$ such that $\mathbf{U}^*\mathbf{U} = \mathbf{I}_b = \mathbf{V}^*\mathbf{V}$ and positive definite diagonal matrices $\mathbf{D}_1$, $\mathbf{D}_2$ of degree a, $b - a$, respectively.

Lemma 2.2. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ and let $a = \mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B}) = b$. Then $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ if and only if*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^* \text{ and } \mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{D}_1 + \mathbf{R}\mathbf{D}_2\mathbf{S} & \mathbf{R}\mathbf{D}_2\\ \mathbf{D}_2\mathbf{S} & \mathbf{D}_2\end{pmatrix}\mathbf{V}^*, \tag{2.2}$$

for some $\mathbf{U} \in \mathbb{C}_{m,b}$, $\mathbf{V} \in \mathbb{C}_{n,b}$ such that $\mathbf{U}^*\mathbf{U} = \mathbf{I}_b = \mathbf{V}^*\mathbf{V}$, positive definite diagonal matrices $\mathbf{D}_1$, $\mathbf{D}_2$ of degree a, $b - a$, respectively, and arbitrary $\mathbf{R} \in \mathbb{C}_{a,b-a}$, $\mathbf{S} \in \mathbb{C}_{b-a,a}$.

It appears that similar characterizations can be obtained for the left-star and right-star partial orderings. They constitute the first original result of this paper. A proof is given for the left-star version only, as the right-star counterpart follows analogously due to the symmetry of definitions (1.3) and (1.4) documented in (1.6). This rule is adopted throughout the entire paper.

Theorem 2.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ and let $a = \mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B}) = b$. Then $\mathbf{A} * \!\leqslant \mathbf{B}$ if and only if*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^* \text{ and } \mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{D}_2\mathbf{S} & \mathbf{D}_2\end{pmatrix}\mathbf{V}^*, \tag{2.3}$$

while $\mathbf{A} \leqslant * \mathbf{B}$ *if and only if*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^* \text{ and } \mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{D}_1 & \mathbf{R}\mathbf{D}_2 \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix}\mathbf{V}^* \tag{2.4}$$

for some $\mathbf{U} \in \mathbb{C}_{m,b}$, $\mathbf{V} \in \mathbb{C}_{n,b}$ *such that* $\mathbf{U}^*\mathbf{U} = \mathbf{I}_b = \mathbf{V}^*\mathbf{V}$, *positive definite diagonal matrices* $\mathbf{D}_1$, $\mathbf{D}_2$ *of degree* a, $b - a$, *respectively, and arbitrary* $\mathbf{R} \in \mathbb{C}_{a,b-a}$, $\mathbf{S} \in \mathbb{C}_{b-a,a}$.

Proof. If $\mathbf{A} * \leqslant \mathbf{B}$, then on account of the first part of (1.8) and Lemma 2.2 it is clear that $\mathbf{A}$ and $\mathbf{B}$ must be of the forms given in (2.2). Consequently,

$$\mathbf{A}^*\mathbf{A} = \mathbf{V}\begin{pmatrix} \mathbf{D}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^* \quad \text{and} \quad \mathbf{A}^*\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{D}_1^2 + \mathbf{D}_1\mathbf{R}\mathbf{D}_2\mathbf{S} & \mathbf{D}_1\mathbf{R}\mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*.$$

It is clear, therefore, that $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B}$ entails $\mathbf{D}_1\mathbf{R}\mathbf{D}_2 = \mathbf{0}$, and hence $\mathbf{R} = \mathbf{0}$. On account of (1.3), this establishes the "only if part". The "if part" follows by noting that matrices (2.3) obviously satisfy the first condition on the right-hand side in (1.3) and, in view of the equality

$$\mathbf{A} = \mathbf{B}\mathbf{V}\begin{pmatrix} \mathbf{I}_a & \mathbf{0} \\ -\mathbf{S} & \mathbf{0} \end{pmatrix}\mathbf{V}^*,$$

the second condition as well. □

In what follows, we will refer to some properties of the $b \times b$ matrix

$$\mathbf{W} = \mathbf{V}^*\mathbf{U} = \begin{pmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{pmatrix}, \tag{2.5}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the matrices occurring in the representations (2.1)–(2.4) (with $m = n$). Notice that if $\mathbf{U}$ and $\mathbf{V}$ are partitioned as $\mathbf{U} = (\mathbf{U}_1 : \mathbf{U}_2)$ and $\mathbf{V} = (\mathbf{V}_1 : \mathbf{V}_2)$, where $\mathbf{U}_1, \mathbf{V}_1 \in \mathbb{C}_{n,a}$ and $\mathbf{U}_2, \mathbf{V}_2 \in \mathbb{C}_{n,b-a}$, then $\mathbf{W}_{ij} = \mathbf{V}_i^*\mathbf{U}_j$, $i, j = 1, 2$, and thus $\mathbf{W}_{11} \in \mathbb{C}_{a,a}$, $\mathbf{W}_{12} \in \mathbb{C}_{a,b-a}$, $\mathbf{W}_{21} \in \mathbb{C}_{b-a,a}$, and $\mathbf{W}_{22} \in \mathbb{C}_{b-a,b-a}$. According to Lemmas 2.1, 2.2 and Theorem 2.1, the representation of a predecessor matrix $\mathbf{A}$ is identical in all partial orderings considered in the present paper. This motivates collecting in one place specific properties of the matrix $\mathbf{W}$ defined in (2.5) corresponding to $\mathbf{A}$ being EP, normal, or Hermitian matrix.

Theorem 2.2. *For any* $\mathbf{A} \in \mathbb{C}_{n,n}$ *of the form as in* (2.1)–(2.4) *and* $\mathbf{W} \in \mathbb{C}_{b,b}$ *of the form* (2.5):

$$\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}} \Leftrightarrow \mathbf{W}_{11} \in \mathbb{C}_a^{\mathsf{U}},\ \ \mathbf{W}_{12} = \mathbf{0},\ \ \mathbf{W}_{21} = \mathbf{0}, \tag{2.6}$$

$$\mathbf{A} \in \mathbb{C}_n^{\mathsf{N}} \Leftrightarrow \mathbf{W}_{11} \in \mathbb{C}_a^{\mathsf{U}},\ \ \mathbf{W}_{12} = \mathbf{0},\ \ \mathbf{W}_{21} = \mathbf{0},\ \ \mathbf{W}_{11}\mathbf{D}_1^2 = \mathbf{D}_1^2\mathbf{W}_{11}, \tag{2.7}$$

$$\mathbf{A} \in \mathbb{C}_n^{\mathsf{H}} \Leftrightarrow \mathbf{W}_{11} \in \mathbb{C}_a^{\mathsf{U}},\ \ \mathbf{W}_{12} = \mathbf{0},\ \ \mathbf{W}_{21} = \mathbf{0},\ \ \mathbf{W}_{11}\mathbf{D}_1 = \mathbf{D}_1\mathbf{W}_{11}^*. \tag{2.8}$$

Moreover, the last condition in (2.7) *may be replaced by* $\mathbf{W}_{11}\mathbf{D}_1 = \mathbf{D}_1\mathbf{W}_{11}$.

Proof. Since

$$\mathbf{A}^{+} = \mathbf{V}\begin{pmatrix}\mathbf{D}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

it follows that the equality $\mathbf{AA}^{+} = \mathbf{A}^{+}\mathbf{A}$, which is equivalent to $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$, holds if and only if

$$\mathbf{U}\begin{pmatrix}\mathbf{I}_a & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^* = \mathbf{V}\begin{pmatrix}\mathbf{I}_a & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*. \tag{2.9}$$

Premultiplying both sides of (2.9) by $\mathbf{V}^*$, postmultiplying them by $\mathbf{U}$, and adopting notation (2.5) shows that (2.9) immediately entails $\mathbf{W}_{12} = \mathbf{0}$ and $\mathbf{W}_{21} = \mathbf{0}$. The additional condition in (2.6) is a simple consequence of the fact that with the use of notation related to (2.5) $\mathbf{A}$ is expressible as $\mathbf{A} = \mathbf{U}_1\mathbf{D}_1\mathbf{V}_1^*$. Consequently, since $\mathbf{P}_{\mathbf{U}_1} = \mathbf{U}_1\mathbf{U}_1^*$ and $\mathbf{P}_{\mathbf{V}_1} = \mathbf{V}_1\mathbf{V}_1^*$ are the orthogonal projectors onto $\mathscr{R}(\mathbf{U}_1) = \mathscr{R}(\mathbf{A})$ and $\mathscr{R}(\mathbf{V}_1) = \mathscr{R}(\mathbf{A}^*)$, respectively, it follows that

$$\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}} \Leftrightarrow \mathbf{U}_1\mathbf{U}_1^* = \mathbf{V}_1\mathbf{V}_1^*. \tag{2.10}$$

Premultiplying the equality in (2.10) by $\mathbf{V}_1^*$ and postmultiplying it by $\mathbf{V}_1$ shows that

$$\mathbf{W}_{11}\mathbf{W}_{11}^* = \mathbf{I}_a, \tag{2.11}$$

i.e., $\mathbf{W}_{11} \in \mathbb{C}_a^{\mathsf{U}}$. Conversely, premultiplying and postmultiplying (2.11) by $\mathbf{V}_1$ and $\mathbf{V}_1^*$, respectively, yields

$$\mathbf{P}_{\mathbf{V}_1}\mathbf{P}_{\mathbf{U}_1}\mathbf{P}_{\mathbf{V}_1} = \mathbf{P}_{\mathbf{V}_1}. \tag{2.12}$$

Hence it is seen that the product $\mathbf{P}_{\mathbf{V}_1}\mathbf{P}_{\mathbf{U}_1}$ of two orthogonal projectors is idempotent, for which it is necessary and sufficient that

$$\mathbf{P}_{\mathbf{V}_1}\mathbf{P}_{\mathbf{U}_1} = \mathbf{P}_{\mathbf{U}_1}\mathbf{P}_{\mathbf{V}_1}; \tag{2.13}$$

cf., e.g., Theorem 1 in [1] and a much more general result in [2]. Consequently, from (2.12) and (2.13) it follows that

$$\mathscr{R}(\mathbf{A}^*) = \mathscr{R}(\mathbf{P}_{\mathbf{V}_1}) = \mathscr{R}(\mathbf{P}_{\mathbf{V}_1}\mathbf{P}_{\mathbf{U}_1}\mathbf{P}_{\mathbf{V}_1}) \subseteq \mathscr{R}(\mathbf{P}_{\mathbf{V}_1}\mathbf{P}_{\mathbf{U}_1}) = \mathscr{R}(\mathbf{P}_{\mathbf{U}_1}\mathbf{P}_{\mathbf{V}_1}) \subseteq \mathscr{R}(\mathbf{A}).$$

Combining the inclusion $\mathscr{R}(\mathbf{A}^*) \subseteq \mathscr{R}(\mathbf{A})$ with $\mathrm{r}(\mathbf{A}^*) = \mathrm{r}(\mathbf{A})$ leads to the equality of these two subspaces, thus proving that $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$.

Further, since $\mathbb{C}_n^{\mathsf{N}} \subseteq \mathbb{C}_n^{\mathsf{EP}}$, the problem in establishing (2.7) is what should be added to the right-hand side of (2.6) to obtain a set of necessary and sufficient conditions for the normality of $\mathbf{A}$, i.e., for

$$\mathbf{U}\begin{pmatrix}\mathbf{D}_1^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^* = \mathbf{V}\begin{pmatrix}\mathbf{D}_1^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*. \tag{2.14}$$

Premultiplying and postmultiplying (2.14) by $\mathbf{V}^*$ and $\mathbf{U}$, respectively, leads immediately to $\mathbf{W}_{11}\mathbf{D}_1^2 = \mathbf{D}_1^2\mathbf{W}_{11}$. The converse follows by noting that, on account of the condition on the right-hand side of (2.10), the equality $\mathbf{U}_1\mathbf{D}_1^2\mathbf{U}_1^* = \mathbf{V}_1\mathbf{D}_1^2\mathbf{V}_1^*$, constituting the reduced version of (2.14), is equivalent to $\mathbf{V}_1^*\mathbf{U}_1\mathbf{D}_1^2 = \mathbf{D}_1^2\mathbf{V}_1^*\mathbf{U}_1$, i.e., to the last condition in (2.7).

Similarly it follows that if $\mathbf{A} \in \mathbb{C}_n^{\mathsf{H}}$, i.e., if

$$\mathbf{U}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^* = \mathbf{V}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{2.15}$$

then premultiplying and postmultiplying by $\mathbf{V}^*$ and $\mathbf{V}$, respectively, entails $\mathbf{W}_{11}\mathbf{D}_1 = \mathbf{D}_1\mathbf{W}_{11}^*$. Adding this equality to the conditions in (2.6) yields the right-hand side of (2.8). The converse implication is obtained similarly as above by noting that $\mathbf{U}_1\mathbf{D}_1\mathbf{V}_1^* = \mathbf{V}_1\mathbf{D}_1\mathbf{U}_1^*$, which represents a reduced version of (2.15), is equivalent to $\mathbf{V}_1^*\mathbf{U}_1\mathbf{D}_1 = \mathbf{D}_1\mathbf{U}_1^*\mathbf{V}_1$.

Replacing the condition $\mathbf{W}_{11}\mathbf{D}_1^2 = \mathbf{D}_1^2\mathbf{W}_{11}$ by $\mathbf{W}_{11}\mathbf{D}_1 = \mathbf{D}_1\mathbf{W}_{11}$ is possible on account of the fact that, for any $\mathbf{K} = (k_{ij}) \in \mathbb{C}_{n,n}$ and any diagonal matrix $\mathbf{D} \in \mathbb{C}_{n,n}$ with the diagonal elements $d_i > 0$, the square of $\mathbf{D}$ in $\mathbf{K}\mathbf{D}^2 = \mathbf{D}^2\mathbf{K}$ may be replaced by $\mathbf{D}$ itself. In fact, it can easily be verified that $\mathbf{K}\mathbf{D}^2 = \mathbf{D}^2\mathbf{K}$ corresponds to

$$(d_i + d_j)(d_i - d_j)k_{ij} = 0, \quad i, j = 1, \ldots, n,$$

which in view of $d_i + d_j > 0$ is equivalent to $(d_i - d_j)k_{ij} = 0$; $i, j = 1, \ldots, n$, i.e., to $\mathbf{KD} = \mathbf{DK}$. □

In this section, the usefulness of Theorem 2.2 is shown in the context of a specific characterization of normal matrices. It is easily seen that if $\mathbf{A} \in \mathbb{C}_{n,n}$ is nonsingular, then it is normal if and only if $\mathbf{A}^*\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A}^*$. A natural generalization of this condition to the form

$$\mathbf{A}^*\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}^*, \tag{2.16}$$

although remains necessary, is no longer sufficient. For example, if

$$\mathbf{A} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

then $\mathbf{A}^* = \mathbf{A}^+$, and therefore (2.16) holds trivially, but $\mathbf{AA}^* \neq \mathbf{A}^*\mathbf{A}$. It appears, however, that (2.16) forces $\mathbf{A}$ to be normal within the set of EP matrices.

Corollary 2.1. *A matrix $\mathbf{A} \in \mathbb{C}_{n,n}$ is normal if and only if it is an EP matrix satisfying $\mathbf{A}^*\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}^*$.*

Proof. As pointed out above, for a nonsingular $\mathbf{A} \in \mathbb{C}_{n,n}$ the result is trivial. Moreover, there is no loss in generality when a singular matrix $\mathbf{A}$ is assumed to have a representation as in (2.1)–(2.4). Then, after taking the conjugate transposes on both sides, premultiplying by $\mathbf{U}^*$, and postmultiplying by $\mathbf{V}$, (2.16) takes the form

$$\begin{pmatrix}\mathbf{D}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{W}_{11} & \mathbf{W}_{12}\\ \mathbf{W}_{21} & \mathbf{W}_{22}\end{pmatrix}\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{W}_{11} & \mathbf{W}_{12}\\ \mathbf{W}_{21} & \mathbf{W}_{22}\end{pmatrix}\begin{pmatrix}\mathbf{D}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}.$$

It can straightforwardly be verified that this equality is equivalent to $\mathbf{W}_{11}\mathbf{D}_1^2 = \mathbf{D}_1^2\mathbf{W}_{11}$. Consequently, comparing characterizations (2.6) and (2.7) concludes the proof. □

An analogue of Theorem 2.2 concerning matrices $\mathbf{B}$ involved in Lemmas 1.1, 1.2 and Theorem 2.1 refers to the entire matrix $\mathbf{W} = \mathbf{V}^*\mathbf{U}$ instead of submatrices occurring in its partitioned form (2.5).

Theorem 2.3. *Let* $\mathbf{B} \in \mathbb{C}_{n,n}$ *of rank* $\mathrm{r}(\mathbf{B}) = b$ *be of the form* $\mathbf{B} = \mathbf{UNV}^*$, *with* $\mathbf{U}^*\mathbf{U} = \mathbf{I}_b = \mathbf{V}^*\mathbf{V}$ *and* $\mathbf{N} \in \mathbb{C}_{b,b}$ *representing any of the matrices whose forms are seen in* (2.1)–(2.4), *and let* $\mathbf{W} = \mathbf{V}^*\mathbf{U}$. *Then*:

$$\mathbf{B} \in \mathbb{C}_n^{\mathsf{EP}} \Leftrightarrow \mathbf{W} \in \mathbb{C}_b^{\mathsf{U}}, \tag{2.17}$$

$$\mathbf{B} \in \mathbb{C}_n^{\mathsf{N}} \Leftrightarrow \mathbf{W} \in \mathbb{C}_b^{\mathsf{U}}, \quad \mathbf{WNN}^* = \mathbf{N}^*\mathbf{NW}, \tag{2.18}$$

$$\mathbf{B} \in \mathbb{C}_n^{\mathsf{H}} \Leftrightarrow \mathbf{W} \in \mathbb{C}_b^{\mathsf{U}}, \quad \mathbf{WN} = \mathbf{N}^*\mathbf{W}^*. \tag{2.19}$$

Proof. In each of the cases (2.1)–(2.4) the matrix $\mathbf{N}$ in $\mathbf{B} = \mathbf{UNV}^*$ is nonsingular. Thus it follows that the orthogonal projectors onto $\mathscr{R}(\mathbf{B})$ and $\mathscr{R}(\mathbf{B}^*)$ are expressible as $\mathbf{UU}^*$ and $\mathbf{VV}^*$. Premultiplying and postmultiplying $\mathbf{UU}^* = \mathbf{VV}^*$ by $\mathbf{V}^*$ and $\mathbf{V}$ leads to $\mathbf{WW}^* = \mathbf{I}_b$, thus establishing the "⇒ part" of (2.17). Applying the same procedure to the equalities

$$\mathbf{UNN}^*\mathbf{U}^* = \mathbf{VN}^*\mathbf{NV}^* \quad \text{and} \quad \mathbf{UNV}^* = \mathbf{VN}^*\mathbf{U}^*, \tag{2.20}$$

which correspond to $\mathbf{B} \in \mathbb{C}_n^{\mathsf{N}}$ and $\mathbf{B} \in \mathbb{C}_n^{\mathsf{H}}$, respectively, and adding $\mathbf{W} \in \mathbb{C}_b^{\mathsf{U}}$ from (2.17) proves the "⇒ parts" of (2.18) and (2.19). The converse implications follow by arguments similar to those used in the proof of Theorem 2.2. Premultiplying and postmultiplying $\mathbf{WW}^* = \mathbf{I}_b$ by $\mathbf{V}$ and $\mathbf{V}^*$, respectively, yields $\mathbf{P_V P_U P_V} = \mathbf{P_V}$, where $\mathbf{P_U} = \mathbf{UU}^*$ and $\mathbf{P_V} = \mathbf{VV}^*$. Hence $\mathbf{P_V P_U} = \mathbf{P_U P_V}$, and thus

$$\mathscr{R}(\mathbf{B}^*) = \mathscr{R}(\mathbf{P_V}) = \mathscr{R}(\mathbf{P_V P_U P_V}) \subseteq \mathscr{R}(\mathbf{P_U P_V}) \subseteq \mathscr{R}(\mathbf{P_U}) = \mathscr{R}(\mathbf{B}),$$

which combined with $\mathrm{r}(\mathbf{B}^*) = \mathrm{r}(\mathbf{B})$ shows that $\mathbf{B} \in \mathbb{C}_n^{\mathsf{EP}}$. The "⇐ parts" of (2.18) and (2.19) are simple consequences of the fact that, on account of $\mathbf{UU}^* = \mathbf{VV}^*$, the equalities in (2.20) are equivalent to $\mathbf{WNN}^* = \mathbf{N}^*\mathbf{NW}$ and $\mathbf{WN} = \mathbf{N}^*\mathbf{W}^*$. □

3. Equivalence of partial orderings and inheritance-type properties

A natural problem which arises in the context of definitions (1.2)–(1.5), relationships (1.7) and (1.8), and characterizations (2.1)–(2.4) is to describe situations where all the four orderings considered in the present paper, or some of them, become equivalent. The number of known results concerning this problem seems to be rather limited. Hartwig and Styan [13, Theorem 2.3] pointed out that the minus and star orderings are equivalent within the set of matrices representing orthogonal projectors. In view of (1.7) and (1.8), this means that

$$\mathbf{A} \overset{-}{\leqslant} \mathbf{B} \Leftrightarrow \mathbf{A} *\!\leqslant \mathbf{B} \Leftrightarrow \mathbf{A} \leqslant\!* \mathbf{B} \Leftrightarrow \mathbf{A} \overset{*}{\leqslant} \mathbf{B} \tag{3.1}$$

for any $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\mathsf{HI}}$. It appears that the same conclusion is valid for substantially wider classes of matrices.

Theorem 3.1. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$ *and* $\mathbf{B} \in \mathbb{C}_{m,n}^{\mathsf{PI}}$ *of ranks* $a = \mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B}) = b$ *be minus-ordered as* $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$. *If their representations of the forms given in* (2.2) *are such that all the diagonal elements* $(\mathbf{D}_1)_i$ $(i = 1, \ldots, a)$ *of* $\mathbf{D}_1$ *are not greater than one or all the diagonal elements* $(\mathbf{D}_2)_j$ $(j = 1, \ldots, b - a)$ *of* $\mathbf{D}_2$ *are not less than one, then* $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$.

Proof. Referring to conditions (1.1), it can straightforwardly be verified that the Moore–Penrose inverse of a matrix $\mathbf{B}$ of the form as in (2.2) admits a representation

$$\mathbf{B}^{+} = \mathbf{V}\begin{pmatrix} \mathbf{D}_1^{-1} & -\mathbf{D}_1^{-1}\mathbf{R} \\ -\mathbf{S}\mathbf{D}_1^{-1} & \mathbf{D}_2^{-1} + \mathbf{S}\mathbf{D}_1^{-1}\mathbf{R} \end{pmatrix}\mathbf{U}^{*}.$$

Consequently, since the set of partial isometries may alternatively be specified as $\mathbb{C}_{m,n}^{\mathsf{PI}} = \{\mathbf{K} \in \mathbb{C}_{m,n} \colon \mathbf{K}^{+} = \mathbf{K}^{*}\}$, it follows that $\mathbf{B} \in \mathbb{C}_{m,n}^{\mathsf{PI}}$ if and only if

$$\mathbf{D}_1^{-1} = \mathbf{D}_1 + \mathbf{S}^{*}\mathbf{D}_2\mathbf{R}^{*}, \quad -\mathbf{D}_1^{-1}\mathbf{R} = \mathbf{S}^{*}\mathbf{D}_2, \tag{3.2}$$

$$-\mathbf{S}\mathbf{D}_1^{-1} = \mathbf{D}_2\mathbf{R}^{*}, \quad \mathbf{D}_2^{-1} + \mathbf{S}\mathbf{D}_1^{-1}\mathbf{R} = \mathbf{D}_2. \tag{3.3}$$

Combining the two equalities in (3.2) leads to $\mathbf{D}_1^{-1} = \mathbf{D}_1 - \mathbf{D}_1^{-1}\mathbf{R}\mathbf{R}^{*}$, and hence to

$$\mathbf{R}\mathbf{R}^{*} = \mathbf{D}_1^{2} - \mathbf{I}_a. \tag{3.4}$$

Since $\mathbf{R}\mathbf{R}^{*}$ is nonnegative definite and the condition $(\mathbf{D}_1)_i \leqslant 1$, $i = 1, \ldots, a$, means that $\mathbf{D}_1^2 - \mathbf{I}_a$ is nonpositive definite, it follows from (3.4) that $\mathbf{R}\mathbf{R}^{*} = \mathbf{0}$. Hence $\mathbf{R} = \mathbf{0}$ and, consequently, $\mathbf{S} = \mathbf{0}$. Similarly, combining the two equalities in (3.3) leads to $\mathbf{D}_2^{-1} - \mathbf{D}_2\mathbf{R}^{*}\mathbf{R} = \mathbf{D}_2$, and hence to $\mathbf{R}^{*}\mathbf{R} = \mathbf{D}_2^{-2} - \mathbf{I}_{b-a}$. Analogous argumentation as above shows that if $(\mathbf{D}_2)_j \geqslant 1$, $j = 1, \ldots, b - a$, then $\mathbf{R}^{*}\mathbf{R} = \mathbf{0}$, and hence $\mathbf{R} = \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$. In view of Lemma 2.1, this establishes the required order $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$. □

In view of (1.7), (1.8), and the fact that $\mathbf{D}_1$ in a representation of $\mathbf{A} \in \mathbb{C}_{m,n}^{\mathsf{PI}}$ in (2.2) must be the identity matrix of order a, Theorem 3.1 leads immediately to the following version of Lemma 2 in [9].

Corollary 3.1. *The equivalences* (3.1) *hold for any* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}^{\mathsf{PI}}$.

The next two results concerning the problem of equivalence of partial orderings deal with square matrices. They reveal a special role of the property that predecessors in the orders considered are EP matrices.

Theorem 3.2. *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$ *and* $\mathbf{B} \in \mathbb{C}_n^{\mathsf{N}}$. *Then*

$$\mathbf{A} \ast\!\leqslant \mathbf{B} \Leftrightarrow \mathbf{A} \overset{*}{\leqslant} \mathbf{B} \quad \textit{and} \quad \mathbf{A} \leqslant\!\ast\, \mathbf{B} \Leftrightarrow \mathbf{A} \overset{*}{\leqslant} \mathbf{B} \tag{3.5}$$

in each case where the submatrices $\mathbf{W}_{12}$ *and* $\mathbf{W}_{21}$ *of the matrix* $\mathbf{W}$ *in* (2.5), *generated by* $\mathbf{U}$ *and* $\mathbf{V}$ *occurring in the representations* (2.3) *and* (2.4), *respectively*, *are null matrices.*

Proof. In view of (1.7), (1.6), and (2.6), the proof of (3.5) reduces to establishing that if $\mathbf{B} \in \mathbb{C}_n^{\mathsf{N}}$, then $\mathbf{A} \ast\!\leqslant \mathbf{B}$ entails $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ whenever $\mathbf{W}_{12} = \mathbf{0}$ and $\mathbf{W}_{21} = \mathbf{0}$. According to Theorem 2.1, the order $\mathbf{A} \ast\!\leqslant \mathbf{B}$ means that $\mathbf{A}$ and $\mathbf{B}$ have representations such as in (2.3). Consequently, on account of (2.6) and (2.18), if $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$, then the normality of $\mathbf{B}$ implies

$$\begin{pmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{D}_1^2 & \mathbf{D}_1\mathbf{S}^*\mathbf{D}_2 \\ \mathbf{D}_2\mathbf{S}\mathbf{D}_1 & \mathbf{D}_2\mathbf{S}\mathbf{S}^*\mathbf{D}_2 + \mathbf{D}_2^2 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{D}_1^2 + \mathbf{S}^*\mathbf{D}_2^2\mathbf{S} & \mathbf{S}^*\mathbf{D}_2^2 \\ \mathbf{D}_2^2\mathbf{S} & \mathbf{D}_2^2 \end{pmatrix} \begin{pmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{pmatrix},$$

and hence, in particular,

$$\mathbf{W}_{22}\mathbf{D}_2\mathbf{S}\mathbf{S}^*\mathbf{D}_2 + \mathbf{W}_{22}\mathbf{D}_2^2 = \mathbf{D}_2^2\mathbf{W}_{22}. \tag{3.6}$$

Since $\mathbf{W}_{12} = \mathbf{0}$ and $\mathbf{W}_{21} = \mathbf{0}$, the first condition on the right hand side of (2.18) ensures that $\mathbf{W}_{22} \in \mathbb{C}_{b-a}^{\mathsf{U}}$, and thus postmultiplying (3.6) by $\mathbf{W}_{22}^*$ leads to

$$\begin{aligned} \operatorname{tr}(\mathbf{D}_2^2) &= \operatorname{tr}(\mathbf{W}_{22}\mathbf{D}_2\mathbf{S}\mathbf{S}^*\mathbf{D}_2\mathbf{W}_{22}^*) + \operatorname{tr}(\mathbf{W}_{22}\mathbf{D}_2^2\mathbf{W}_{22}^*) \\ &= \operatorname{tr}(\mathbf{D}_2\mathbf{S}\mathbf{S}^*\mathbf{D}_2) + \operatorname{tr}(\mathbf{D}_2^2), \end{aligned} \tag{3.7}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix argument. But (3.7) actually means that the Frobenius norm of $\mathbf{D}_2\mathbf{S}$ is equal to zero, which on account of the nonsingularity of $\mathbf{D}_2$ yields $\mathbf{S} = \mathbf{0}$. After substituting $\mathbf{S} = \mathbf{0}$ to (2.3) the representations of $\mathbf{A}$ and $\mathbf{B}$ take the forms as in (2.1), thus showing that $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$. □

Theorem 3.3. *The equivalences* (3.1) *hold for any* $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$ *and any idempotent* $\mathbf{B} \in \mathbb{C}_{n,n}$.

Proof. In view of (1.7) and (1.8), the proof reduces to showing that $\mathbf{A} \overset{-}{\leqslant} \mathbf{B}$ implies $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$. On account of the equalities $\mathbf{W}_{12} = \mathbf{0}$ and $\mathbf{W}_{21} = \mathbf{0}$, which according to

(2.6) are consequences of the assumption that **A** is an EP matrix, it follows that **B** of the form as in (2.2) is idempotent if and only if

$$(\mathbf{D}_1 + \mathbf{RD}_2\mathbf{S})\mathbf{W}_{11}(\mathbf{D}_1 + \mathbf{RD}_2\mathbf{S}) + \mathbf{RD}_2\mathbf{W}_{22}\mathbf{D}_2\mathbf{S} = \mathbf{D}_1 + \mathbf{RD}_2\mathbf{S}, \tag{3.8}$$

$$(\mathbf{D}_1 + \mathbf{RD}_2\mathbf{S})\mathbf{W}_{11}\mathbf{RD}_2 + \mathbf{RD}_2\mathbf{W}_{22}\mathbf{D}_2 = \mathbf{RD}_2, \tag{3.9}$$

$$\mathbf{D}_2\mathbf{SW}_{11}(\mathbf{D}_1 + \mathbf{RD}_2\mathbf{S}) + \mathbf{D}_2\mathbf{W}_{22}\mathbf{D}_2\mathbf{S} = \mathbf{D}_2\mathbf{S}, \tag{3.10}$$

$$\mathbf{D}_2\mathbf{SW}_{11}\mathbf{RD}_2 + \mathbf{D}_2\mathbf{W}_{22}\mathbf{D}_2 = \mathbf{D}_2. \tag{3.11}$$

Combining (3.11) first with (3.9), and then with (3.10) leads to the equalities

$$\mathbf{D}_1\mathbf{W}_{11}\mathbf{RD}_2 = \mathbf{0} \quad \text{and} \quad \mathbf{D}_2\mathbf{SW}_{11}\mathbf{D}_1 = \mathbf{0}.$$

Hence, on account of the nonsingularity if $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{W}_{11}$, the last being ascertained by (2.6), it follows that $\mathbf{R} = \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$. In view of Lemmas 2.1 and 2.2, this means that $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$, as desired. □

Theorem 3.3 generalizes the statement of Hartwig and Styan [12, Section 3F], where both **A** and **B** are assumed to be Hermitian. Notice also that substituting $\mathbf{R} = \mathbf{0}$ or $\mathbf{S} = \mathbf{0}$ to (3.8) yields $\mathbf{D}_1\mathbf{W}_{11}\mathbf{D}_1 = \mathbf{D}_1$, which is a necessary and sufficient condition for **A** of the form given in (2.1)–(2.4) to be idempotent. Actually, the idempotency of a matrix **A** being a minus-predecessor of an idempotent matrix **B** holds without any restriction on **A** (cf. Proposition 1.8(a) in [6]). We call implications of such a type "inheritance properties". To be more precise, if matrices **A** and **B**, with **B** having a property $\pi(\mathbf{B})$ are the predecessor and successor, respectively, according to a given partial ordering, then we say that **A** inherits the property $\pi(\cdot)$ under this ordering whenever $\pi(\mathbf{B})$ implies $\pi(\mathbf{A})$. Several results of this type are known in the literature (cf. Theorem 2.1 in [5] and references to [3,6–8,10–13] in its proof). Theorems 3.4 and 3.5 below are additions to this collection. They assert that an EP matrix **A** cannot be a star-predecessor of a normal or Hermitian matrix **B** unless it is itself normal or Hermitian, respectively.

Theorem 3.4. *Within the set* $\mathbb{C}_n^{\mathsf{EP}}$, *if* $\mathbf{A} \ast\!\leqslant \mathbf{B}$ *or* $\mathbf{A} \leqslant\!\ast\, \mathbf{B}$ *and* **B** *is normal, then* **A** *must also be normal.*

Proof. From Theorem 3.2 it is clear that the star order $\mathbf{A} \overset{*}{\leqslant} \mathbf{B}$ can be considered instead of weaker (here seemingly only) orders $\mathbf{A} \ast\!\leqslant \mathbf{B}$ and $\mathbf{A} \leqslant\!\ast\, \mathbf{B}$. In view of (2.6) and (2.7), the proof reduces to establishing the equality $\mathbf{W}_{11}\mathbf{D}_1^2 = \mathbf{D}_1^2\mathbf{W}_{11}$ when it is known that $\mathbf{W}_{11} \in \mathbb{C}_a^{\mathsf{U}}$, $\mathbf{W}_{12} = \mathbf{0}$, $\mathbf{W}_{21} = \mathbf{0}$, and **B** is normal. On account of (2.18) and Lemma 2.1 the last condition leads to

$$\begin{pmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{D}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2^2 \end{pmatrix} = \begin{pmatrix} \mathbf{D}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2^2 \end{pmatrix} \begin{pmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{pmatrix},$$

and hence the required equality is seen immediately. □

Theorem 3.5. *Within the set* $\mathbb{C}_n^{\mathsf{EP}}$, *if* $\mathbf{A} \mathbin{*\leqslant} \mathbf{B}$ *or* $\mathbf{A} \mathbin{\leqslant *} \mathbf{B}$ *and* $\mathbf{B}$ *is Hermitian, then* $\mathbf{A}$ *must also be Hermitian.*

Proof. From the equality

$$\begin{pmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix} \begin{pmatrix} \mathbf{W}_{11}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22}^* \end{pmatrix},$$

which is a version of the condition $\mathbf{WN} = \mathbf{N}^*\mathbf{W}^*$ from (2.19), corresponding to $\mathbf{N}$ of the form revealed in Lemma 2.1, it is seen that $\mathbf{W}_{11}\mathbf{D}_1 = \mathbf{D}_1\mathbf{W}_{11}^*$. In view of (3.4), combining this condition with those in (2.6) shows that $\mathbf{A} \in \mathbb{C}_n^{\mathsf{H}}$. □

Our last result is concerned with the class of orthogonal projectors which, in addition to the specification given in Section 1, can also be characterized as $\mathbb{C}_n^{\mathsf{HI}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathbf{K} = \mathbf{KK}^*\}$ or as $\mathbb{C}_n^{\mathsf{HI}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \mathbf{K} = \mathbf{K}^*\mathbf{K}\}$. It is known that if a given matrix is a star-predecessor of an orthogonal projector, then it must be an orthogonal projector as well (cf. Theorem 3 in [8], Lemma 2 in [11], and a generalization of this statement given in Theorem 1 in [3]). The theorem below, although not in full accordance with the concept of inheritance specified above, describes another situation where the condition $\mathbf{B} \in \mathbb{C}_n^{\mathsf{HI}}$ implies that $\mathbf{A} \in \mathbb{C}_n^{\mathsf{HI}}$.

Theorem 3.6. *Within the set* $\mathbb{C}_n^{\mathsf{EP}}$, *if* $\mathbf{A}^2 = \mathbf{AB}$ *or* $\mathbf{A}^2 = \mathbf{BA}$ *and* $\mathbf{B}$ *is an orthogonal projector, then* $\mathbf{A}$ *must also be an orthogonal projector.*

Proof. If $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$ and $\mathbf{A}^2 = \mathbf{AB}$ for some $\mathbf{B} \in \mathbb{C}_n^{\mathsf{HI}}$, then

$$\mathbf{A} = \mathbf{A}^+\mathbf{A}^2 = \mathbf{A}^+\mathbf{AB} = \mathbf{A}^+\mathbf{ABB}^* = \mathbf{A}^+\mathbf{A}^2\mathbf{B}^* = \mathbf{AB}^*,$$

and hence

$$\mathbf{AA}^* = \mathbf{AB}^*\mathbf{A}^* = \mathbf{A}(\mathbf{AB})^* = \mathbf{A}(\mathbf{A}^2)^*. \tag{3.12}$$

Postmultiplying (3.12) by $(\mathbf{A}^+)^*$ yields $\mathbf{A} = \mathbf{AA}^*$, which means that $\mathbf{A} \in \mathbb{C}_n^{\mathsf{HI}}$. In the case where $\mathbf{A}^2 = \mathbf{BA}$, this conclusion follows similarly by utilizing the representation $\mathbf{B} = \mathbf{B}^*\mathbf{B}$ to show that $\mathbf{A}^*\mathbf{A} = (\mathbf{A}^*)^2\mathbf{A}$, which after premultiplying by $(\mathbf{A}^+)^*$ yields $\mathbf{A} = \mathbf{A}^*\mathbf{A}$. □

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