Journal of Pure and Applied Algebra 215 (2011) 866–884

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Irreducible morphisms in the bounded derived category

Raymundo Bautista [a,*], María José Souto Salorio [b]

[a] *Instituto de Matemáticas, UNAM, Unidad Morelia, A.P. 61-3, Xangari, C.P. 58089, Morelia, Michoacán, Mexico*

[b] *Facultade de Informatica, Campus de Elviña, Universidade da Coruña, CP 15071, A Coruña, Espagne*

ARTICLE INFO

Article history:
Received 20 February 2009
Received in revised form 7 May 2010
Available online 15 July 2010
Communicated by I. Reiten

MSC: 16G70; 18G35; 16G10; 18E30

ABSTRACT

We study irreducible morphisms in the bounded derived category of finitely generated modules over an Artin algebra Λ, denoted $\mathbf{D}^b(\Lambda \text{ mod})$, by means of the underlying category of complexes showing that, in fact, we can restrict to the study of certain subcategories of finite complexes. We prove that as in the case of modules there are no irreducible morphisms from X to X if X is an indecomposable complex. In case Λ is a selfinjective Artin algebra we show that for every irreducible morphism f in $\mathbf{C}^b(\Lambda \text{ proj})$ either f^j is split monomorphism for all $j \in \mathbb{Z}$ or split epimorphism, for all $j \in \mathbb{Z}$. Moreover, we prove that all the non-trivial components of the Auslander–Reiten quiver of $\mathbf{C}^b(\Lambda \text{ proj})$ are of the form $\mathbb{Z}A_\infty$.

0. Introduction

The successful concepts of irreducible morphisms and almost split sequences were introduced by Auslander and Reiten in the category of finitely generated modules over an Artin algebra, Λ mod. Moreover, they proved the existence of such sequences (see [1]). These notions have been studied in a more general context. D. Happel gave the notion of Auslander–Reiten triangles for the derived category of bounded complexes over Λ mod, $\mathbf{D}^b(\Lambda \text{ mod})$ (see [6]). Later, Krause generalized these ideas to compactly generated triangulated categories (see [8]).

On the other hand, the existence of irreducible morphisms is not trivial in general, but we know that the almost split sequences provide a wealth of irreducible morphisms.

In the recent paper [7], the authors investigate irreducible morphisms in $\mathbf{D}^b(\Lambda \text{ mod})$, in case Λ is a finite dimensional algebra over a field. They obtain their results as a consequence of a careful study of the known Happel's functor which provides an embedding of $\mathbf{D}^b(\Lambda \text{ mod})$ into the stable category $\underline{\text{mod}}\ \hat{\Lambda}$ of finite dimensional modules over the repetitive algebra. This functor becomes an equivalence if and only if Λ has finite global dimension. In case Λ is selfinjective, they construct, for any subset $I \subset \mathbb{Z}$, triangulated subcategories $\underline{\mathcal{C}}_I$ of $\underline{\text{mod}}\ \hat{\Lambda}$ containing $\mathbf{D}^b(\Lambda \text{ mod})$ such that if $I \subset I'$ then $\underline{\mathcal{C}}_I \subset \underline{\mathcal{C}}_{I'}$. Moreover, they give certain conditions on the subsets I under which the intersection of a subfamily of such subcategories coincides with $\mathbf{D}^b(\Lambda \text{ mod})$.

The Auslander–Reiten theory was also studied in certain subcategories of complexes (see [9,2]). Namely, in the last paper and in order to study Auslander–Reiten triangles in the bounded derived category of finitely generated modules over an Artin algebra, the authors introduced certain subcategories of complexes and proved that they have almost split sequences.

More exactly, for any interval I, they denote by $\mathbf{C}_I(\Lambda \text{ proj})$ the full subcategory of $\mathbf{C}(\Lambda \text{ proj})$ whose objects are the I-complexes. These categories are exact with enough projective and injective objects and they have finite global dimension. For each interval I one can consider the left triangulated category $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$(see 1.1 below). In particular, if the cardinal of I is two then $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$ is equivalent to Λ mod and if the cardinal is one then $\mathbf{C}_I(\Lambda \text{ proj})$ coincides with Λ proj, and $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$ is equivalent to the additive category consisting of only the zero object.

* Corresponding author.
E-mail addresses: raymundo@matmor.unam.mx (R. Bautista), mariaj@udc.es (M.J. Souto Salorio).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2010.06.031

We have that, in general, for each interval $I = [a, b]$ with $b - a \geq 2$, $\overline{\mathbf{C}}_I(\Lambda\,\text{proj})$ is equivalent to the full subcategory $\mathbf{U}_I$ of $\mathbf{D}^b(\Lambda\,\text{mod})$ whose objects are the complexes X such that $H^i(X) = 0$ for i outside the interval $[a+1, b]$. Note that we can recover $\mathbf{D}^b(\Lambda\,\text{mod})$ as the union set of all these categories $\mathbf{U}_I$. Moreover, if f is an irreducible map in the bounded derived category then there is some interval I such that f is an irreducible morphism in $\mathbf{U}_I$ and then f corresponds to an irreducible morphism in $\mathbf{C}_I(\Lambda\,\text{proj})$. Similarly an Auslander–Reiten triangle in $\mathbf{D}^b(\Lambda\,\text{mod})$ can be seen into some $\mathbf{U}_I$ and it corresponds to an almost split sequence in $\mathbf{C}_I(\Lambda\,\text{proj})$ (see [2]).

Conversely, given an irreducible map in $\mathbf{C}_I(\Lambda\,\text{proj})$, it represents a morphism in $\mathbf{U}_I$, we shall see under which conditions this map is irreducible in the whole category $\mathbf{D}^b(\Lambda\,\text{mod})$.

The above comments show that the study of morphisms in the category $\mathbf{D}^b(\Lambda\,\text{mod})$ can be replaced by the study of morphisms between complexes of $\mathbf{C}_I(\Lambda\,\text{proj})$.

In our considerations the shape of irreducible maps between complexes play an important role. We recall that an irreducible morphism in the category of finitely generated Λ-modules, where Λ is an Artin algebra, is either a monomorphism or an epimorphism. This simple but useful fact was generalized in [4] for irreducible morphisms between complexes.

The techniques introduced in [2], are strongly used along this paper which is organized as follows. After preliminaries we prove some results about the shape of irreducible maps by using the above setting of I-complexes. In particular, in Section 2 we show that there are no irreducible morphisms from a complex to itself.

Section 3 is devoted to investigate irreducible morphisms in the category $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, which leads us to the knowledge of irreducible maps in the bounded derived category. We establish the relationship between irreducible maps in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ and the ones in $\mathbf{C}_I(\Lambda\,\text{proj})$. We also investigate the behaviour of the middle term of almost split sequences and give necessary conditions for the existence of irreducible maps between two modules in the category of complexes.

In Section 4 we show that irreducible maps in $\mathbf{D}^b(\Lambda\,\text{mod})$ ending in a perfect complex Y (that means $Y \in \mathbf{K}^b(\Lambda\,\text{proj})$) are completely determined by irreducible morphisms in $\mathbf{C}_I(\Lambda\,\text{proj})$ ending in Y where $I = [a, b]$ is an interval such that $Y^a = 0 = Y^b$.

In Section 5, we focus our attention to the case Λ is a selfinjective Artin algebra showing, in particular, that an irreducible map f in $\mathbf{C}^b(\Lambda\,\text{proj})$ is such that either all the f^j (for all $j \in \mathbb{Z}$) are split monomorphism or all of them are split epimorphism in $\Lambda\,\text{proj}$. Moreover, all the non-trivial components of the Auslander–Reiten quiver of $\mathbf{C}^b(\Lambda\,\text{proj})$ are of the form $\mathbb{Z}A_\infty$. This fact was first proved in [12] for Λ a finite dimensional algebra over a field and later with a different proof in Theorem 5.4 of [7].

Finally, Section 6 is devoted to the case of irreducible maps in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ involving a non-perfect complex. We apply the results to the Gorenstein case.

1. Preliminaries

We start giving notations and basic facts which we will use in the subsequent sections.

Let k be a commutative Artinian ring, Λ an Artin k-algebra. We denote by $\Lambda\,\text{mod}$ and $\Lambda\,\text{proj}$ the category of finitely generated left Λ-modules and the full subcategory of $\Lambda\,\text{mod}$ consisting of the finitely generated projective Λ-modules. In general, $\mathcal{A}$ will denote an additive k-category.

1.1. Complexes

We recall that a complex $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ over $\mathcal{A}$ is a family of morphisms $d_X^i : X^i \to X^{i+1}$, $i \in \mathbb{Z}$, such that $d_X^{i+1} d_X^i = 0$ for all $i \in \mathbb{Z}$. If X and Y are complexes over $\mathcal{A}$, a morphism f of degree l from X to Y is given by a family of morphisms $f^i : X^i \to Y^{i+l}$, $i \in \mathbb{Z}$ such that $d_Y^{i+l} f^i = (-1)^l f^{i+1} d_X^i$. We denote by $\mathbf{C}(\mathcal{A})$ the category whose objects are the complexes over $\mathcal{A}$ and the morphisms between two complexes are the degree zero morphisms.

In the category of cochain complexes, $\mathbf{C}(\mathcal{A})$, we consider the class $\mathcal{E}$ of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that for all $n \in \mathbb{Z}$, the sequence $0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0$ is split exact. Sequences in $\mathcal{E}$ are called conflations and it is known that $(\mathbf{C}(\mathcal{A}), \mathcal{E})$ is an exact category in the sense of [10] or equivalently [3] (see for instance Example 3.2 in [5]). Moreover, $\mathbf{C}(\mathcal{A})$ is a Frobenius category with respect to $\mathcal{E}$. The maps factoring through an injective–projective object are the null homotopic maps. Then, the stable category of $\mathbf{C}(\mathcal{A})$ coincides with the homotopy category $\mathbf{K}(\mathcal{A})$.

Take $I = [a, b]$, $I = (-\infty, b]$ or $I = [a, \infty)$ an interval in $\mathbb{Z}$ or $I = \mathbb{Z}$, an I-complex is a complex X such that $X^i = 0$ for all i outside the interval I.

We denote by $\mathbf{C}_I(\mathcal{A})$ the full subcategory of $\mathbf{C}(\mathcal{A})$ whose objects are the I-complexes. If X is a complex and I is an interval we denote by X_I the complex such that $X_I^i = X^i$ if i lies in I and $X_I^i = 0$ in case i is not in I. Moreover, $d_{X_I}^i = d_X^i$ if i and $i+1$ are in I and zero otherwise. If $f : X \to Y$ is a morphism of complexes then $f_I : X_I \to Y_I$ is the morphism of complexes such that $f_I^i = f^i$ if i lies in I and $f^i = 0$ otherwise.

We will use some results of $\mathbf{C}_I(\mathcal{A})$ which can be found in [2].

$(\mathbf{C}_I(\mathcal{A}), \mathcal{E}_I)$ is an exact category with $\mathcal{E}_I$ the class of composable morphisms in $\mathbf{C}_I(\mathcal{A})$ which are in $\mathcal{E}$.

Now, assume $I = [m, n]$ is a finite interval. For $M \in \mathcal{A}$ we consider the following complexes:

- $J_i(M) = (J^s, d^s)$ with $J^s = 0$ if $s \neq i, s \neq i+1$, $J^i = J^{i+1} = M$, $d^i = id_M$.

- $S(M) = (X^i, d^i)_{i\in\mathbb{Z}}$ with $X^i = 0$ for $i \neq m$, $X^m = M$, $d^i = 0$.
- $T(M) = (Y^i, d^i)_{i\in\mathbb{Z}}$ with $Y^i = 0$ for $i \neq n$, $Y^n = M$, $d^i = 0$.

The objects $T(M), J_i(N)$ for $i = m, \ldots, n-1$ are $\mathcal{E}_I$-projective in $\mathbf{C}_I(\mathcal{A})$. The objects $S(M), J_i(N)$ for $i = m, \ldots, n-1$ are $\mathcal{E}_I$-injective in $\mathbf{C}_I(\mathcal{A})$ (see [2]).

We consider $\mathbf{C}^{\leq n}(\mathcal{A})$ the full subcategory of $\mathbf{C}(\mathcal{A})$ whose objects are those $X \in \mathbf{C}(\mathcal{A})$ such that $X^i = 0$ for $i > n$. Similarly, $\mathbf{C}^{\geq m}(\mathcal{A})$ is the full subcategory of $\mathbf{C}(\mathcal{A})$ whose objects are those $X \in \mathbf{C}(\mathcal{A})$ such that $X^i = 0$ for $i < m$.

There is a functor $F_I : \mathbf{C}^{\leq n}(\mathcal{A}) \to \mathbf{C}_I(\mathcal{A})$ given in objects, by $F_I(X) = X_I$ and in morphisms, $f : X \to Y$ by $F_I(f) = f_I$.

We now consider $\mathcal{M}_I$, the full subcategory of $\mathbf{C}^{\leq n}(\Lambda \text{ proj})$ whose objects are those complexes X with $H^j(X) = 0$ for $j \leq m$. By $\mathcal{L}_I$ we denote the full subcategory of $\mathbf{K}(\Lambda \text{ proj})$ whose objects are in $\mathcal{M}_I$.

In the following we denote by $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$ the category with the same objects as $\mathbf{C}_I(\Lambda \text{ proj})$ and the morphisms are the morphisms of this category modulo those which factor through $\mathcal{E}_I$-injective complexes.

Proposition 1.2. *If $I = [m, n]$ is a finite interval we have the following:*

(a) *The functor F_I induces a full functor from $\mathcal{M}_I$ to $\mathbf{C}_I(\Lambda \text{ proj})$. Moreover, if $X \in \mathcal{M}_I$ has no $\mathcal{E}$-projective direct summands, then X_I has no $\mathcal{E}_I$-injective direct summands.*
(b) *If $X \in \mathbf{C}_I(\Lambda \text{ proj})$, there is a $\hat{X} \in \mathcal{M}_I$ such that $\hat{X}_I = X$.*
(c) *For any W complex in $\mathbf{C}^{\leq n}(\Lambda \text{ proj})$, W_I is a subcomplex of W, and if we denote by $\sigma_W : W_I \to W$ the inclusion, then for any morphism $f : W \to Z$ in $\mathbf{C}^{\leq n}(\Lambda \text{ proj})$, we have $f\sigma_W = \sigma_Z f_I$.*
(d) *The functor F_I induces an equivalence between the category $\mathcal{L}_I$ and the category $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$. Moreover, if $X, Y \in \mathbf{C}_I(\Lambda \text{ proj})$ have no $\mathcal{E}_I$-injective direct summands and $X \cong Y$ in $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$ then $X \cong Y$ in $\mathbf{C}_I(\Lambda \text{ proj})$.*

Proof. The first part of (a) follows from Lemma 5.3 of [2]. For the second part recall that a complex W of projective Λ-modules has no $\mathcal{E}$-projective direct summands if and only if $\mathrm{Im} d^i_W \subset \mathrm{rad} W^{i+1}$ for all $i \in \mathbb{Z}$. Then if X has this last property the complex X_I also has this property. Therefore if X has no $\mathcal{E}$-projective direct summands, then X_I has no direct summands of the form $J_i(P)$. Suppose X_I has a direct summand of the form $S(P)$, then $X^m = P \oplus Q$ with $d^m(P) = 0$. But $H^m(X) = 0$, then $P \subset \mathrm{Im} d^{m-1}_X$, so $\mathrm{Im} d^{m-1}_X$ is not contained in $\mathrm{rad} X^m$, which cannot be if X has not $\mathcal{E}$-projective direct summands. This proves (a).

For X an I-complex, take

$$\cdots \to P^{-1} \to P^0 \xrightarrow{\eta} \mathrm{Ker} d^m_X \to 0$$

a minimal projective resolution, then if $i : \mathrm{Ker} d^m_X \to X^m$ is the inclusion, take the complex

$$\hat{X} : \cdots \to P^{-1} \to P^0 \xrightarrow{i\eta} X^m \xrightarrow{d^m_X} \cdots \to X^{n-1} \xrightarrow{d^{n-1}_X} X^n \to 0 \cdots .$$

It is easy to verify that $\hat{X} \in \mathcal{M}_I$ and $\hat{X}_I = X$. This proves (b).

Statement (c) is clear.

Statement (d) follows from Corollary 5.7 of [2]. □

1.3. Factorization of morphisms

Let I be an interval in $\mathbb{Z}$ and let X be a complex.

(1) If $I = [a, b]$ with $a \leq b$ we denote by $I(-) = (-\infty, a-1]$, $I(+) = [b+1, \infty)$.
We have the following one degree morphisms:

$$d^{I(-)}_X : X_{I(-)} \to X_I$$
$$d^I_X : X_I \to X_{I(+)}$$

given by $(d^{I(-)}_X)^i = 0$ if $i \neq a-1$ and $(d^{I(-)}_X)^{a-1} = d^{a-1}_X$; $(d^I_X)^i = 0$ for $i \neq b$, $(d^I_X)^b = d^b_X$. Clearly $d^I_X d^{I(-)}_X = 0$.

(2) If $I = [a, \infty)$ we denote by $I(-) = \{n \in \mathbb{Z}, n < a\}$ and $I(+) = \emptyset$. We have the following one degree morphisms:

$$d^{I(-)}_X : X_{I(-)} \to X_I$$

given by $(d^{I(-)}_X)^i = 0$ if $i \neq a-1$ and $(d^{I(-)}_X)^{a-1} = d^{a-1}_X$;

$$d^I_X : X_I \to X_{I(+)}$$

is the zero morphism.

(3) If $I = (-\infty, b]$ we denote by $I(+) = \{n \in \mathbb{Z}, n > b\}$ and $I(-) = \emptyset$. We have the following one degree morphisms:

$$d^I_X : X_I \to X_{I(+)}$$

given by $(d^I_X)^i = 0$ for $i \neq b$, $(d^I_X)^b = d^b_X$ and

$$d^{I(-)}_X : X_{I(-)} \to X_I$$

the zero morphism.

Conversely, if X_1 is an $I(-)$-complex, X_2 is an I-complex, X_3 is an $I(+)$-complex and we have one degree morphisms, $u : X_1 \to X_2$, $v : X_2 \to X_3$ with $vu = 0$, then we may construct a complex X such that $X_{I(-)} = X_1, X_I = X_2, X_{I(+)} = X_3$ and $u = d_X^{I(-)}$, $v = d_X^I$.

In the following if I is a fixed interval we identify the complex X with the "complex"

$$X_{I(-)} \xrightarrow{d_X^{I(-)}} X_I \xrightarrow{d_X^I} X_{I(+)}.$$

Observe that if $f : X \to Y$ is a morphism of complexes, then we have the following commutative diagram:

$$\begin{array}{ccccc} X_{I(-)} & \xrightarrow{d_X^{I(-)}} & X_I & \xrightarrow{d_X^I} & X_{I(+)} \\ {\scriptstyle f_{I(-)}}\downarrow & & {\scriptstyle f_I}\downarrow & & {\scriptstyle f_{I(+)}}\downarrow \\ Y_{I(-)} & \xrightarrow{d_Y^{I(-)}} & Y_I & \xrightarrow{d_Y^I} & Y_{I(+)}. \end{array}$$

Proposition 1.4 (*See [4]*). *Let $f : X \to Y$ be a morphism of complexes and let I be an interval such that in $\mathbf{C}_I(\mathcal{A})$, $f_I = vu$ with $u : X_I \to Z$ and $v : Z \to Y_I$, then we have the following factorization of f in $\mathbf{C}(\mathcal{A})$:*

$$\begin{array}{ccccc} X_{I(-)} & \xrightarrow{d_X^{I(-)}} & X_I & \xrightarrow{d_X^I} & X_{I(+)} \\ {\scriptstyle id}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle f_{I(+)}}\downarrow \\ X_{I(-)} & \xrightarrow{ud_X^{I(-)}} & Z & \xrightarrow{d_Y^I v} & Y_{I(+)} \\ {\scriptstyle f_{I(-)}}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle id}\downarrow \\ Y_{I(-)} & \xrightarrow{d_Y^{I(-)}} & Y_I & \xrightarrow{d_Y^I} & Y_{I(+)}. \end{array}$$

In the following we put $\hat{u} = (id, u, f_{I(+)})$ and $\hat{v} = (f_{I(-)}, v, id)$.

Remark 1.5. If X and Y are in $\mathbf{C}_J(\mathcal{A})$ for some interval J of $\mathbb{Z}$ containing I, then $\hat{u}$ and $\hat{v}$ are morphisms of J-complexes.

2. Irreducible morphisms in $\mathbf{C}(\mathcal{A})$

In the first part of this section we collect, for an easy reference, some results on irreducible morphisms with their full proofs from [4]. In the second part we study irreducible morphisms from some indecomposable to itself in the category $\mathbf{C}_I(\Lambda \text{ proj})$. We will prove that there are no irreducible morphisms from some indecomposable to itself in this category if $I = [a, b]$ with $b - a > 1$. This result will be used to prove that there are no irreducible morphisms from some complex to itself in the category $\mathbf{C}(\Lambda \text{ proj})$.

Throughout the paper we identify the category $\mathcal{A}$ with the full subcategory of $\mathbf{C}(\mathcal{A})$ whose objects are those X such that $X^i = 0$ if $i \neq 0$.

Definition 2.1. If $\mathcal{C}$ is an additive category, a morphism $f : X \to Y$ in this category is called irreducible if it is neither a retraction nor a section and $f = vu$ with $u : X \to Z$, $v : Z \to Y$ morphisms in $\mathcal{C}$, implies that either u is a section or v is a retraction.

Proposition 2.2. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}_J(\mathcal{A})$ with J an interval or $J = \mathbb{Z}$ and I an interval contained in J, then $f_I : X_I \to Y_I$ is a section or a retraction or an irreducible morphism in $\mathbf{C}_I(\mathcal{A})$.*

Proof. Suppose f_I is not a section, not a retraction, not an irreducible, then $f_I = vu$ with u no section and v no retraction, then by 1.4 $f = \hat{v}\hat{u}$ is a factorization in $\mathbf{C}_J(\mathcal{A})$, since u is not a section then $\hat{u}$ is not a section, here v is not a retraction, so $\hat{v}$ is not a retraction, but this is impossible because f is irreducible. □

2.3

Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$ and let I be an interval in $\mathbb{Z}$. As a consequence of Proposition 2.2, we have that if $f_I : X_I \to Y_I$ is neither a section nor a retraction then, for each interval I' containing I, the morphism $f_{I'}$ is irreducible in $\mathbf{C}_{I'}(\mathcal{A})$.

Proposition 2.4 (*Girardo–Merklen*). *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$ and I some interval of $\mathbb{Z}$. If f_I is not a retraction, then $f_{I(+)}$ is a section. If f_I is not a section, then $f_{I(-)}$ is a retraction.*

Proof. Suppose f_I is not a retraction, consider the factorization $f_I = vu$ with $u = id$, $v = f_I$, then $f = \hat{v}\hat{u}$ with $\hat{u} = (id, id, f_{I(+)})$, $\hat{v} = (f_{I(-)}, f_I, id)$, then $\hat{v}$ is not a retraction, so $\hat{u}$ is a section consequently $f_{I(+)}$ is a section. If f_I is not a section, then take $u = f_I$, $v = id$, then $\hat{u} = (id, f_I, f_{I(+)})$, $\hat{v} = (f_{I(-)}, id, id)$, therefore $\hat{u}$ is not a section, thus $\hat{v}$ is a retraction, then $f_{I(-)}$ is also a retraction. □

Corollary 2.5. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$.*

(1) *For each $i \in \mathbb{Z}$, the morphism f^i is either irreducible or split in $\mathcal{A}$.*
(2) *If there exists an integer i such that f^i is irreducible, then such integer is unique.*

Proof. Consider $I = \{i\}$. By Proposition 2.2 we get (1). Now, assume there is some integer i, such that f^i is an irreducible morphism in $\mathcal{A}$, then f^i is not a retraction and $f_{[i+1,\infty)}$ is a section by Proposition 2.4. Moreover, f^i is not a section and again by Proposition 2.4 , $f_{(-\infty,i-1]}$ is a retraction. Consequently, for each $j \neq i$, f^j is not an irreducible morphism. □

Now we prove the following result. A slight weak version, but essentially the same, is due to Giraldo and Merklen in [4].

Proposition 2.6. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$ and let I be an interval bounded below by the integer a. If f_I is a section which is not an isomorphism and f^{a-1} is an epimorphism then f^{a-1} is an isomorphism and $f_{I(-)\cup I}$ is a section.*

Proof. Take $I' = I(-) \cup I$. We know that $f_{I'}$ is a section, or a retraction or an irreducible morphism in $\mathbf{C}_{I'}(\mathcal{A})$. If $f_{I'}$ is a section we clearly have the conclusion. If $f_{I'}$ is a retraction, then we get a contradiction to the fact that f_I is not an isomorphism. Then assume that $f_{I'}$ is an irreducible morphism.

The morphism f_I is a section then there exists g_I such that $g_I f_I = id$. Note that

$$f^a g^a d_Y^{a-1} f^{a-1} = f^a g^a f^a d_X^{a-1} = f^a d_X^{a-1} = d_Y^{a-1} f^{a-1}.$$

Take $\lambda = g^a d_Y^{a-1}$. We have, $f^a \lambda = f^a g^a d_Y^{a-1} = d_Y^{a-1}$ because f^{a-1} is an epimorphism, and

$$\lambda f^{a-1} = g^a d_Y^{a-1} f^{a-1} = g^a f^a d_X^{a-1} = d_X^{a-1}.$$

Therefore we have the following factorization of $f_{I'}$:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X^{a-2} & \xrightarrow{d_X^{a-2}} & X^{a-1} & \xrightarrow{d_X^{a-1}} & X_I \\
 & & \downarrow{\scriptstyle f^{a-2}} & & \downarrow{\scriptstyle f^{a-1}} & & \downarrow{\scriptstyle id} \\
\cdots & \longrightarrow & Y^{a-2} & \xrightarrow{d_Y^{a-2}} & Y^{a-1} & \xrightarrow{\lambda} & X_I \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f_I} \\
\cdots & \longrightarrow & Y^{a-2} & \xrightarrow{d_Y^{a-2}} & Y^{a-1} & \xrightarrow{d_Y^{a-1}} & Y_I
\end{array}$$

and either $(\cdots, f^{a-2}, f^{a-1}, id)$ is a section or $(\cdots, id, id, f_I)$ is a retraction. In the first case, f^{a-1} is a section so it is an isomorphism. In the second case f_I is a retraction, so it is an isomorphism and we get a contradiction. □

The following result can be proved with similar arguments as in the previous proposition.

Proposition 2.7. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$ and I be an interval bounded above by the integer b. If f_I is a retraction non-isomorphism and f^{b+1} is a monomorphism, then f^{b+1} is an isomorphism and $f_{I\cup I(+)}$ is a retraction.*

Definition 2.8. A morphism $f : X \to Y$ in $\mathbf{C}(\mathcal{A})$ is called $\mathcal{E}$-monomorphism if f^i is a section for all $i \in \mathbb{Z}$. Similarly, $g : Y \to Z$ is called $\mathcal{E}$-epimorphism if g^i is a retraction for all $i \in \mathbb{Z}$.

In an abelian category an irreducible morphism is either an epimorphism or a monomorphism, the following result due to Giraldo and Merklen, is a generalization of this fact for complexes.

Proposition 2.9. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$, then one of the following statements is true:*

(i) *There is an unique integer i such that f^i is an irreducible morphism in the category $\mathcal{A}$.*
(ii) *The morphism f is an $\mathcal{E}$-monomorphism.*
(iii) *The morphism f is an $\mathcal{E}$-epimorphism.*

Proof. By Corollary 2.5, we know that either there is an unique i with f^i an irreducible morphism in $\mathcal{A}$ or f^j splits, for all j. Then we may assume that (i) does not hold and each f^i is either a section or a retraction. Note that, since $f : X \to Y$ is an irreducible morphism, it is not an isomorphism, so assume there exists an i_0 such that f^{i_0} is a section but not an isomorphism. We claim that f is an $\mathcal{E}$-monomorphism. The case $\mathcal{E}$-epimorphism is dual.

If some f^i is a section but not an isomorphism, by Proposition 2.4, $f_{[i+1,\infty)}$ is a section, then f^s is a section for all $s \geq i$. If some f^j is a retraction but not an isomorphism, again by Proposition 2.4, $f_{(-\infty,j-1]}$ is a retraction, so f^t is a retraction for all $t \leq j$. Suppose f is not an $\mathcal{E}$-monomorphism, take i_0 minimal such that f^{i_0} is a section which is not an isomorphism and j_0 maximal such that f^{j_0} is a retraction but not an isomorphism, so $j_0 < i_0$. Observe that f^s is an isomorphism for all $s \in [j_0, i_0]$, so it is an epimorphism. Therefore f^{i_0-1} is an epimorphism. Taking $I = [i_0, i_0]$ in Proposition 2.6, we deduce that $f_{(-\infty,i_0]}$ is a section, therefore f^{j_0} is an isomorphism, a contradiction. Then all the f^i are sections. □

Proposition 2.10. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$ such that f is an $\mathcal{E}$-monomorphism, f^{i-1} is an isomorphism and f^i is not an isomorphism, then $f_{(-\infty,i-1]}$ is an isomorphism.*

Proof. If $f_{[i-1,\infty)}$ is an irreducible morphism then it is not a section, so $f_{(-\infty,i-2]}$ is a retraction therefore an isomorphism, clearly $f_{(-\infty,i-1]}$ is also an isomorphism. If $f_{[i-1,\infty)}$ is not an irreducible morphism, then it is a section, so $f_{[i,\infty)}$ is a section, which is not an isomorphism. Since f^{i-1} is an epimorphism, by Proposition 1.4 we have that $f_{(-\infty,i-1]}$ is a section, so $f_{[i-1,\infty)}$ and $f_{(-\infty,i-1]}$ are sections, now f^{i-1} is an isomorphism this implies that f is a section, which is a contradiction. □

Proposition 2.11. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}(\mathcal{A})$ which is an $\mathcal{E}$-epimorphism such that f^i is an isomorphism and f^{i-1} is not an isomorphism, then $f_{[i,\infty)}$ is an isomorphism.*

Proof. Similar to the proof of the above proposition. □

In the last part of this section we consider the case $\mathcal{A} = \Lambda$ proj, with Λ an Artin algebra over the commutative Artinian ring k.

As in [2] we take for each projective Λ-module P the complex $J_i(P)$ such that $J_i(P)^s = P$ if $s = i, i+1$ and zero otherwise, $d^s_{J_i(P)} = id_P$ if $s = i$ and 0 in case $s \neq i$. Let S be a simple Λ-module and let

$$\dots P^{-s} \overset{d^{-s}}{\to} P^{-s+1} \overset{d^{-s+1}}{\to} P^{-s+2} \to \dots \to P^{-1} \overset{d^{-1}}{\to} P^0 \overset{\eta_S}{\to} S$$

be a minimal projective resolution of S. Let P_S be the complex:

$$\dots P^{-s} \overset{d^{-s}}{\to} P^{-s+1} \overset{d^{-s+1}}{\to} P^{-s+2} \to \dots \to P^{-1} \overset{d^{-1}}{\to} P^0 \to 0 \cdots,$$

clearly P_S is an indecomposable complex.

We have a morphism of complexes: $u : P_S \to J_{-1}(P^0)$ with $u^j = 0$ if $j \neq -1$ and $j \neq 0$, $u^0 = id_{P^0}$, $u^{-1} = d^{-1}$.

The second part of the following result is proved in Theorem 4.4 of [11].

Proposition 2.12. *The morphism $u : P_S \to J_{-1}(P^0)$ is a minimal right almost split morphism in $\mathbf{C}(\Lambda$ proj$)$. In particular, u is an irreducible morphism.*

Proof. Given a complex Y we have a morphism of complexes $\eta : Y \to Y_{(-\infty,0]}$ given by $\eta^j = id_{Y^j}$ if $j \leq 0$ and $\eta^j = 0$ in case $j > 0$. Now we shall prove that u is a minimal right almost split morphism. Clearly u is not a retraction and if $\lambda u = u$ for an endomorphism λ of $J_{-1}(P^0)$, then λ is the identity. Let $h : Y \to J_{-1}(P^0)$ be a morphism which is not a retraction in $\mathbf{C}(\Lambda$ proj$)$, take now $I = [a, 0]$ a finite interval with $a < -1$. From [2] we know that

$$u_I : (P_S)_I \to J_{-1}(P^0)$$

is a minimal right almost split morphism in $\mathbf{C}_I(\Lambda$ proj$)$. Since h is not a retraction , then h_I cannot be a retraction, therefore there exists a morphism $v_1 : Y_I \to (P_S)_I$, such that $u_I v_1 = h_I$. Here $(Y_{(-\infty,0]})_I = Y_I$ and P_S lies in $\mathcal{L}_I$. Then by Lemma 5.3 of [2] there is a morphism $w : Y_{(-\infty,0]} \to P_S$ such that $w_I = v_1$. Take $v = w\eta : Y \to P_S$. Then since $\eta_I = id_{Y_I}$ we have $v_I = v_1$. Therefore $(uv)_I = h_I$, but $h^n = 0$ and $(uv)^n = 0$ for n outside I, so $uv = h$. □

By duality one can prove that if $S \overset{i}{\to} I^0 \overset{d^0}{\to} I^1 \to \cdots$ is a minimal injective co-resolution of the simple S, then there is a minimal left almost split morphism $h : J_0(I^0) \to I_S$ in the category $\mathbf{C}(\Lambda$ inj$)$. Here $I_S = \cdots 0 \to I^0 \overset{d^0}{\to} I^1 \cdots$. The morphism h is given by $h^0 = id_{I^0}$, $h^1 = d^0$ and $h^j = 0$ for $j \neq 0$ and $j \neq 1$. We know that the Nakajama functor ν induces an equivalence of categories $\nu : \mathbf{C}(\Lambda$ proj$) \to \mathbf{C}(\Lambda$ inj$)$. Then there is a morphism $v' : J_0(P^0) \to Q_S$ such that $\nu(v') = h$ with $\nu(Q_S) = I_S$. Therefore, taking $v = v'[1]$, we obtain the following result:

Proposition 2.13. *There is a morphism $v : J_{-1}(P^0) \to Q_S[1]$ which is a minimal left almost split morphism in $\mathbf{C}(\Lambda$ proj$)$, in particular v is an irreducible morphism.*

We finish this section by showing that there are no loops of irreducible morphisms in the category of complexes. First, we see the following useful Lemma.

Lemma 2.14. *Let $f : M \to N$ be a morphism in $\mathcal{A}$, and let I be either any interval of $\mathbb{Z}$ containing $[-1, 0]$ or $I = \mathbb{Z}$.*

(1) *If f is an irreducible morphism in $\mathbf{C}_I(\mathcal{A})$ then f is irreducible in $\mathcal{A}$.*
(2) *In case $\mathcal{A} = \Lambda$ proj we have:*
 (a) *If f is irreducible in $\mathbf{C}_I(\mathcal{A})$ then f is an irreducible monomorphism in $\mathcal{A}$.*
 (b) *If $M = N$ then f is not an irreducible morphism in $\mathbf{C}_I(\mathcal{A})$.*

Proof. By Proposition 2.2 with $J = I$ and $I = [0, 0]$, if f is irreducible in $\mathbf{C}_I(\mathcal{A})$ then f is irreducible or it is a retraction or a section in $\mathcal{A}$. But if f is a retraction or a section in $\mathcal{A}$, then f is also a retraction or a section in $\mathbf{C}_I(\mathcal{A})$.

(2) (a) In case f is not a monomorphism we can take a projective cover of Ker$f$ with $\eta \neq 0$: $Q \stackrel{\eta}{\to}$ Ker$f$. We have a factorization of complexes $f = vu$ through the complex $\cdots 0 \to Q \to M \to 0$ where $u = (\cdots, 0, id, 0, \cdots)$ and $v = (\cdots, 0, f, 0, \cdots)$. But v is not a retraction because f is irreducible in $\mathcal{A}$. Moreover, if u is a section we get a contradiction.

(2) (b) If $M = N$ and f is an irreducible in $\mathbf{C}_I(\mathcal{A})$ then using the above, we infer that f is an irreducible monomorphism. But M is a module having finite length and then f is an isomorphism in contradiction to the fact that f is irreducible. □

Proposition 2.15. *Let X be an indecomposable complex in $\mathbf{C}_I(\Lambda \text{ proj})$ with $I = [a, b]$ a finite interval and $b - a \geq 2$, then there are no irreducible morphisms $f : X \to X$ in $\mathbf{C}_I(\Lambda \text{ proj})$.*

Proof. Note that, since all the Λ-modules X^i have finite length over k, then in case all the f^i are either sections or retractions, we get that all the f^i are isomorphisms and f is an isomorphism. Therefore, if f is irreducible, there is an $i \in \mathbb{Z}$ with f^i an irreducible morphism. Then since f^i is not a section, by Proposition 2.4 we have that $f_{[a,i-1]}$ is a retraction, but then $f_{[a,i-1]}$ is an isomorphism. Similarly, since f^i is not a retraction, then again by Proposition 2.4, $f_{[i+1,b]}$ is an isomorphism.

On the other hand, f irreducible implies that f is nilpotent in the local algebra $\mathrm{End}_{\mathbf{C}_I(\Lambda \text{ proj})}(X)$ and there is a natural n such that $f^n = 0$. Hence, for all $j \neq i$ the isomorphisms f^j are zero. Thus we can assume $f = (\cdots, 0, f^i, 0, \cdots)$ with f^i irreducible in Λ proj. If $i > a$, then by Lemma 2.14, f^i is a monomorphism and therefore an isomorphism, in contradiction to the fact that f is irreducible. Therefore $i = a$ and $f = u[-a]$, for some irreducible morphism $u : M \to M$ in Λ proj. Then $X = M[-a]$, with X indecomposable consequently, M is indecomposable. Therefore the morphism u is in the radical. So there is a non-zero morphism $v : M \to M$ such that $vu = 0$. Now $b - a \geq 2$, we obtain a non-trivial factorization of f,

$$\begin{array}{ccccccc}
\cdots 0 & \longrightarrow & M & \xrightarrow{0} & 0 & \longrightarrow & 0 \cdots \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow \\
\cdots 0 & \xrightarrow{0} & M & \xrightarrow{v} & M & \longrightarrow & 0 \cdots \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle 0} & & \downarrow \\
\cdots 0 & \xrightarrow{0} & M & \xrightarrow{0} & 0 & \longrightarrow & 0 \cdots
\end{array}$$

This contradicts our assumption that f is irreducible. □

Now we will prove that there are no irreducible morphisms $X \to X$ in the category $\mathbf{C}(\Lambda \text{ proj})$, for this we need the following results.

We recall that a morphism $g : Z \to W$ in a Krull–Schmidt category is called radical if for any section $\sigma : Z_1 \to Z$ and any retraction $\tau : W \to W_1$ the composition $\tau g \sigma$ is not an isomorphism. Radical morphisms in $\mathcal{A}$ are the morphisms which are in the radical of $\mathcal{A}$.

Lemma 2.16. *Let $\mathcal{A}$ be a Krull–Schmidt category, then any morphism $f : X \to Y$ is isomorphic to a morphism of the form:*

$$f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} : X_1 \oplus X_2 \to Y_1 \oplus Y_2$$

with f_1 an isomorphism and f_2 a radical morphism.

Proof. Suppose f is not a radical morphism, then there is a section $\sigma : X_1 \to X$ and a retraction $\tau : Y \to Y_1$ such that $\tau f \sigma$ is an isomorphism. Then we have for some decomposition of X and Y:

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} : X_1 \oplus X_2 \to Y_1 \oplus Y_2.$$

Take now the isomorphism:

$$\phi : \begin{pmatrix} 1_{X_1} & -f_1^{-1} f_2 \\ 0 & 1_{X_2} \end{pmatrix} : X_1 \oplus X_2 \to X_1 \oplus X_2,$$

then f is isomorphic to the morphism

$$f\phi = \begin{pmatrix} f_1 & 0 \\ f_3 & -f_3 f_1^{-1} f_2 + f_4 \end{pmatrix}.$$

So we may assume $f_2 = 0$. Taking now the isomorphism:

$$\psi : \begin{pmatrix} 1_{Y_1} & 0 \\ -f_3 f_1^{-1} & 1_{Y_2} \end{pmatrix} : Y_1 \oplus Y_2 \to Y_1 \oplus Y_2,$$

we see that f is isomorphic to

$$\psi f = \begin{pmatrix} f_1 & 0 \\ 0 & f_4' \end{pmatrix}.$$

We may repeat our procedure for f_4', the process must finish because each object of our category is decomposable into a finite number of indecomposables in a unique way up to isomorphisms. □

Proposition 2.17. *A morphism*

$$f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} : X_1 \oplus X_2 \to Y_1 \oplus Y_2$$

with f_1 an isomorphism and f_2 a radical morphism is irreducible if and only if f_2 is irreducible.

Proof. Suppose f is irreducible. Then f_2 is neither a section nor a retraction. Take $f_2 = vu$, $u : X_2 \to Z$, $v : Z \to Y_2$ a factorization. Then we obtain the following factorization of $f = VU$, $U : X_1 \oplus X_2 \to X_1 \oplus Z$, $V : X_1 \oplus Z \to Y_1 \oplus Y_2$ with

$$U = \begin{pmatrix} 1_{X_1} & 0 \\ 0 & u \end{pmatrix}; \qquad V = \begin{pmatrix} f_1 & 0 \\ 0 & v \end{pmatrix}.$$

Then, since f is irreducible, either U is a section or V is a retraction, but this implies that either u is a section or v is a retraction, so f_2 is irreducible.

Suppose now that f_2 is irreducible, so f_2 is neither a section, nor a retraction, therefore f is neither a section nor a retraction. Suppose we have a factorization of $f = VU$, with $U : X_1 \oplus X_2 \to Z$, $V : Z \to Y_1 \oplus Y_2$. Then if $U = (u_1, u_2)$, $V = (v_1, v_2)^T$. We have

$$f_1 = v_1u_1, \qquad v_2u_1 = 0, \qquad v_1u_2 = 0, \qquad v_2u_2 = f_2.$$

Here f_2 is irreducible, then either u_2 is a section or v_2 is a retraction. Suppose u_2 is a section, then there is a morphism $t : Z \to X_2$ such that $tu_2 = 1_{X_2}$.

Take $\lambda = (f_1^{-1}v_1, t)^T : Z \to X_1 \oplus X_2$, then

$$\lambda U = \begin{pmatrix} 1_{X_1} & f_1^{-1}v_1u_2 \\ tu_1 & tu_2 \end{pmatrix} = \begin{pmatrix} 1_{X_1} & 0 \\ tu_1 & 1_{X_2} \end{pmatrix}.$$

Therefore λU is an isomorphism, consequently U is a section.

If v_2 is a retraction we proceed in a similar way for proving that V is a retraction. This proves our result. □

Proposition 2.18. *Let $f : X \to Y$ be an irreducible morphism in the radical of a Krull–Schmidt category $\mathcal{A}$, then if $p : Y \to Z$ is a non-zero retraction the morphism $pf : X \to Z$ is irreducible, similarly if $s : W \to X$ is a non-zero section then $fs : W \to Y$ is an irreducible morphism.*

Proof. Here $p : Y \to Z$ is a retraction, thus there is a morphism $i : Z \to Y$ with $pi = id_Z$. Since ip is an idempotent and idempotents split in $\mathcal{A}$, then there is a retraction $p' : Y \to Z'$ and $i' : Z' \to Y$ such that $p'i' = id_{Z'}$, $ip + i'p' = id_Y$ and $pi' = 0$, $p'i = 0$. We claim that if f is an irreducible morphism in the radical of $\mathcal{A}$, then pf is also irreducible. Since f is in the radical of $\mathcal{A}$, pf is neither a section nor a retraction. Suppose $pf = vu$ with $u : X \to W$ and $v : W \to Z$. This factorization of pf gives a factorization of $f = gh$, with

$$h = \begin{pmatrix} u \\ p'f \end{pmatrix} : X \to W \oplus Z',$$

$$g = (iv, i') : W \oplus Z' \to Y.$$

Then either h is a section or g is a retraction, in the first case there is a morphism

$$\lambda = (\lambda_1, \lambda_2) : W \oplus Z' \to X$$

with $id_X = \lambda h = \lambda_1 u + \lambda_2 p'f$. Since f is in the radical of $\mathcal{A}$, then $\lambda_2 p'f$ is also in the radical of $\mathcal{A}$, this implies that $id_X - \lambda_2 p'f$ is an isomorphism, consequently in this case u is a section. If g is a retraction then there is a morphism

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} : W \oplus Z' \to Y$$

such that $id_Y = iv\mu_1 + i'\mu_2$. So $p = piv\mu_1 = v\mu_1$, and then $id_Z = pi = v\mu_1 i$, therefore v is a retraction. This implies that pf is irreducible. The second part of the proposition is proved in a similar way. □

Theorem 2.19. *Let X be a complex in $\mathbf{C}(\Lambda\ \text{proj})$, then there are no irreducible morphisms $f : X \to X$.*

Proof. As in the proof of Proposition 2.15 , there is an i with f^i irreducible in Λ proj and f^j is an isomorphism, for all $j \neq i$. Take now a finite interval I containing the interval $[i-1, i]$. By Proposition 2.2, f_I is irreducible in $\mathbf{C}_I(\Lambda \text{ proj})$, then by 2.16 and 2.17,

$$f_I = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} : W_1 \oplus W_2 \to Y_1 \oplus Y_2,$$

with f_1 an isomorphism and f_2 a radical irreducible morphism in $\mathbf{C}_I(\Lambda \text{ proj})$, $X_I = W_1 \oplus W_2 = Y_1 \oplus Y_2$. Here $\mathbf{C}_I(\Lambda \text{ proj})$ is a Krull–Schmidt category, so $W_2 \cong Y_2$ and then we obtain a radical irreducible morphism $W_2 \to W_2$, but composition of radical irreducible morphisms with sections or retractions are irreducible, therefore we obtain an irreducible morphisms $W_2' \to W_2'$ in $\mathbf{C}_I(\Lambda \text{ proj})$ for W_2' an indecomposable direct summand of W_2, such that $(W_2')^i \neq 0$, but this is not possible by Proposition 2.15. □

3. Irreducibles in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$

In this section we study irreducible morphisms in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$, the full subcategory of $\mathbf{C}(\Lambda \text{ proj})$, whose objects are the complexes bounded above with bounded cohomology. As a consequence of the fact that any complex is quasi-isomorphic to another one without direct summands homotopic to zero, we have that the results in this section apply to the homotopy category $\mathbf{K}^{-,b}(\Lambda \text{ proj})$ and hence to the bounded derived category $\mathbf{D}^b(\Lambda \text{ mod})$ if we assume this additional condition.

We start noting that we have analogous versions of the results in the previous section for $\mathbf{C}^{-,b}(\Lambda \text{ proj})$.

The next results show the relationship between irreducible maps in the categories $\mathbf{C}^{-,b}(\Lambda \text{ proj})$, $\mathbf{K}^{-,b}(\Lambda \text{ proj})$ and $\mathbf{C}_I(\Lambda \text{ proj})$.

First we consider irreducible morphisms $f : X \to Y$, where either X or Y have some $\mathcal{E}$-projective direct summand.

Remark 3.1. There are no irreducible morphism between $\mathcal{E}$-projective complexes in the category $\mathbf{C}^{-,b}(\Lambda \text{ proj})$.

Proof. Suppose $f : X \to Y$ is an irreducible morphism between projective complexes in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$. Since X is $\mathcal{E}$-injective and Y is $\mathcal{E}$-projective, then f is neither an $\mathcal{E}$-monomorphism nor an $\mathcal{E}$-epimorphism. Therefore by 2.2, there is an integer i, such that f^i is an irreducible morphism in Λ proj. Take $J = [-i-1, i+1]$, then f_J is an irreducible morphism in $\mathbf{C}_J(\Lambda \text{ proj})$. This last category is a Krull–Schmidt category, then by 2.16 there is a radical irreducible morphism $u : X_J \to Y_J$ in $\mathbf{C}_J(\Lambda \text{ proj})$. Here $X^i \neq 0$ and $Y^i \neq 0$, there is an indecomposable direct summand of X_J of the form $J_u(P)$ with $u \in [i-1, i]$ and there is an indecomposable direct summand of Y_J of the form $J_v(Q)$. Then by 2.18 there is an irreducible morphism $J_u(P) \to J_v(Q)$ in $\mathbf{C}_J(\Lambda \text{ proj})$. By Proposition 8.5 of [2] we have a minimal right almost split morphism $(P_S[-v-1])_J \to J_v(Q)$ in $\mathbf{C}_J(\Lambda \text{ proj})$, with $S = Q/\text{rad}Q$. But this implies that $J_u(P)$ is a direct summand of $(P_S[-v-1])_J$ which is not possible. □

In the next Proposition we use the notation of 2.13.

Proposition 3.2. *Suppose $f : J_{-1}(P^0) \to Z$ is an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$. Then Q_S is a finite complex and $Z \cong Q_S[1]$.*

Proof. First observe that if $g : J_{-1}(P^0) \to W$ is any irreducible morphism in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$ or in $\mathbf{C}(\Lambda \text{ proj})$, then $g_{[-1,0]}$ is an irreducible morphism in $\mathbf{C}_{[-1,0]}(\Lambda \text{ proj})$. Indeed, since $J_{-1}(P^0)$ is an $\mathcal{E}$-injective complex, then g is not an $\mathcal{E}$-monomorphism. Therefore by 2.2 either g is an $\mathcal{E}$-epimorphism or there is an $i \in [-1, 0]$ such that g^i is an irreducible morphism in Λ proj. In both cases $g_{[-1,0]}$ is an irreducible morphism in $\mathbf{C}_{[-1,0]}(\Lambda \text{ proj})$. From 2.13 we know that there is $v : J_{-1}(P^0) \to Q_S[1]$, a minimal left almost split morphism in $\mathbf{C}(\Lambda \text{ proj})$. Therefore there is a morphism $\lambda : Q_S[1] \to Z$ such that $\lambda v = f$. Now if L is any interval containing $[0, 1]$, then $(I_S)_L$ is indecomposable. Therefore if $[a, b]$ contains the interval $[-1, 0]$, then $\nu(Q_S[1])_{[a,b]} = (I_S)[1]_{[a,b]} = (I_S)_{[a+1,b+1]}[1]$, so $(Q_S[1])_{[a,b]}$ is indecomposable. Since $Z \in \mathbf{C}^{-,b}(\Lambda \text{ proj})$, there is an integer l such that $Z^j = 0$ for $j > l$. Choose now $L = [s, t]$ with $s < -1$ and $t > l$. Therefore $\lambda_L v_L = f_L$. Now f_L and v_L are irreducible morphisms in $\mathbf{C}_L(\Lambda \text{ proj})$, consequently λ_L is a retraction, so Z_L is a direct summand of $(Q_S[1])_L$, this implies that $Z_L \cong (Q_S[1])_L$. Then $Q_S[1]^t = 0$ and $Z^s = 0$ for all $s \leq -2$ and $t \geq l$. So Q_S is finite and $Z \cong Q_S[1]$. From this we obtain our result. □

In the following if $u : X \to Y$ is a morphism in $\mathbf{C}_I(\Lambda \text{ proj})$, we denote by $\underline{u}$ its image in the category $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$.

Proposition 3.3. *Suppose X and Y are complexes in $\mathbf{C}_I(\Lambda \text{ proj})$, without $\mathcal{E}_I$-injective direct summands. Then:*

(a) *$u : X \to Y$ is a section (respectively, retraction) if and only if $\underline{u}$ is a section (respectively, retraction).*
(b) *$u : X \to Y$ is an irreducible morphism if and only if $\underline{u}$ is irreducible.*

Proof. Clearly if u is a section, then also $\underline{u}$ is a section. Conversely suppose $\underline{u}$ is a section, then there is a morphism $v : Y \to X$ such that $vu = id_X + \lambda$ with λ a morphism which factorizes through some $\mathcal{E}_I$-injective. Since X has no $\mathcal{E}_I$-injective direct summands, it follows that λ is a radical morphism and so vu is an isomorphism, consequently u is a section. The corresponding statement for a retraction is proved in a similar way.

Now assume $u : X \to Y$ is an irreducible morphism, then by (a), $\underline{u}$ is neither a section nor a retraction. Suppose $\underline{u} = \underline{gf}$ is a factorization of $\underline{u}$ in $\overline{\mathbf{C}}_I(\Lambda \text{ proj})$, where $f : X \to Z$, $g : Z \to Y$. Then $u = gf + rs$ with $s : X \to W$, $r : W \to Y$, and W is an $\mathcal{E}_I$-injective complex. But then $u = (g, r)(f, s)^T$, $(f, s)^T : X \to Z \oplus W$ and $(g, r) : Z \oplus W \to Y$. Therefore either $(f, s)^T$ is a section, which implies by (a), that $\underline{f}$ is a section or (g, r) is a retraction which again by (a) implies that $\underline{g}$ is a retraction.

Now suppose $\underline{u}$ is an irreducible morphism, as before, by (a), this implies u is neither a section nor a retraction. Then if $u = gf$ is a factorization of u, we have $\underline{u} = \underline{gf}$. So either $\underline{f}$ is a section, which by (a) implies that f is a section or $\underline{g}$ is a retraction which by (a) implies that g is a retraction. This proves our result. □

Proposition 3.4. *If $f : X \to Y$ is a morphism in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$ with X and Y having no $\mathcal{E}$-projective direct summands then:*

(a) *f is a section (respectively, retraction) if and only if its image in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ is a section (respectively, retraction);*
(b) *f is irreducible if and only if its image in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ is irreducible.*

Proof. Similar to the proof of 3.3.

Theorem 3.5. *Let $f : X \to Y$ be a radical irreducible morphism between non-$\mathcal{E}$-projective complexes in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$. Then, there is a finite interval I_0 such that for all interval I containing I_0 we have the following.*

(1) *The morphism f_I is irreducible in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$.*
(2) *If $Z \in \{X, Y\}$ and Z has no $\mathcal{E}$-projective direct summands, then Z is indecomposable if and only if Z_I is indecomposable.*

Proof. We recall that $\mathcal{L}_{[a,b]}$ is the full subcategory of $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ whose objects are the complexes W with $W^j = 0$ for $j > b$ and $H^j(W) = 0$ for $j \le a$. Take $I_0 = [a, b]$ a finite interval such that $X, Y \in \mathcal{L}_{I_0}$. Then for all interval I containing I_0 we have $X, Y \in \mathcal{L}_I$.

We have an equivalence of categories:

$$F_I : \mathcal{L}_I \to \overline{\mathbf{C}}_I(\Lambda\,\mathrm{proj}).$$

Now take $X = X_0 \oplus T$ and $Y = Y_0 \oplus T'$ with X_0, Y_0 without $\mathcal{E}$-projective direct summands and T, T', $\mathcal{E}$-projective complexes. Then by (a) of 1.2, $(X_0)_I$ and $(Y_0)_I$ have no $\mathcal{E}_I$-injective direct summands. Clearly T_I and $(T')_I$ are $\mathcal{E}_I$-injective complexes.

(1) Consider the morphism $f_0 = pfi$ where i is the inclusion of X_0 in X and p is the projection of Y onto Y_0. By 2.18, f_0 is an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$. Then by 3.4, f_0 is an irreducible morphism in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$, so in the full subcategory $\mathcal{L}_I$. Using the equivalence F_I, we see that $(f_0)_I$ is an irreducible morphism in $\overline{\mathbf{C}}_I(\Lambda\,\mathrm{proj})$. Thus, by 3.3 $(f_0)_I$ is an irreducible morphism in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$.

Then

$$f_I = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} : X_0 \oplus T \to Y_0 \oplus T'$$

with $u_1 = (f_0)_I$ a radical irreducible morphism, u_2 and u_3 radical morphisms, so f_I is neither a section nor a retraction. Therefore, by 2.2, f_I is an irreducible morphism.

(2) The functor F_I induces an epimorphism of rings

$$\eta : E_Z = \mathrm{End}_{\mathbf{C}(\Lambda\,\mathrm{proj})}(Z) \to \mathrm{End}_{\mathbf{C}(\Lambda\,\mathrm{proj})}(Z_I) = E_{Z_I}.$$

Since Z is not $\mathcal{E}$-projective, the kernel of η is contained in $\mathrm{rad}E_Z$, therefore

$$E_Z/\mathrm{rad}E_Z \cong E_{Z_I}/\mathrm{rad}E_{Z_I}$$

so E_Z is a local ring if and only if E_{Z_I} is also local, this proves our claim. □

As a consequence of the above we get the following result and its dual.

Proposition 3.6. *Let $f : X \to Y$ be an irreducible morphism in the category $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, with X and Y without $\mathcal{E}$-projective direct summands. Assume that f is an $\mathcal{E}$-monomorphism and I is a finite interval with f_I irreducible, then $f_{I(-)}$ is an isomorphism.*

Proof. By the above result 3.5, we know that there is a finite interval I, with f_I irreducible. Then f_I is not a section, consequently $f_{I(-)}$ is a retraction. But f is an $\mathcal{E}$-monomorphism, therefore $f_{I(-)}$ is an isomorphism. □

Corollary 3.7. *If (x) $\quad 0 \to X \xrightarrow{f} E \xrightarrow{g} Y \to 0$ is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, then Y is a finite complex.*

Proof. Here X is not an $\mathcal{E}$-projective complex. If E has an $\mathcal{E}$-projective direct summand we obtain an irreducible morphism from some indecomposable $\mathcal{E}$-projective complex to Y, then by 3.2, Y is a finite complex and we have proved the proposition in this case. So we may assume X and Y have no $\mathcal{E}$-projective direct summands. Since the sequence (x) is a sequence of complexes of projective Λ-modules, then it is a conflation, so f is an $\mathcal{E}$-monomorphism. By 3.5 there is a finite interval I such that f_I is irreducible, then by the above proposition $f_{I(-)}$ is an isomorphism. But the exact sequence (x) restricted to the interval $I(-)$ is exact, therefore $Y_{I(-)} = 0$. Since Y is a bounded above complex, then it is a finite complex. □

It is well known that in $\Lambda\,\mathrm{mod}$, there are no irreducible morphisms from an indecomposable object into itself, the following shows that the same holds in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$ and therefore also in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ and in $\mathbf{D}^b(\Lambda\,\mathrm{mod})$. In case $\mathbf{D}^b(\Lambda\,\mathrm{mod})$ has Auslander–Reiten triangles this has been proved in [13].

Proposition 3.8. *If X is an indecomposable complex in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, then there are no irreducible morphisms $X \to X$.*

Proof. By 3.1 we may assume $f : X \to X$ is an irreducible morphism and X is not $\mathcal{E}$-projective. Then by 3.5 there is a finite interval I_0 of $\mathbb{Z}$ such that for all interval I of $\mathbb{Z}$ containing I_0, f_I is an irreducible morphism in $\mathbf{C}_I(\Lambda\,\text{proj})$ and X_I is indecomposable. If $I_0 = [a, b]$, take $I = [a-1, b+1]$, then we obtain an irreducible morphism $f_I : X_I \to X_I$ in $\mathbf{C}_I(\Lambda\,\text{proj})$, but this is not possible by Proposition 2.15. □

Now we look for the middle term of an almost split sequence and we prove the following:

Proposition 3.9. *Let $0 \to X \xrightarrow{f} E_1 \oplus \cdots \oplus E_n \xrightarrow{g} Y \to 0$ be an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, with E_i indecomposable objects which are not $\mathcal{E}$-projective, and X an infinite complex. Then there is at most one E_i such that the irreducible morphism $f_i = \pi_i f : X \to E_i$ is an $\mathcal{E}$-monomorphism.*

Proof. If there are two f_i, f_j which are $\mathcal{E}$-monomorphisms, then also the irreducible morphism $f_{i,j} = (f_i, f_j)^T : X \to E_i \oplus E_j$ is an $\mathcal{E}$-monomorphism, but then there is a finite interval I such that $(f_i)_I$, $(f_j)_I$ and $(f_{i,j})_I$ are irreducible morphisms, so $(f_i)_{I(-)}$, $(f_j)_{I(-)}$ and $(f_{i,j})_{I(-)}$ are isomorphisms, consequently $X_{I(-)} \cong (E_i)_{I(-)} \cong (E_j)_{I(-)} \cong (E_i)_{I(-)} \oplus (E_j)_{I(-)}$, a contradiction. □

Now, we give necessary conditions for the existence of irreducible morphisms between two modules in the category of complexes. The result is a generalization of Proposition 6.2 in [7].

In the following if M is a Λ-module and

$$\cdots \to P^{-3} \to P^{-2} \to P^{-1} \to P^0 \to M$$

is a minimal projective resolution of M we denote by P_M the complex:

$$\cdots \to P^{-3} \to P^{-2} \to P^{-1} \to P^0 \to 0 \to 0 \cdots.$$

Proposition 3.10. *Let X, Y be finitely generated Λ-modules. Suppose that the irreducible morphism $u : X \to Y$ induces an irreducible morphism $f : P_X \to P_Y$ in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, then $\text{pd}Y \geq 3$ implies $\text{pd}Y \leq \text{pd}X$.*

Proof. Take the interval $I = [-1, 0]$. By Proposition 2.2 we have that f_I is an irreducible morphism in $\mathbf{C}_I(\Lambda\,\text{proj})$ and by Proposition 2.4, $f_{I(-)}$ is a retraction, from here follows our result. □

Proposition 3.11. *Let $f : X \to Y$ be an irreducible map in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ such that f^i is an irreducible morphism in $\Lambda\,\text{proj}$. If $f_{(-\infty,i-1]}$ is not an isomorphism then*

$$X_{(-\infty,i-1]} \cong Y_{(-\infty,i-1]} \oplus P_U[-i+1],$$

with U a non-zero submodule of $\text{Ker}f^i$.

Proof. Here f^i is irreducible, so it is not a section, then by Proposition 2.4, f_J is a retraction with $J = (-\infty, i-1]$, therefore $X_J = Y_J \oplus Z$. Put $I = [i+1, \infty)$, the morphism $f : X \to Y$ is described by the following diagram:

$$\begin{array}{ccccc} Y_J \oplus Z & \xrightarrow{(\lambda,\mu)} & X^i & \xrightarrow{d_X^i} & X_I \\ \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f_I} \\ Y_J & \xrightarrow{d_Y^J} & Y^i & \xrightarrow{d_Y^i} & Y_I. \end{array}$$

Since $f^i\mu = 0$, then $U = \text{Im}\mu \subset \text{Ker}f^i$, we have $\mu = \mu_2\mu_1$ with $\mu_1 : Z^{i-1} \to U$ an epimorphism and $\mu_2 : U \to X^i$ the inclusion.

Take a minimal projective resolution of U:

$$\cdots \to P^{i-3} \to P^{i-2} \to P^{i-1} \xrightarrow{\eta} U \to 0$$

and the complex :

$$P_U[-i+1] : \cdots \to P^{i-3} \to P^{i-2} \to P^{i-1} \to 0 \to \cdots.$$

Here Z^{i-1} is projective and η is an epimorphism, then there is a morphism $\rho^{i-1} : Z^{i-1} \to P^{i-1}$ such that $\eta\rho^{i-1} = \mu_1$. We have $\mu_2\eta\rho^{i-1}d_Z^{i-2} = \mu d_Z^{i-2} = 0$. Therefore $\eta\rho^{i-1}d_Z^{i-2} = 0$, so there is a $\rho^{i-2} : Z^{i-2} \to P^{i-2}$ such that

$$\rho^{i-1}d_Z^{i-2} = d_{P_U}^{i-2}\rho^{i-2}.$$

Following this procedure we obtain a morphism of complexes $\rho : Z \to P_U[-i+1]$ such that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & X^i \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle id} \\ P_U[-i+1] & \xrightarrow{\mu_2\eta} & X^i. \end{array}$$

Then we have the following factorization of f:

$$\begin{array}{ccccc}
Y_J \oplus Z & \xrightarrow{(\lambda,\mu)} & X^i & \xrightarrow{d_X^i} & X_I \\
{\scriptstyle u}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\
Y_J \oplus P_U[-i+1] & \xrightarrow{(\lambda,\mu_2\eta)} & X^i & \xrightarrow{d_X^i} & X_I \\
{\scriptstyle (1,0)}\downarrow & & {\scriptstyle f^i}\downarrow & & {\scriptstyle f_I}\downarrow \\
Y_J & \xrightarrow{d_Y^J} & Y^i & \xrightarrow{d_Y^i} & Y_I
\end{array}$$

with $u = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$. Here f^i is not a retraction, then u is a section, therefore ρ is a section, in particular $\rho^{i-1} : Z^{i-1} \to P^{i-1}$ is a section. Now $\eta : P_U \to U$ is a minimal projective cover, so η is an essential epimorphism and $\eta\rho^{i-1} = \mu_1$ is an epimorphism, therefore ρ^{i-1} is an epimorphism, which shows that ρ^{i-1} is an isomorphism and we have

$$P_U[-i+1] \cong Z \oplus L,$$

an isomorphism of complexes with $L^{i-1} = 0$, so L is an acyclic complex in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$, this implies that L is a null homotopic complex, then since P_U is a minimal projective resolution, $L = 0$. This proves that ρ is an isomorphism and our result is proved. □

Let S be a simple module, and let P_S be the complex associated to a minimal projective resolution of S. Similarly, let I_S be the complex associated to a minimal injective co-resolution of S. We may assume $I_S = \nu(Q_S)$, with Q_S a complex of projective modules not necessarily in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$. We can take $P_S^0 = Q_S^0$. Consider the following complex B_S given by $B_S^j = P_S^j$ for $j < 0$ and $B_S^j = Q_S^{j+1}$ for $j \geq 0$. The differential of B_S is given by $d_{B_S}^j = d_{P_S}^j$ for $j < -1$, $d_{B_S}^{-1} = d_{Q_S}^0 d_{P_S}^{-1}$, $d_{B_S}^j = d_{Q_S}^{j+1}$ for $j \geq 0$. We have a morphism $\sigma : P_S \to B_S$ given by $\sigma^j = id_{P_S^j}$ for $j < 0$, $\sigma^0 = d_{Q_S}^0$ and $\sigma^j = 0$ for $j > 0$.

Now define a morphism $\rho : B_S \to Q_S[1]$ as follows $\rho^j = 0$ for $j < -1$, $\rho^{-1} = -d_{P_S}^{-1}$ and $\rho^j = (-1)^j id_{Q_S^{j+1}}$ for $j \geq 0$. It is easy to verify that σ and ρ are morphisms of complexes. We recall from 2.12 and 2.13 that we have a minimal right almost split morphism $u : P_S \to J_{-1}(P_S^0)$, and a minimal left almost split morphism $v : J_{-1}(P_S^0) \to Q_S[1]$ given by $v^j = 0$ for $j < -1$, $v^{-1} = id_{Q_S^0}$, $v^0 = -d_{Q_S}^0$ and $v^j = 0$ for $j > 0$.

We obtain the following conflation in $\mathbf{C}(\Lambda \text{ proj})$:

$$(s) \quad 0 \to P_S \overset{(\sigma,u)^T}{\to} B_S \oplus J_{-1}(P_S^0) \overset{(\rho,v)}{\to} Q_S[1] \to 0.$$

The above sequence in general is not an almost split sequence, however its restriction to all intervals L containing a certain fixed interval L_0 give almost split sequences in the corresponding category of L-complexes.

Proposition 3.12. *The above sequence* (s) *has the following properties:*

(a) *If* $h : P_S \to P_S$ *is a morphism of complexes which is not an isomorphism, then there is a morphism* $g : B_S \oplus J_{-1}(P_S^0) \to P_S$ *such that* $g(\sigma, u)^T = h$.
(b) *If* L *is a finite interval containing* $[-2, 0]$*, then the restriction of* (s) *to* L *is an almost split sequence in* $\mathbf{C}_L(\Lambda \text{ proj})$.
(c) *The sequence* (s) *is an almost split sequence in* $\mathbf{C}^{-,b}(\Lambda \text{ proj})$ *if and only if* Q_S *is a finite complex, this is, if and only if* S *has finite injective dimension.*

Proof. (a) By the properties of projective resolutions, h is an isomorphism if and only if h induces a non-zero endomorphism of S. Therefore, h is not an isomorphism if and only if h is null homotopic, this is $h = h_2h_1$ for some $h_1 : P_S \to T$, $h_2 : T \to P_S$, for some $\mathcal{E}$-injective T. Since (s) is an $\mathcal{E}$-sequence h_1 factorizes through $(\sigma, u)^T$, this implies (a).

(b) Here L contains $[-1, 0]$, so $P_S \in \mathcal{L}_I$. Moreover P_S gives a minimal projective resolution of S, then P_S does not have $\mathcal{E}$-projective direct summands. Therefore as in the proof of (2) of 3.5 we conclude that $(P_S)_L$ is indecomposable. Now $(I_S[1])_L = (I_S)_{L[-1]}[1]$, where if $L = [a, b]$, $L[-1] = [a + 1, b + 1]$. The interval $L[-1]$ contains the interval $[0, 1]$, so by duality $(I_S)_{L[-1]}$ is indecomposable. Therefore $(I_S[1])_L$ is indecomposable. The functor ν induces an equivalence $\nu : \mathbf{C}(\Lambda \text{ proj}) \to \mathbf{C}(\Lambda \text{ inj})$, then $Q_S[1]_L$ is an indecomposable complex. Suppose $L = [a, b]$, then $a \leq -2$ and $b \geq 0$. Here the differentials of I_S are radical morphisms, the same is true for $Q_S[1]$. Consequently $(Q_S[1])_L$ is not of the form $J_i(P)$. Since $Q_S[1]^a = 0$, $(Q_S[1])_L$ is not of the form $S(P)$. We infer that $Q_S[1]$ is not an $\mathcal{E}_I$-injective complex. By Proposition 6.12 of [2] there is an almost split sequence in $\mathbf{C}_L(\Lambda \text{ proj})$ ending in $(Q_S[1])_L$ and starting from Z_L, where Z is an indecomposable complex in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$ which is quasi-isomorphic to $\tau^{\leq b}\nu((Q_S[1])_L[-1]) = \tau^{\leq b}(I_S)_{L[-1]}$. The interval $L[-1]$ contains the interval $[0, 1]$, then there is a quasi-isomorphism from S to $\tau^{\leq b}(I_S)[-1]$. We have also a quasi-isomorphism from P_S to S, so we have a quasi-isomorphism from P_S to $\tau^{\leq b}\nu((Q_S[1])_L[-1])$. Then we may assume $Z = P_S$.

Now we are going to prove that the restriction of (s) to L is an almost split sequence. First observe that if the restriction of (s) to L splits then σ_I is a section, but σ^0 is not a section. Therefore the restriction of (s) to L gives a non-zero element z

of $\mathrm{Ext}_{\mathbf{C}_I(\Lambda\,\mathrm{proj})}((Q_S[1])_L, (P_S)_L)$. Now we know that there is an almost split sequence starting from $(P_S)_I$ and ending in $(Q_S)_I$. Then Theorem 9.3 of [3] implies that the socle of $\mathrm{Ext}_{\mathbf{C}_I(\Lambda\,\mathrm{proj})}((Q_S[1])_L, (P_S)_L)$ as $\mathrm{End}_{\mathbf{C}_I(\Lambda\,\mathrm{proj})}((P_S)_L)$-module is simple and any non-trivial element of this socle is an almost split sequence. So for proving our claim we only need to prove that z lies in the socle of $\mathrm{Ext}_{\mathbf{C}_I(\Lambda\,\mathrm{proj})}((Q_S[1])_L, (P_S)_L)$ as $\mathrm{End}_{\mathbf{C}_L(\Lambda\,\mathrm{proj})}((P_S)_L)$-module. Take any morphism $u : (P_S)_L \to (P_S)_L$ which is not an isomorphism. By 1.2 there is a morphism $v : P_S \to P_S$ such that $v_L = u$. By (a) of 3.3 and (a) of 3.4 v is not an isomorphism, then by (a) there is a morphism $g : B_S \oplus J_{-1}(P_S^0) \to P_S$ such that $v = g(\sigma, u)^T$. Therefore $v = g_I((\sigma, u)^T)_I$, and then $vz = 0$. This proves that z is in the socle and consequently (s) restricted to L is an almost split sequence.

(c) If (s) is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, then by 3.7 $Q_S[1]$ is finite. Conversely if Q_S is finite, then (s) is a sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, whose restriction to any interval L which contains $[-2, 0]$ is an almost split sequence in $\mathbf{C}_L(\Lambda\,\mathrm{proj})$. This implies that (s) is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$. □

4. Irreducible morphisms ending in a perfect complex

In this section we consider irreducible morphisms ending in a perfect complex in $\mathbf{D}^b(\Lambda\,\mathrm{mod})$. We recall that a perfect complex Y is a complex isomorphic to one $Y' \in \mathbf{K}^b(\Lambda\,\mathrm{proj})$, where this last category is the homotopy category of $\mathbf{C}^b(\Lambda\,\mathrm{proj})$, the category of bounded complexes.

We first consider the close relation between Auslander–Reiten triangles in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ and almost split sequences in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$.

We recall that in the category $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ a triangle

$$X \xrightarrow{u} E \xrightarrow{v} Y \xrightarrow{w} X[1]$$

is called an Auslander–Reiten triangle if:

$(AR1)$ X and Y are indecomposable

$(AR2)$ $w \neq 0$

$(AR3)$ If $f : W \to Y$ is not a retraction, then there exists $f' : W \to E$ such that $vf' = f$.

Observe that $(AR3)$ is equivalent to

$(AR3')$ If $f : W \to Y$ is not a retraction, then $wf = 0$. Moreover by Lemma 4.2 of [H] it follows that if $g : X[1] \to W$ is not a section then $gw = 0$.

Proposition 4.1 (*See Theorem 2.7 of [11]*). *If*

$$X \to E \to Y \xrightarrow{w} X[1]$$

is an Auslander–Reiten triangle in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$, *then it is isomorphic to a triangle of the form:*

$$X \xrightarrow{f} F \xrightarrow{g} Y \xrightarrow{w} X[-1]$$

where

$$0 \to X \xrightarrow{f} F \xrightarrow{g} Y \to 0$$

is a conflation and an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$. *The equivalence class of this last sequence in the $\mathcal{E}$-extension group corresponds to $-w$ under the natural isomorphism* $\psi : \mathrm{Ext}_{\mathcal{E}}(Y, X) \to \mathrm{Hom}_K(Y, X[1])$.

Proof. Suppose (a) $X \to E \to Y \xrightarrow{w} X[1]$ is an Auslander–Reiten triangle, take $x = \psi^{-1}(-w)$ and the corresponding conflation:

$$(x) \quad 0 \to X \xrightarrow{f} F \xrightarrow{g} Y \to 0.$$

Suppose $s : W \to Y$ is not a retraction. Consider the extension given by $\mathrm{Ext}(s, id)(x) = y$. Then $\psi(y) = -ws = 0$. Therefore $y = 0$, this implies that there is a morphism $t : W \to F$ with $gt = s$. Similarly, if $u : X \to W$ is not a section, take $\mathrm{Ext}(id, u)(x) = z$, then $\psi(z) = -uw = 0$, so $z = 0$. This implies that there is a morphism $v : F \to W$ with $vf = u$. Therefore (x) is an almost split sequence and clearly

$$X \xrightarrow{f} F \xrightarrow{g} Y \xrightarrow{w} X[1]$$

is a triangle isomorphic to (a). □

Proposition 4.2. *Suppose*

$$(x) \quad 0 \to Z \xrightarrow{f} E \xrightarrow{g} Y \to 0$$

is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$ *and $I = [a, b]$ is an interval such that Y is an I-complex, with $Y^a = 0$, $Y^b = 0$, then Z and E are in $\mathcal{L}_I$ and*

$$(y) \quad 0 \to Z_I \xrightarrow{f_I} E_I \xrightarrow{g_I} Y \to 0$$

is an almost split sequence in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$.

The complex E has an indecomposable direct summand $J_l(P)$ if and only if $Z \cong P_S[-l-1]$, with $S = P/\mathrm{rad}P$. In this case, with the notation of Proposition 3.12, (x) is isomorphic to a shift of (s) and $E_I = (B_S[-l-1])_I \oplus J_l(P)$.

Proof. We know from [5] that Z is quasi-isomorphic to $\nu(Y)[-1]$. Moreover $\nu(Y)[-1]^a = \nu(Y^{a-1}) = 0$, so $H^a(Z) \cong H^a(\nu(Y)[-1]) = 0$. For $j > b$, $\nu(Y)[-1]^j = \nu(Y^{j-1}) = 0$, then $Z \in \mathbf{C}^{\leq b}(\Lambda\,\mathrm{proj})$, so $Z \in \mathcal{L}_I$.

Since (x) is an exact sequence we deduce that E is in $\mathbf{C}^{\leq b}(\Lambda\,\mathrm{proj})$. From the exact sequence

$$H^a(Z) \to H^a(E) \to H^a(Y)$$

we obtain that $H^a(E) = 0$, so $E \in \mathcal{L}_I$.

By (c) of 1.2 there is an inclusion of complexes $\sigma_E : E_I \to E$ such that $g_I = g\sigma_E$, we know from (d) of 1.2 that F_I induces an equivalence of categories $\mathcal{L}_I \to \overline{\mathbf{C}}_I(\Lambda\,\mathrm{proj})$. Therefore, if $g_I : E_I \to Y$ is a retraction then $g : E \to Y$ is a retraction in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$, which is not the case, thus $g_I : E_I \to Y$ is not a retraction, so we have the non-splittable exact sequence:

$$(y) \quad 0 \to Z_I \xrightarrow{f_I} E_I \xrightarrow{g_I} Y \to 0.$$

We are going to prove that (y) is an almost split sequence in the category $\mathbf{C}_I(\Lambda\,\mathrm{proj})$.

First observe that g_I is an irreducible morphism, indeed we know from 2.2 that g_I is a section or a retraction or an irreducible. We already saw that g_I is not a retraction. Now since all the g^i are epimorphisms, if g_I is a section then it is an isomorphism so a retraction which is not the case, therefore g_I is an irreducible morphism.

For proving that (y) is an almost split sequence it is enough to prove that if $h : W \to Y$ is not a retraction in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$ then there is a $v : W \to E_I$ with $g_I v = h$. Now h is not a retraction in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, so there is a morphism $v' : W \to E$ with $gv' = h$. But $W \in \mathbf{C}_I(\Lambda\,\mathrm{proj})$, thus there is a $v : W \to E_I$ such that $v' = \sigma_E v$. Therefore $g_I v = g\sigma_E v = gv' = h$. This proves that (y) is an almost split sequence.

Suppose E has a direct summand of the form $J_i(P)$ with P indecomposable projective Λ-module. Then there is an irreducible morphism $Z \to J_i(P)$, by 2.12 there is a minimal right almost split morphism $P_S[-i-1] \to J_{-1}(P)[-i-1] = J_i(P)$ with $S = P/\mathrm{rad}P$. This implies that Z is a direct summand of $P_S[-i-1]$ which is indecomposable, so $Z \cong P_S[-i-1]$. Therefore (x) is isomorphic to a shift of the sequence (s) of 3.12. Conversely if $Z \cong P_S[j]$ for some j, then (x) is isomorphic to a shift of the sequence (s) of 3.12, therefore $J_{-j-1}(P)$ is a direct summand of E. □

Corollary 4.3. *Let $u : X \to Y$ be a morphism of complexes whose homotopy class is an irreducible morphism in $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ with X, Y indecomposable complexes. Suppose $I = [a, b]$ is an interval such that $Y^a = 0 = Y^b$ and $Y \in \mathbf{C}_I(\Lambda\,\mathrm{proj})$ then $X \in \mathcal{L}_I$ and $u_I : X_I \to Y_I$ is an irreducible morphism in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$.*

Proof. By [5] we know that there is an Auslander–Reiten triangle in the category $\mathbf{K}^{-,b}(\Lambda\,\mathrm{proj})$ ending in Y. Then by 4.1 there exists an almost split sequence in the category $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$:

$$0 \to Z \xrightarrow{f} E \xrightarrow{g} Y \to 0.$$

We know from (b) of 3.4 that u is an irreducible morphism in the category $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$. Then there is a section $s : X \to E$ such that $gs = u$. Therefore there is an isomorphism $h : X \oplus E' \to E$ such that $g' = gh = (u, u') : X \oplus E' \to Y$. Then we have the almost split sequence:

$$(x) \quad 0 \to Z \xrightarrow{f'} X \oplus E' \xrightarrow{g'} Y \to 0.$$

By 4.2 the restriction of (x) to I is an almost split sequence, so g'_I is an irreducible morphism in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$. Therefore u_I is also an irreducible morphism in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$. Finally by the first part of 4.2, $X \oplus E'$ is in $\mathcal{L}_I$ and consequently $X \in \mathcal{L}_I$. □

For the statement of the following proposition we recall from 1.2 that if $X \in \mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, and $I = [a, b]$, then X_I is a subcomplex of X. By $\sigma_X : X_I \to X$ we denote the inclusion.

Proposition 4.4. *Let $v : W \to Y$ be an irreducible morphism in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$, where W and Y have no $\mathcal{E}_I$-injective direct summands, Y is an indecomposable, and let $I = [a, b]$, with $Y^a = 0 = Y^b$. Then there is an irreducible morphism $u : X \to Y$ in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$ such that $X_I = W$ and $u\sigma_X = v$, where $\sigma_X : W = X_I \to X$.*

Proof. As in the proof of 4.3 there is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$:

$$0 \to Z \xrightarrow{f} E \xrightarrow{g} Y \to 0,$$

such that its restriction to I is an almost split sequence in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$. Since v is an irreducible morphism in $\mathbf{C}_I(\Lambda\,\mathrm{proj})$, there is a section $s : W \to E_I$ such that $g_I s = v$. Now by (b) of 1.2 there is a complex X in the category $\mathcal{M}_I$ such that $X_I = W$ and by (a) of 1.2 there is a morphism of complexes $t : X \to E$ such that $t_I = s$. Observe that we may assume that X has not $\mathcal{E}$-projective direct summands, indeed $X = X_0 \oplus T$ where T is $\mathcal{E}$-projective and X_0 has not $\mathcal{E}$-projective direct summands. Then $W = (X_0)_I \oplus T_I$, here T_I is an $\mathcal{E}_I$-injective complex, but W has no $\mathcal{E}_I$-injective direct summands, so $T_I = 0$, and we can take X_0 instead of X.

The morphism s is a section, so there is a morphism $s' : E_I \to W$ with $s's = id_W$ and the functor F_I induces an equivalence between the category $\mathcal{L}_I$ (the homotopy category of $\mathcal{M}_I$) and the category $\overline{\mathbf{C}}_I(\Lambda\,\mathrm{proj})$. Then there is a morphism $t' : E \to X$ with $t'_I = s'$ such that $t't = id_X + \lambda$ where λ is a morphism which factorizes through some $\mathcal{E}$-projective. Since X has no $\mathcal{E}$-projective summands, then we have that λ is in the radical of the endomorphism ring of W, so $t't$ is an isomorphism and consequently t is a section. Then $u = gt : X \to Y$ is an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$.

We have $u\sigma_X = gt\sigma_X = g\sigma_E t_I = g\sigma_E s = g_I s = v$. This proves our result. □

Corollary 4.5. *Let $v : W \to Y$ be an irreducible morphism in the category $\mathbf{C}_I(\Lambda\,\text{proj})$ with the conditions of 4.4. Then v is an irreducible morphism in $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$ if and only if d_W^a is a monomorphism.*

Proof. By Proposition 4.4, there is an irreducible morphism $u : X \to Y$ in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, with $X_I = W$ and $u\sigma_X = v$. If v is irreducible in $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$ then σ_X is a section, this implies $d_X^{a-1} = 0$ or equivalently, $\text{Ker}d_X^a = \text{Ker}d_W^a = 0$.

Note that if $\text{Ker}d_X^a = 0$ then $X = W$ and $v = u$ is irreducible. □

Proposition 4.6. *Let Y be an indecomposable complex in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, which is not $\mathcal{E}$-projective. If Y is perfect then there is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ ending in Y. If Y is isomorphic in $\mathbf{D}^b(\Lambda\,\text{mod})$ to a finite complex of injectives, then there is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ starting from Y.*

Proof. Here Y is not $\mathcal{E}$-projective so it is indecomposable in the category $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$. Then if Y is perfect we know from [5] that there exists an Auslander–Reiten triangle in $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$ ending in Y, therefore by 4.1 there is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ ending in Y. Now if Y is isomorphic in $\mathbf{D}^b(\Lambda\,\text{mod})$, to a finite complex of injective Λ-modules W, we may assume W is an indecomposable complex of finitely generated injective Λ-modules, then $W \cong \nu(Z)$ with Z an indecomposable finite complex of finitely generated projective Λ-modules. We know that Y is not homotopically trivial, so Z is not an $\mathcal{E}$-projective complex, therefore there is an almost split sequence ending in $Z[1]$ and starting from L, an indecomposable complex in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ which is quasi-isomorphic to $\nu(Z) \cong W$. Here Y and L are indecomposable complexes in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, they are isomorphic in $\mathbf{D}^b(\Lambda\,\text{mod})$, so L and Y are isomorphic complexes and we obtain an almost split sequence in $\mathbf{D}^b(\Lambda\,\text{mod})$ starting from Y. □

5. The selfinjective case

In this section we assume that Λ is a selfinjective Artin k-algebra, k a commutative Artinian ring. We are going to study irreducible morphisms in $\mathbf{C}^b(\Lambda\,\text{proj})$.

In the following if U is a Λ-module and $U \to P^0 \to P^1 \to \cdots$ is a minimal injective co-resolution, denote by I_U the complex:

$$I_U : \cdots \to 0 \to P^0 \to P^1 \to \cdots.$$

We have the following dual of Proposition 3.11.

Proposition 5.1. *Let Λ be a selfinjective algebra and let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$ such that f^i is an irreducible morphism in $\Lambda\,\text{proj}$. If $f_{[i+1,\infty)}$ is not an isomorphism we have*

$$Y_{[i+1,\infty)} \cong X_{[i+1,\infty)} \oplus I_U[-i-1],$$

with U a non-zero submodule of $\text{Coker}f^i$.

Before to state the main result of this section we need to notice the following property of the irreducible morphisms in the category $\Lambda\,\text{proj}$.

Proposition 5.2. *Let Λ be a selfinjective algebra and $f : P \to Q$ an irreducible morphism in $\Lambda\,\text{proj}$, then the cokernel of f has no projective submodules and the kernel of f has no projective submodules.*

Proof. By 2.16 and 2.17 there are decompositions of $P = P_1 \oplus Z$ and of $Q = Q_1 \oplus W$ such that with respect to these decompositions

$$f = \begin{pmatrix} g & 0 \\ 0 & s \end{pmatrix}$$

with $g : P_1 \to Q_1$ an irreducible morphism in the radical and s an isomorphism. Clearly $\text{Ker}f = \text{Ker}g$ and $\text{Coker}f = \text{Coker}g$, so we may assume that f is a radical irreducible morphism. Let L be a projective submodule of $\text{Coker}f$. Here Λ is a selfinjective algebra, then L is an injective module and then a direct summand of $\text{Coker}f$. We have a retraction $v : \text{Coker}f \to L$, and then an epimorphism $\eta : Q \to L$ with $\eta f = 0$. Since L is projective, then η is a retraction. But f is irreducible then ηf is irreducible, a contradiction. Similarly one can prove that $\text{Ker}f$ has not projective submodules. □

Lemma 5.3. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, with X and Y complexes in $\mathbf{C}_{[a,b]}(\Lambda\,\text{proj})$. We have the following:*

(1) *If $f^i : X^i \to Y^i$ is an irreducible morphism in $\Lambda\,\text{proj}$, then $f_{(-\infty,i-1]}$ and $f_{[i+1,\infty)}$ are isomorphisms.*
(2) *If $X^a \neq 0$ and $Y^a = 0$, then f is an $\mathcal{E}$-epimorphism. If $X^a = 0$ and $Y^a \neq 0$, then f is an $\mathcal{E}$-monomorphism.*
(3) *If $X^b \neq 0$ and $Y^b = 0$, then f is an $\mathcal{E}$-epimorphism. If $X^b = 0$ and $Y^b \neq 0$, then f is an $\mathcal{E}$-monomorphism.*

Proof. (1) If $f_{[i+1,\infty)}$ is not an isomorphism then $Y_{[i+1,\infty)}$ has $I_U[-i-1]$ as a direct summand, with U a non-trivial submodule of $\text{Coker}f^i$. By Proposition 5.2, U is not an injective Λ-module. Since Λ is selfinjective, I_U is not a perfect complex, which implies that Y is not perfect, a contradiction, therefore $f_{[i+1,\infty)}$ is an isomorphism. In a similar way we can prove that $f_{(-\infty,i-1]}$ is an isomorphism.

(2) $X^a \neq 0$, $Y^a = 0$. If for some s, f^s is an irreducible morphism in $\Lambda\,\text{proj}$, then since $f^a = 0$, $s > a$, then by (1), $f_{(-\infty,s-1]}$ is an isomorphism which implies that $f^a = 0$ is an isomorphism which is not the case. Then by the version of Proposition 2.9 for $\mathbf{C}^{-,b}(\Lambda\,\text{proj})$, we have that f is either an $\mathcal{E}$-epimorphism or an $\mathcal{E}$-monomorphism. But f^a is not a section, therefore f is an $\mathcal{E}$-epimorphism. The other cases are proved in a similar way. □

We have already seen that given a perfect complex Y in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, there is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, ending in Y and starting from a complex Z of projective Λ-modules quasi-isomorphic to $\nu(Y)[-1]$. In our case $\nu(Y)[-1]$ is a finite complex of projective Λ-modules. Therefore we may assume $Z = \nu(Y)[-1]$, this implies that the almost split sequence ending in Y and starting from $\nu(Y)[-1]$ is in $\mathbf{C}^b(\Lambda\,\mathrm{proj})$. Moreover, since we have an equivalence of categories $\nu : \mathbf{C}^b(\Lambda\,\mathrm{proj}) \to \mathbf{C}^b(\Lambda\,\mathrm{proj})$, the complex $Y \cong \nu(Z)[-1]$ for Z some finite complex of projective Λ-modules. Consequently, there is an almost split sequence in $\mathbf{C}^b(\Lambda\,\mathrm{proj})$, also starting from Y.

However, observe that in general (by 2.12) there are no minimal right almost split morphisms in $\mathbf{C}^b(\Lambda\,\mathrm{proj})$ ending in indecomposable $\mathcal{E}$-projective complexes. Similarly, (by 2.13) in general there are no minimal left almost split morphisms $\mathbf{C}^b(\Lambda\,\mathrm{proj})$ starting from indecomposable $\mathcal{E}$-injective complexes.

Now we are ready to give the following property of almost split sequences in $\mathbf{C}^b(\Lambda\,\mathrm{proj})$, for Λ a selfinjective Artin algebra.

In the following for $M \in \Lambda$-mod we denote by $|M|$ the length of M as a k-module. If X is a finite complex of finitely generated Λ-modules we put $|X| = \sum_{i\in\mathbb{Z}} |X^i|$. For X a complex $\mathrm{supp}X = \{i \in \mathbb{Z} | X^i \neq 0\}$. Clearly if X is an indecomposable finite complex, then $\mathrm{supp}X = [a, b]$ for some a and b.

Theorem 5.4. *Let Λ be a selfinjective Artin algebra and let X be an indecomposable complex in $\mathbf{C}^b(\Lambda\,\mathrm{proj})$ with $\mathrm{supp}X = [a, b]$. If*

$$(a)\quad 0 \to X \overset{u=(u_1,\ldots,u_n)}{\to} E_1 \oplus \cdots \oplus E_n \overset{v=(v_1,\ldots,v_n)^T}{\to} Y \to 0$$

is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$ with all the E_i indecomposable complexes, then $n \leq 2$. For some i, u_i is an $\mathcal{E}$-monomorphism and v_i is an $\mathcal{E}$-epimorphism. Moreover, each irreducible morphism between indecomposable objects in the category $\mathbf{C}^b(\Lambda\,\mathrm{proj})$ is either an $\mathcal{E}$-monomorphism or an $\mathcal{E}$-epimorphism.

Proof. Claim 1 Suppose

$$(z)\quad 0 \to Z \overset{(u,s)^T}{\to} U \oplus V \overset{(v,t)}{\to} W \to 0$$

is an almost split sequence in $\mathbf{C}^{-,b}(\Lambda\,\mathrm{proj})$, with $\mathrm{supp}W = [c, d]$, then $\mathrm{supp}Z = [c + 1, d + 1]$. Moreover, if $U^c \neq 0$, then $V^c = 0$ and u is an $\mathcal{E}$-monomorphism with u^{d+1} an isomorphism. The morphism $s : Z \to V$ is an $\mathcal{E}$-epimorphism.

In fact we may assume $Z = \nu(W)[-1]$ so $\mathrm{supp}Z = [c + 1, d + 1]$. Since $W^{d+1} = 0$, the exactness of (z) implies that

$$(u^{d+1}, s^{d+1})^T : Z^{d+1} \to U^{d+1} \oplus V^{d+1}$$

is an isomorphism. Therefore $|Z^{d+1}| = |V^{d+1}| + |U^{d+1}|$.

Suppose now that $U^c \neq 0$ and $V^c \neq 0$. Since $Z^c = 0$, (2) of 5.3, implies that u and s are $\mathcal{E}$-monomorphisms. Consequently, u^{d+1} and s^{d+1} are sections, which implies that $|Z^{d+1}| \leq |U^{d+1}|$ and $|Z^{d+1}| \leq |V^{d+1}|$. Then $|Z^{d+1}| = |V^{d+1}| + |U^{d+1}| \geq 2|Z^{d+1}|$, a contradiction, proving that $V^c = 0$.

Here $V^c = 0$ and $W^c \neq 0$, then by (2) of 5.3, t is an $\mathcal{E}$-monomorphism. Since $W^{d+1} = 0$, then $V^{d+1} = 0$. We have that $Z^{d+1} \neq 0$, then (3) of 5.3 implies that s is an $\mathcal{E}$-epimorphism.

Finally, from $|Z^{d+1}| = |V^{d+1}|$ we deduce that $u^{d+1} : Z^{d+1} \to V^{d+1}$ is an isomorphism.

Claim 2 Suppose $u : Z \to U$ is an irreducible morphism between indecomposable complexes in $\mathbf{C}^b(\Lambda\,\mathrm{proj})$, such that Z is not $\mathcal{E}$-projective and u is an $\mathcal{E}$-monomorphism. Then $\mathrm{supp}Z = [a, b]$, implies $\mathrm{supp}U = [a - 1, b]$. Moreover the morphism $u^b : Z^b \to U^b$ is an isomorphism.

Indeed there is an almost split sequence starting from Z of the form (z). Using the above notation we have $[a, b] = [c + 1, d + 1]$. If $U^c = 0$, then u is an $\mathcal{E}$-epimorphism which is not the case, so $U^c \neq 0$, thus by Claim 1, we have that $u^{d+1} = u^b$ is an isomorphism.

Now, take an almost split sequence (a) starting from X. Suppose $\mathrm{supp}X = [a, b]$, then $\mathrm{supp}Y = [a - 1, b - 1]$. We are going to prove that $n \leq 2$.

There is some i with $E_i^{a-1} \neq 0$, we may assume $i = 1$. By Claim 1, $(E_2 \oplus \cdots \oplus E_n)^{a-1} = 0$. Here $(v_2, \ldots, v_n) : E_2 \oplus \cdots \oplus E_n \to Y$ is an irreducible morphism and $Y^{a-1} \neq 0$, then by (2) of 5.3, $(v_2, \ldots, v_n)$ is an $\mathcal{E}$-monomorphism. Therefore

$$|E_2^{b-1}| + \cdots + |E_n^{b-1}| \leq |Y^{b-1}|.$$

Suppose $n \geq 3$, then as before v_2 and v_3 are $\mathcal{E}$-monomorphisms. By 4.2, E_2 and E_3 are indecomposable which are not $\mathcal{E}$-projective complexes. Claim 2, shows that $|E_2^{b-1}| = |Y^{b-1}|$ and $|E_3^{b-1}| = |Y^{b-1}|$, this implies that $2|Y^{b-1}| = |E_2^{b-1}| + |E_3^{b-1}| \leq |Y^{b-1}|$, a contradiction. Consequently, $n \leq 2$. Then in case $n = 2$, u_1 and v_1 are $\mathcal{E}$-monomorphisms and u_2 and v_2 are $\mathcal{E}$-epimorphisms.

Now, if $f : Z \to W$ is an irreducible morphism between indecomposable perfect complexes, they are part of an almost split sequence (a). The above implies that f is an $\mathcal{E}$-monomorphism or an $\mathcal{E}$-epimorphism. This completes the proof. □

Theorem 5.5. *If Λ is a selfinjective Artin algebra, then the non-trivial components of the Auslander–Reiten quiver of $\mathbf{C}^b(\Lambda\,\mathrm{proj})$ are of the form $\mathbb{Z}A_\infty$.*

Proof. Let X be an indecomposable complex which is not $\mathcal{E}$-projective in $\mathbf{C}^b(\Lambda\,\text{proj})$, then by Proposition 4.6 there are almost split sequences in $\mathbf{C}^b(\Lambda\,\text{proj})$, ending in and starting from X, so this last category has almost split sequences and we can consider its Auslander–Reiten quiver. Let $\mathcal{C}$ be a non-trivial component of the Auslander–Reiten quiver of $\mathbf{C}^b(\Lambda\,\text{proj})$. By Theorem 5.4 given any indecomposable $Y \in \mathcal{C}$, there is an irreducible $\mathcal{E}$-monomorphism $Y \to Y'$ with Y' indecomposable, we have $|Y| < |Y'|$. Therefore we may find a sequence of irreducible $\mathcal{E}$-monomorphisms:

$$Y_0 \to Y_1 \to Y_2 \to Y_3 \to \cdots$$

such that there are not any irreducible $\mathcal{E}$-monomorphism ending in Y_0.

Clearly, we have an almost split sequence

$$\tau Y_0 \to \tau Y_1 \to Y_0,$$

where $\tau(-) = \nu(-)[-1]$.

Note that for all $i > 0$ we have irreducible morphisms $\tau Y_{i+1} \to Y_i$ and $Y_i \to Y_{i+1}$ where the last one is an $\mathcal{E}$-monomorphism and then again by Theorem 5.4 the first morphism is an $\mathcal{E}$-epimorphism. Therefore, we have that $|Y_{i-1}| < |Y_i| < |\tau Y_{i+1}|$ so Y_{i-1} is not isomorphic to τY_{i+1} and we get the almost split sequence:

$$\tau Y_i \longrightarrow \tau Y_{i+1} \oplus Y_{i-1} \longrightarrow Y_i.$$

Moreover, we have a sequence of irreducible $\mathcal{E}$-monomorphisms:

$$\tau Y_0 \to \tau Y_1 \to \tau Y_2 \to \tau Y_3 \to \cdots$$

such that there are not any irreducible $\mathcal{E}$-monomorphism ending in τY_0.

Applying repeatedly this procedure, we have, for all $s \in \mathbb{Z}$ and $i > 0$ the almost split sequence:

$$(x(s,i)) \quad \tau^s Y_i \longrightarrow \tau^s Y_{i+1} \oplus \tau^{s-1} Y_{i-1} \longrightarrow \tau^{s-1} Y_i,$$

with $\tau^s Y_i \to \tau^s Y_{i+1}$ an irreducible morphism which is an $\mathcal{E}$-monomorphism and $\tau^s Y_i \to \tau^{s-1} Y_{i-1}$ an irreducible morphism which is an $\mathcal{E}$-epimorphism. For $i = 0$ we have the almost split sequence:

$$(x(s,0)) \quad \tau^s Y_0 \longrightarrow \tau^s Y_1 \longrightarrow \tau^{s-1} Y_0,$$

for all $s \in \mathbb{Z}$.

Then if Z is an indecomposable in the component $\mathcal{C}$ we have that $Z = \tau^s Y_i$, for some s and i. In order to prove that $\mathcal{C} = \mathbb{Z}A_\infty$, it is enough to prove that $\tau^s Y_i \cong \tau^t Y_j$ implies $i = j$ and $s = t$. Suppose i and j are both greater than 0. Then the sequences $(x(s,i))$ and $(x(t,j))$ are isomorphic, and since the irreducible morphism $\tau^t Y_j \to \tau^{t-1} Y_{j-1}$ is an $\mathcal{E}$-epimorphism, then $\tau^{s-1} Y_{i-1} \cong \tau^{t-1} Y_{j-1}$. Following in this way we can find some l such that $i - l = 0$ or $j - l = 0$. But then the almost split sequence starting from $\tau^{s-l} Y_{i-l}$ has an indecomposable middle term, so $i - l = 0$ and $j - l = 0$. Consequently, $i = l = j$ and then $t = s$, because the complexes Y_i are not τ-periodic. □

Proposition 5.6. *If $f : X \to Y$ is an irreducible morphism in the category $\mathbf{C}^b(\Lambda\,\text{proj})$, then f is either an $\mathcal{E}$-epimorphism or an $\mathcal{E}$-monomorphism.*

Proof. By Lemma 2.16 and Proposition 2.17 we may assume f is a radical morphism. Then if X and Y are indecomposable our result follows from 5.4. If both X and Y are decomposable we should have irreducible morphisms $X_1 \to Y_1, X_2 \to Y_2, X_2 \to Y_1$ and $X_2 \to Y_2$, for pairwise non-isomorphic indecomposable objects X_1, X_2, Y_1 and Y_2. But this is impossible in $\mathbb{Z}A_\infty$. If X is decomposable and Y indecomposable f is a minimal right almost split morphism, so it is an $\mathcal{E}$-epimorphism. In case X is indecomposable and Y decomposable f is a minimal left almost split morphism, so it is an $\mathcal{E}$-monomorphism. □

6. Irreducible morphisms involving non-perfect complexes

This last part is devoted to the study of irreducible morphisms between indecomposable complexes $f : X \to Y$ in the category $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$, where either X or Y is a non-perfect complex.

We first consider the case in which Y is a non-perfect complex, as we see later this implies that X is also a non-perfect complex.

In case Λ is Gorenstein all Auslander–Reiten triangles in $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$ consist of perfect complexes, so in this case if $f : X \to Y$ is an irreducible morphism in $\mathbf{K}^{-,b}(\Lambda\,\text{proj})$ involving non-perfect complexes both X and Y are non-perfect complexes. This situation is considered in Proposition 5.5 of [7] for finite dimensional Gorenstein algebras over a field. It is proved that in this case $X \cong \nu Y[-1]$. For the general case of an Artin algebra we obtain a generalization of the above mentioned result.

Finally we consider the case in which Y is a perfect complex and X is non-perfect. This situation occurs when there is an Auslander–Reiten triangle ending in Y and νY is a non-perfect complex.

In order to prove our results in this section, we use the following Lemma. We also need to recall that for every complex $X \in \mathbf{C}_l(\Lambda\,\text{proj})$ there exist H an $\mathcal{E}_l$-injective complex and X_0, a complex without $\mathcal{E}_l$-injective direct summands, such that $X \simeq X_0 \oplus H$ in $\mathbf{C}_l(\Lambda\,\text{proj})$.

Lemma 6.1. *Let $f : X \to Y$ be an irreducible morphism in $\mathbf{K}^{-,b}(\Lambda\text{ proj})$. If Y is a non-perfect complex then X is also a non-perfect complex.*

Proof. We may assume $f : X \to Y$ is an irreducible morphism in the category $\mathbf{C}^{-,b}(\Lambda\text{ proj})$. By Theorem 3.5 there is an interval I such that f_I is an irreducible morphism and then f_I is not a section. So by Proposition 2.4, $f_{I(-)}$ is a retraction. This implies that for all $j \in I(-)$, the map f^j is not zero and so $X^j \neq 0$. So X is not a perfect complex. □

For the proof of next proposition we recall that a complex $X \in \mathbf{C}(\Lambda\text{ mod})$ is called q-projective if $\mathrm{Hom}_{\mathbf{K}(\Lambda\text{ mod})}(X, C) = 0$ for any acyclic complex $C \in \mathbf{C}(\Lambda\text{ mod})$. Moreover if $s_1 : X_1 \to Y_1$ and $s_2 : X_2 \to Y_2$ are quasi-isomorphisms with X_1, X_2 q-projective then given a morphism $f : Y_1 \to Y_2$ in $\mathbf{C}(\Lambda\text{ mod})$ there is a unique morphism up to homotopy $h : X_1 \to X_2$ such that $fs_1 = s_2h$ in $\mathbf{K}(\Lambda\text{ mod})$. Observe that if $Y_1 = Y_2$ then there is an isomorphism $u : X_1 \to X_2$ in the homotopy category such that $s_2u = s_1$ in $\mathbf{K}(\Lambda\text{ mod})$. If X is a complex in $\mathbf{C}^{\leq m}(\Lambda\text{ proj})$ for some m, then X is a q-projective complex.

Proposition 6.2. *Let $W \in \mathbf{C}^{\leq n}(\Lambda\text{ mod})$ and let $q : Z \to W$ be a quasi-isomorphism with Z a complex in $\mathbf{C}^{\leq n}(\Lambda\text{ proj})$ without $\mathcal{E}$-projective direct summands.*

Suppose $L = [a, n]$ is an interval of $\mathbb{Z}$. If $q_1 : Z' \to W_L$ is a quasi-isomorphism with $Z' \in \mathbf{C}^{\leq n}(\Lambda\text{ proj})$ then there is an isomorphism of complexes:

$$(Z_L)_0 \cong (Z'_L)_0.$$

Proof. There is a morphism of complexes $W_{L(-)} \xrightarrow{d} W_L[1]$, given by $d^j = 0$ if $j \neq a-1$, $d^{a-1} = d_W^{a-1} : W_{L(-)}^{a-1} = W^{a-1} \to W^a = W_L[1]^{a-1}$. Observe that $W = \mathrm{Con}(d[-1])$. Therefore in $\mathbf{K}(\Lambda\text{ mod})$ we have the triangle:

$$(a) \quad W_{L(-)}[-1] \xrightarrow{d[-1]} W_L \xrightarrow{u} W \xrightarrow{v} W_{L(-)}$$

and we have the exact sequence of complexes:

$$(x) \quad 0 \to W_L \xrightarrow{u} W \xrightarrow{v} W_{L(-)} \to 0.$$

Let $q_2 : R \to W_{L(-)}$ be a quasi-isomorphism with $R \in \mathbf{C}^{\leq a-1}(\Lambda\text{ proj})$.

There exists a morphism of complexes $h : R[-1] \to Z'$ such that $q_1h = d[-1]q_2[-1]$ in the homotopy category. Consider now the triangle:

$$(b) \quad R[-1] \xrightarrow{h} Z' \to \mathrm{Con}(h) \to R.$$

The triple $(q_2[-1], q_1, \lambda)$ is a morphism from the triangle (b) to the triangle (a). Since q_2 and q_1 are quasi-isomorphisms then $\lambda : \mathrm{Con}(h) \to W$ is also a quasi-isomorphism. We have also a quasi-isomorphism $q : Z \to W$, therefore there is an isomorphism in the homotopy category $v : \mathrm{Con}(h) \to Z$ such that $qv = \lambda$. Since Z has no $\mathcal{E}$-projective direct summands, then $\mathrm{Con}(h) \cong Z \oplus T$ as complexes, with T an $\mathcal{E}$-projective complex.

We have the exact sequence

$$0 \to Z' \to \mathrm{Con}(h) \to R \to 0.$$

Since $R_L = 0$, we obtain

$$Z_L \oplus T_L \cong \mathrm{Con}(h)_L \cong (Z')_L.$$

Note that T_L is $\mathcal{E}_L$-injective and then the result holds. □

Recall that if X, Y belong to $\mathbf{C}^{-,b}(\Lambda\text{ proj})$ we can take $I_{X,Y} = [m, l]$ a finite interval such that $X, Y \in \mathcal{L}_{I_{X,Y}}$, with $Y^l = 0$.

Proposition 6.3. *Let X, Y be indecomposable complexes which are not $\mathcal{E}$-projective, and let $f : X \to Y$ be an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda\text{ proj})$.*

Suppose Y is a non-perfect complex and $q : Z \to \nu Y[-1]$ is a quasi-isomorphism with Z a complex in $\mathbf{C}^{\leq n}(\Lambda\text{ proj})$ without $\mathcal{E}$-projective direct summands, then for $I = I_{X,Y}$:

(1) *For all interval L containing I, X_L is a direct summand of Z_L.*
(2) *If Z_L is indecomposable for all interval L containing I then $X \cong Z$. In particular $Z \in \mathbf{C}^{-,b}(\Lambda\text{ proj})$.*
(3) *If $Z \in \mathbf{C}^{-,b}(\Lambda\text{ proj})$, then $X \cong Z$.*

Proof. (1) Since Y is indecomposable which is not $\mathcal{E}$-projective in the category $\mathbf{C}^{-,b}(\Lambda\text{ proj})$, then by (2) of 3.5 for all interval $L = [a, n]$ containing I, the complex Y_L is indecomposable. Moreover, the Nakayama functor ν induces an equivalence $\nu : \mathbf{C}_L(\Lambda\text{ proj}) \to \mathbf{C}_L(\Lambda\text{ inj})$ and $\nu(Y)_L = \nu(Y_L)$ is indecomposable.

Consider the intervals $I_1 = [a-2, n], I_0 = (-\infty, n]$. By Theorem 3.5, f_L is an irreducible morphism for all interval L containing I, in particular, f_{I_1}, f_{I_0} are irreducible morphisms between indecomposable complexes. Take the almost split sequence in $\mathbf{C}_{I_1}(\Lambda\text{ proj})$ (see [2]):

$$(a) \quad 0 \to A_{I_1}(Y_{I_1}) \xrightarrow{u} E \xrightarrow{v} Y_{I_1} \to 0.$$

We have the inclusion $\sigma : Y_L \to Y_{I_1}$, this morphism is not a retraction, so there is a morphism $\lambda : Y_L \to Y_{I_1}$ such that $\upsilon\lambda = \sigma$. But this implies that the restriction of the above sequence to L splits.

Here $Y^n = 0$, then $A_{I_1}(Y_{I_1})$ coincides with Z'_{I_1}, where Z' is a complex without $\mathcal{E}$-projective direct summands in $\mathbf{C}^{\leq n}(\Lambda \text{ proj})$ which is quasi-isomorphic to $\nu Y_{I_1}[-1] = (\nu Y[-1])_{[a-1,n]}$ (see Proposition 6.12 in [2]).

By Proposition 6.2, if Z is a complex without $\mathcal{E}$-projective direct summands in $\mathbf{C}^{\leq n}(\Lambda \text{ proj})$ quasi-isomorphic to $\nu Y[-1]$ then

$$(Z_{[a-1,n]})_0 \cong (Z'_{[a-1,n]})_0.$$

Note that for any complex C, $C_L = (C_{[a-1,n]})_L$. Moreover $C_{[a-1,n]} = (C_{[a-1,n]})_0 \oplus T$, with T an $\mathcal{E}_{[a-1,n]}$-injective. Clearly T_L is an $\mathcal{E}_L$-injective, so

$$(*) \quad (C_L)_0 \cong [((C_{[a-1,n]})_0)_L]_0.$$

Using the isomorphism $(*)$, we obtain the following isomorphisms of complexes:

$$(Z_L)_0 \cong [((Z_{[a-1,n]})_0)_L]_0 \cong [((Z'_{[a-1,n]})_0)_L]_0 \cong (Z'_L)_0.$$

Here $f_{I_1} : X_{I_1} \to Y_{I_1}$ is an irreducible morphism, then there exists a complex isomorphism $E \cong E' \oplus X_{I_1}$. But the sequence (a) restricted to L splits, then $(Z'_{I_1})_L \oplus Y_L \cong E'_L \oplus X_L$. Note that X_L is not $\mathcal{E}_L$-injective because by Lemma 6.1 X is a non-perfect complex. Using Proposition 2.15 we know that X_L is not isomorphic to Y_L and then, by Krull–Schmidt Theorem, we conclude that X_L is a direct summand of $(Z'_{I_1})_L \cong Z'_L$, and then X_L is a direct summand of $(Z'_L)_0 \cong (Z_L)_0$ and we obtain our first statement.

(2) If Z_L is indecomposable for all interval L containing I, then $X_L \cong Z_L$, for all interval L containing I. In particular, for all $i \in \mathbb{Z}$, $H^i(X) \cong H^i(Z)$ and hence $Z \in \mathbf{C}^{-,b}(\Lambda \text{ proj})$.

(3) Suppose $Z \in \mathbf{C}^{-,b}(\Lambda \text{ proj})$. Take $L = [t, n]$ an interval containing I such that $H^i(X) = 0$ and $H^i(Z) = 0$ for i outside of $[t+1, n]$ and $Z \in \mathbf{C}^{\leq n}(\Lambda \text{ proj})$. Then X and Z are in $\mathcal{L}_L$, by (2) of Theorem 3.5, Z_L is indecomposable, so by (1) we have $X_L \cong Z_L$. Then, from (d) of 1.2 we deduce that $X \cong Z$. □

As a consequence of the above we obtain the following result in case of Gorenstein Artin algebras. This was proved in Proposition 5.5 of [7] for finite dimensional algebras over a field.

Corollary 6.4. *If Λ is a Gorenstein Artin algebra, and there is an irreducible morphism in $\mathbf{C}^{-,b}(\Lambda \text{ proj})$, $f : X \to Y$ with X, Y indecomposable, Y a non-perfect complex, then X is quasi-isomorphic to $\nu Y[-1]$.*

Proof. In case Λ is Gorenstein, if $Y \in \mathbf{C}^{-,b}(\Lambda \text{ proj})$ then $\nu(Y) = D(\Lambda) \otimes_\Lambda Y$ is a complex quasi-isomorphic to a complex $Z \in \mathbf{C}^{-,b}(\Lambda \text{ proj})$, then we apply Proposition 6.3. □

Proposition 6.5. *If $f : X \to Y$ is an irreducible morphism between indecomposable complexes in $\mathbf{K}^{-,b}(\Lambda \text{ proj})$ with Y a perfect complex and X a non-perfect complex then $H^j(\nu X) \neq 0$ for infinitely many integers $j \in \mathbb{Z}$, or $\nu(X) \cong \nu(Y)$ in $\mathbf{D}^b(\Lambda \text{ mod})$.*

Proof. Since Y is a perfect complex, there is an Auslander–Reiten triangle in $\mathbf{K}^{-,b}(\Lambda \text{ proj})$:

$$W \to E \to Y \to W[1],$$

so X is a direct summand of E and then there is an irreducible morphism $u : W \to X$.

Now, suppose the integers $j \in \mathbb{Z}$ such that $H^j(\nu X) \neq 0$ form a finite set. Take $Z \in \mathbf{C}^{\leq n}(\Lambda \text{ proj})$ an indecomposable complex quasi-isomorphic to νX for some n. Then by our hypothesis $Z \in \mathbf{C}^{-,b}(\Lambda \text{ proj})$. Therefore by (3) of Proposition 6.3, $W \simeq Z[-1]$, so $W \simeq \nu X[-1]$ in the bounded derived category. But $W \simeq \nu Y[-1]$ in $\mathbf{D}^b(\Lambda \text{ mod})$ (see [5]), then $\nu(X) \cong \nu(Y)$ in $\mathbf{D}^b(\Lambda \text{ mod})$. □

Acknowledgement

The authors are grateful to the referee for many useful suggestions and corrections.

References

[1] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Cambridge University Press, Cambridge, 1995.
[2] R. Bautista, M.J. Souto Salorio, R. Zuazua, Almost split sequences for complexes of fixed size, J. Algebra 287 (2005) 140–168.
[3] P. Gabriel, A.V. Roiter, Representations of finite dimensional algebras, in: A.I. Kostrikin, I.V. Shafarevich (Eds.), Encyclopaedia of the Mathematical Sciences, in: Algebra VIII, vol. 73, Springer, 1992.
[4] H. Giraldo, H. Merklen, Irreducible morphisms of categories of complexes, J. Algebra 321 (2009) 2716–2736.
[5] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, in: London Mathematical Society, in: Lecture Note Series, vol. 119, 1988.
[6] D. Happel, On the derived category of a finite-dimensional algebra, Comment. Math. Helv. 62 (1987) 339–389.
[7] D. Happel, B. Keller, I. Reiten, Bounded derived categories and repetitive algebras, J. Algebra 319 (2008) 1611–1635.
[8] H. Krause, Auslander–Reiten theory via Brown representability, K-theory 20 (2000) 331–344.
[9] H. Krause, J. Le, Auslander–Reiten formula for complexes of modules, Adv. Math. 207 (2006) 133–148.
[10] D. Quillen, Higher Algebraic K-Theory I, in: Lecture Notes in Math., vol. 341, Springer, Berlin, 1973, pp. 85–147.
[11] S. Scherotzke, Finite and bounded Auslander–Reiten components in the Derived Category. arXiv:0902.0518v1 (2009).
[12] W. Wheeler, The triangulated structure of the stable derived category, J. Algebra 165 (1994) 23–40.
[13] J. Xiao, B. Zhu, Locally finite triangulated categories, J. Algebra 290 (2005) 473–490.