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Some Existence Theorems for nth-Order Boundary Value Problems*

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1. INTRODUCTION

A number of authors have employed variations of the Ważewski retract method (see [12]) to study second-order boundary value problems. In the area of ordinary differential equations, these include Hukuhara [6, 7], Jackson and Klaasen [8], and Kaplan, Lasota, and Yorke [9]. Bebernes and Kelley [1] have extended the results of Jackson and Klaasen to contingent equations.

In this paper, we use a variation of the Ważewski method similar to those developed by Kluczny [11] and Hukuhara [6] to prove existence theorems for some nth-order boundary value problems, where $n \geqslant 3$. Section 2 contains the basic topological results. In Section 3, we show that an existence theorem of Klaasen [10] extends to nth-order equations under slightly weaker hypotheses. The results of Sections 2 and 3 are brought together in Section 4 to produce existence theorems for a class of boundary value problems. We conclude the paper with an example of a third-order equation from boundary-layer theory. Coppel [2] has also used a topological argument to prove existence for a particular boundary value problem associated with this equation.

2. TOPOLOGICAL PRELIMINARIES

Let $\mathbf{R}^n$ denote n-dimensional Euclidean space and let V be an open set in $\mathbf{R}^1 \times \mathbf{R}^n$. We shall consider the ordinary differential equation

$$x' = f(t, x), \tag{1}$$

where f is a continuous function on V with values in $\mathbf{R}^n$. Let W be a subset of V and ∂W be the topological boundary of W.

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For $P \in W$, the *right zone of emission from* P is the set $E^+(P) = \{(\tau, x(\tau)): \exists \sigma \leqslant \tau$ such that $P = (\sigma, x(\sigma))$, $x(t)$ is a solution of (1) on $[\sigma, \tau]$ and $(t, x(t)) \in W$ for $t \in [\sigma, \tau]\}$. If $A \subset W$, we define $E^+(A) = \bigcup_{P \in A} E^+(P)$. The *right trace of emission from* A is defined to be the set $T^+(A) = E^+(A) \cap \partial W$.

The following criterion for compactness of the right zone of emission will be useful.

LEMMA 2.1. *Suppose A is a compact subset of W and W is a closed subset of V. Then $E^+(A)$ is compact if and only if there is no solution $x(t)$ of (1) emanating from A with $(t, x(t)) \in W$ on its right maximal interval of existence.*

Proof. If there is such a solution $x(t)$, then $E^+(A)$ is not bounded, so it is not compact.

Suppose there is no such solution emanating from A. Let $\langle P_k \rangle_{k=1}^{\infty}$ be a sequence of points in $E^+(A)$. For each k, let $x_k(t)$ be a solution of (1) and let σ_k and τ_k be real numbers such that $(\sigma_k, x_k(\sigma_k)) \in A$, $P_k = (\tau_k, x_k(\tau_k))$ and $(t, x_k(t)) \in W$ for $t \in [\sigma_k, \tau_k]$. Furthermore, suppose each $x_k(t)$ is defined on its maximal interval of existence (α_k, ω_k). Since A is compact, we may assume $\sigma_k \to \sigma$, $x_k(\sigma_k) \to x_0$ as $k \to \infty$, and we have $(\sigma, x_0) \in A$.

By Theorem 3.2 of [4, p. 14] there is a solution $x(t)$ of (1) defined on its maximal interval of existence (α, ω) with $x(\sigma) = x_0$ and a subsequence which we again name $\langle x_k(t) \rangle_{k=1}^{\infty}$ which converges uniformly to $x(t)$ on compact subintervals of (α, ω). Choose $t_0 \in (\alpha, \omega)$ so that $(t_0, x(t_0)) \notin W$. For k sufficiently large, $(t_0, x_k(t_0)) \notin W$ and thus $\sigma_k \leqslant \tau_k < t_0$. Choose $\epsilon > 0$ so that $\alpha < \sigma - \epsilon$. Then $\sigma - \epsilon < \tau_k < t_0$ and $x_k(t) \to x(t)$ uniformly on $[\sigma - \epsilon, t_0]$ for k sufficiently large, so $\langle P_k \rangle_{k=1}^{\infty}$ is bounded.

Thus some subsequence of $\langle P_k \rangle_{k=1}^{\infty}$ converges to a point P in W. By the uniform convergence, P lies on the graph of $x(t)$, so $P \in E^+(A)$. Hence $E^+(A)$ is compact. Q.E.D.

The assumption that W is closed can be weakened. See Theorem 4.1 of [11] for details.

Next, we generalize the idea of a strict egress point in the fashion of Hukuhara [6]. Define

$$S = \{P = (t_0, x_0) \in \partial W: \exists \epsilon > 0 \text{ such that, for } 0 < \tau - t_0 < \epsilon,$$
$$E^+(P) \cap (\{(\tau, x): x \in \mathbf{R}^n\} \cup (\partial W \cap \{(t, x): t \leqslant \tau, x \in \mathbf{R}^n\})) \text{ is connected}\}.$$

The points of S are the strict egress points of W relative to Eq. (1).

The following lemma is very much the same as Proposition 4.2 of [6], and we give only an indication of the proof.

LEMMA 2.2. *Suppose A is a connected subset of W and $E^+(A)$ is compact. If $T^+(A) \subset S$, then $T^+(A)$ is connected.*

Proof. Suppose $T^+(A)$ is not connected. Then we can write $T^+(A) = T_1 \cup T_2$, where T_1 and T_2 are nonempty and compact, and $T_1 \cap T_2 = \varnothing$. Define, for $i = 1, 2$,

$$E_i = \{Q \in E^+(A) : E^+(Q) \cap T_i \neq \varnothing\}.$$

Then each E_i is a nonempty closed subset of $E^+(A)$ and $E_1 \cup E_2 = E^+(A)$. Since A is connected, $E^+(A)$ is connected, so $E_1 \cap E_2$ is nonempty and compact. Let $P = (t_0, x_0)$ belong to $E_1 \cap E_2$ and have the property that no point in $E_1 \cap E_2$ has t component greater than t_0.

Suppose $P \in \partial W$. We can assume $P \in T_1$. There is an $\epsilon > 0$ such that for $0 < \tau - t_0 < \epsilon$,

$$F_\tau \equiv E^+(P) \cap (\{(\tau, x) : x \in \mathbf{R}^n\} \cup (\partial W \cap \{(t, x) : t \leqslant \tau, x \in \mathbf{R}^n\}))$$

is connected. Fix $\tau \in (t_0, t_0 + \epsilon)$ and define, for $i = 1, 2$,

$$F_{\tau i} = \{Q \in F_\tau : E^+(Q) \cap T_i \neq \varnothing\}.$$

These sets are closed and $F_{\tau 1} \cup F_{\tau 2} = F_\tau$. Furthermore, $F_{\tau 2} - P$ is closed and nonempty since P is not a limit point of $F_{\tau 2}$. Since $F_{\tau 1} \cup (F_{\tau 2} - P) = F_\tau$ is connected, $F_{\tau 1} \cap (F_{\tau 2} - P) \neq \varnothing$. This result implies that there is a point in $E_1 \cap E_2$ with t component greater than t_0 and gives a contradiction.

If P is in the interior of W, one can arrive at a contradiction in a similar manner using the Kneser Theorem for ordinary differential equations. It follows that $T^+(A)$ is connected. Q.E.D.

3. A Boundary Value Problem for nth-Order Equations

We shall now consider the nth-order differential equation

$$y^{(n)} = f(t, y, \ldots, y^{(n-1)}), \tag{2}$$

where $n \geqslant 3$ and f is continuous from $\mathbf{R}^1 \times \mathbf{R}^n$ to $\mathbf{R}^1$. Let $\beta^0, \beta^1, \ldots, \beta^{n-2}$, δ^{n-2} be real constants. The boundary value problem to be considered in this section is

$$\begin{gathered} y^{(n)} = f(t, y, \ldots, y^{(n-1)}), \\ y^{(i)}(a) = \beta^i \quad (i = 0, \ldots, n-2), \qquad y^{(n-2)}(b) = \delta^{n-2}. \end{gathered} \tag{3}$$

The following lemma follows in a routine way from the Schauder fixed-point theorem.

Lemma 3.1. *If f is bounded on $[a, b] \times \mathbf{R}^n$, then* (3) *has a solution.*

It will be necessary in order to prove existence of solutions to various boundary value problems to add an assumption governing the behavior of the $(n - 1)$st derivative of solutions of (2). Let I be an interval of real numbers. We have the following hypothesis.

$H(I)$: Let $y(t)$ be a solution of (2) with maximal interval of existence J with respect to $I \times \mathbf{R}^n$. If $y^{(n-2)}(t)$ is bounded on J, then $y^{(n-1)}(t)$ is bounded, on J.

LEMMA 3.2. *Let Z be a closed subset of $\{a\} \times \mathbf{R}^n$ and suppose $H([a, b])$ is satisfied. If $\langle y_k \rangle_{k=1}^{\infty}$ is a sequence of solutions of $y^{(n)} = f_k(t, y, \ldots, y^{(n-1)})$ with initial points in Z such that $\langle y_k^{(i)} \rangle_{k=1}^{\infty}$ is uniformly bounded on $[a, b]$ for $i = 0, \ldots, n - 2$, where the f_k's are continuous functions on $[a, b] \times \mathbf{R}^n$ which converge uniformly to f on compact subsets, then there is a solution $y(t)$ of (2) on $[a, b]$ and a subsequence of $\langle y_k \rangle_{k=1}^{\infty}$ which converges to y uniformly on $[a, b]$.*

Proof. There is an M such that $| y_k^{(n-2)}(t) | \leqslant M$ for each k and all $t \in [a, b]$. Then

$$| y_k^{(n-1)}(t_k)| = \frac{| y_k^{(n-2)}(b) - y_k^{(n-2)}(a) |}{b - a} \leqslant \frac{2M}{b - a},$$

for some sequence $\langle t_k \rangle_{k=1}^{\infty}$ of points in $[a, b]$.

The sequences $\langle t_k \rangle_{k=1}^{\infty}, \langle y_k(t_k) \rangle_{k=1}^{\infty}, \ldots, \langle y_k^{(n-1)}(t_k) \rangle_{k=1}^{\infty}$ are bounded sequences of real numbers. Let $\langle k_i \rangle_{i=1}^{\infty}$ be a subsequence of $\langle k \rangle_{k=1}^{\infty}$ such that each of the above sequences converges.

By the convergence theorem used in the proof of Lemma 2.1, there is a subsequence of $\langle y_{k_i} \rangle_{i=1}^{\infty}$ which converges uniformly to a solution $y(t)$ of (2) on compact subintervals of the maximal interval of existence of $y(t)$. Hypothesis $H([a, b])$ and the uniform boundedness of $\langle y_k^{(i)} \rangle_{k=1}^{\infty}$ for $i = 0, \ldots, n - 2$ imply that $y(t)$ is defined on $[a, b]$ and that the subsequence converges uniformly to $y(t)$ on $[a, b]$. Q.E.D.

Let $\psi, \phi \in C^n([a, b])$ with $\psi^{(i)}(t) \geqslant \phi^{(i)}(t)$ for $i = 0, \ldots, n - 2$ and all $t \in [a, b]$ and

$$\psi^{(n)}(t) \leqslant f(t, y, \ldots, y^{(n-3)}, \psi^{(n-2)}(t), \psi^{(n-1)}(t)), \tag{4}$$

$$\phi^{(n)}(t) \geqslant f(t, y, \ldots, y^{(n-3)}, \phi^{(n-2)}(t), \phi^{(n-1)}(t)), \tag{5}$$

for $\phi^{(i)}(t) \leqslant y^{(i)} \leqslant \psi^{(i)}(t)$ $(i = 0, \ldots, n - 3)$ and $t \in [a, b]$.

Next, we make some modifications of f. For $i = 0, \ldots, n - 3$ define the variables

$$\bar{y}^{(i)}(t) = \begin{cases} \psi^{(i)}(t) & \text{for } \; y^{(i)} \geqslant \psi^{(i)}(t), \\ y^{(i)} & \text{for } \; \psi^{(i)}(t) > y^{(i)} > \phi^{(i)}(t), \\ \phi^{(i)}(t) & \text{for } \; \phi^{(i)}(t) \geqslant y^{(i)} \qquad (t \in [a, b]). \end{cases}$$

Then we define

$$F(t, y,..., y^{(n-1)}) = \begin{cases} f(t, \bar{y}(t),..., \bar{y}^{(n-3)}(t), \psi^{(n-2)}(t), y^{(n-1)}) + \dfrac{y^{(n-2)} - \psi^{(n-2)}(t)}{1 + y^{(n-2)} - \psi^{(n-2)}(t)} & \text{for } y^{(n-2)} \geqslant \psi^{(n-2)}(t), \\ f(t, \bar{y}(t),..., \bar{y}^{(n-3)}(t), y^{(n-2)}, y^{(n-1)}) & \text{for } \psi^{(n-2)}(t) > y^{(n-2)} > \phi^{(n-2)}(t), \\ f(t, \bar{y}(t),..., \bar{y}^{(n-3)}(t), \phi^{(n-2)}(t), y^{(n-1)}) + \dfrac{y^{(n-2)} - \phi^{(n-2)}(t)}{1 + \phi^{(n-2)}(t) - y^{(n-2)}} & \text{for } \phi^{(n-2)}(t) \geqslant y^{(n-2)}. \end{cases}$$

Finally, let

$$G_j(t, y,..., y^{(n-1)}) = \begin{cases} F(t, y,..., y^{(n-2)}, j) & \text{for } y^{(n-1)} \geqslant j, \\ F(t, y,..., y^{(n-2)}, y^{(n-1)}) & \text{for } j > y^{(n-1)} > -j, \\ F(t, y,..., y^{(n-2)}, -j) & \text{for } -j \geqslant y^{(n-1)}, \end{cases}$$

for integers j satisfying $j \geqslant \max_{t\in[a\ b]} \{\max\{|\phi^{(n-1)}(t)|, |\psi^{(n-1)}(t)|\}\}$.

We now consider the differential equations

$$y^{(n)} = F(t, y,..., y^{(n-1)}), \tag{6}$$

$$y^{(n)} = G_j(t, y,..., y^{(n-1)}). \tag*{(7)$_j$}$$

LEMMA 3.3. *Suppose $\psi(t)$ and $\phi(t)$ are as described above and suppose $y(t)$ is a solution of* (6) *or* (7)$_j$ *on some interval $I \subset [a, b]$. Let $\sigma_1, \sigma_2 \in I$ with $\sigma_2 > \sigma_1$ and suppose $\phi^{(n-2)}(\sigma_1) \leqslant y^{(n-2)}(\sigma_1) \leqslant \psi^{(n-2)}(\sigma_1)$. If $y^{(n-2)}(\sigma_2) > \psi^{(n-2)}(\sigma_2)$, then $y^{(n-2)}(t) > \psi^{(n-2)}(t)$ for $t \in I \cap [\sigma_2, b]$, and if $y^{(n-2}(\sigma_2) < \phi^{(n-2}(\sigma_2)$, then $y^{(n-2)}(t) < \phi^{(n-2)}(t)$ for $t \in I \cap [\sigma_2, b]$.*

Proof. Let $y(t)$ be a solution of (6). If it is a solution of (7)$_j$ for some j, the proof is similar.

Consider the case $y^{(n-2)}(\sigma_2) > \psi^{(n-2)}(\sigma_2)$. Suppose there exists a $t \in I \cap [\sigma_2, b]$ such that $y^{(n-2)}(t) = \psi^{(n-2)}(t)$. Then $y^{(n-2)} - \psi^{(n-2)}$ has a positive maximum at some $t_0 \in (\sigma_1, t)$. We have $y^{(n-1)}(t_0) - \psi^{(n-1)}(t_0) = 0$, $y^{(n-2)}(t_0) - \psi^{(n-2)}(t_0) > 0$, and $y^{(n)}(t_0) - \psi^{(n)}(t_0) \leqslant 0$. But, by (4),

$$\begin{aligned} y^{(n)}(t_0) - \psi^{(n)}(t_0) &= F(t_0, ..., y^{(n-1)}(t_0)) - \psi^{(n)}(t_0) \\ &= f(t_0, \bar{y}(t_0),..., \bar{y}^{(n-3)}(t_0), \psi^{(n-2)}(t_0), \psi^{(n-1)}(t_0)) - \psi^{(n)}(t_0) \\ &\quad + \frac{y^{(n-2)}(t_0) - \psi^{(n-2)}(t_0)}{1 + y^{(n-2)}(t_0) - \psi^{(n-2)}(t_0)} \\ &\geqslant \frac{y^{(n-2)}(t_0) - \psi^{(n-2)}(t_0)}{1 + y^{(n-2)}(t_0) - \psi^{(n-2)}(t_0)} > 0. \end{aligned}$$

This contradiction proves the lemma for the case considered. The case $y^{(n-2)}(\sigma_2) < \phi^{(n-2)}(\sigma_2)$ is similar. Q.E.D.

The next theorem is a generalization of Theorem 7 in [10].

THEOREM 3.1. *Let $\psi(t)$ and $\phi(t)$ be as given above and let $\beta^0,\ldots,\beta^{n-2}$, δ^{n-2} be real numbers satisfying $\phi^{(i)}(a) \leqslant \beta^{(i)} \leqslant \psi^{(i)}(a)$ for $i = 0,\ldots, n-2$ and $\phi^{(n-2)}(b) \leqslant \delta^{n-2} \leqslant \psi^{(n-2)}(b)$. Assume hypothesis $H([a, b])$ is satisfied. Then* (3) *has a solution $y(t)$ such that $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [a, b]$ and $i = 0,\ldots, n-2$.*

Proof. Since $G_j(t, y,\ldots, y^{(n-1)})$ is continuous and bounded on $[a, b] \times \mathbf{R}^n$, the boundary value problem consisting of (7)j and the boundary conditions in (3) has a solution $y_j(t)$ for each value of j by Lemma 3.1. Moreover, by Lemma 3.3 $\phi^{(n-2)}(t) \leqslant y_j^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ for $t \in [a, b]$. Since $\phi^{(i)}(a) \leqslant \beta^i \leqslant \psi^{(i)}(a)$ for $i = 0,\ldots, n-3$, successive integrations yield $\phi^{(i)}(t) \leqslant y_j^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $i = 0,\ldots, n-3$.

Now $\langle G_j \rangle$ converges uniformly to F on compact subsets of $[a, b] \times \mathbf{R}^n$ and F coincides with f for $\phi^{(i)}(t) \leqslant y^{(i)} \leqslant \psi^{(i)}(t)$ and $t \in [a, b]$ $(i = 0,\ldots, n-2)$. By Lemma 3.2, a subsequence of $\langle y_j \rangle$ converges uniformly to a solution $y(t)$ of (3) on $[a, b]$, and $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [a, b]$ and $i = 0,\ldots, n-2$. Q.E.D.

4. A Class of Boundary Value Problems

We can use the results of Sections 2 and 3 to analyze other types of boundary value problems for (2) with slightly stronger assumptions about ψ, ϕ, and f. Let $\psi, \phi \in C^n(\mathbf{R}^1)$ with $\psi^{(i)}(t) \geqslant \phi^{(i)}(t)$ for $i = 0,\ldots, n-3$, $\psi^{(n-2)}(t) > \phi^{(n-2)}(t)$ and

$$\psi^{(n)}(t) \leqslant f(t, \psi(t),\ldots, \psi^{(n-1)}(t)), \tag{8}$$

$$\phi^{(n)}(t) \geqslant f(t, \phi(t),\ldots, \phi^{(n-1)}(t)), \tag{9}$$

for $t \in \mathbf{R}^1$. Furthermore, we assume f is nonincreasing in $y^{(i)}$ for $\phi^{(i)}(t) \leqslant y^{(i)} \leqslant \psi^{(i)}(t)$ and fixed values of $t \in \mathbf{R}^1$, $\phi^{(k)}(t) \leqslant y^{(k)} \leqslant \psi^{(k)}(t)$, where $k = 0,\ldots, n-2$, $k \neq i$, and $y^{(n-1)} \in \mathbf{R}^1$ $(i = 0,\ldots, n-3)$. Note that (8), (9) and the additional assumption on f imply that ψ and ϕ satisfy (4) and (5), respectively.

Now (2) can be thought of as a special case of (1), and we shall use the terminology of Section 2, where $V = \mathbf{R}^1 \times \mathbf{R}^n$ and $W = \{(t, y,\ldots, y^{(n-1)})$: $\phi^{(n-2)}(t) \leqslant y^{(n-2)} \leqslant \psi^{(n-2)}(t)$ and t, $y,\ldots, y^{(n-3)}, y^{(n-1)}$ any real numbers$\}$. It will be useful to single out two subsets of ∂W. Let B_1 be the set of points in the boundary $\{y^{(n-2)} = \psi^{(n-2)}(t)\}$ with $y^{(n-1)} \geqslant \psi^{(n-1)}(t)$ and let B_2 be the set of points in $\{y^{(n-2)} = \phi^{(n-2)}(t)\}$ with $y^{(n-1)} \leqslant \phi^{(n-1)}(t)$. Clearly, if $Q \in (\text{interior } W) \cup B_1 \cup B_2$, then $T^+(Q) \subset B_1 \cup B_2$. The following result will allow us to apply Lemma 2.2 below.

LEMMA 4.1. *Suppose there is an* $\epsilon_1 > 0$ *so that* $H([c, d])$ *is satisfied whenever* $d - c < \epsilon_1$. *If* $P = (\gamma, \alpha^0,..., \alpha^{n-1}) \in B_1 \cup B_2$ *and* $\phi^{(i)}(\gamma) \leqslant \alpha^i \leqslant \psi^{(i)}(\gamma)$ *for* $i = 0,..., n-3$, *then* $P \in S$.

Proof. We shall consider only the case $P \in B_1$, for the case $P \in B_2$ is handled in the same way.

Suppose $\alpha^{n-1} > \psi^{(n-1)}(\gamma)$. If $y(t)$ is a solution of (2) passing through P, then $y^{(n-1)}(\gamma) > \psi^{(n-1)}(\gamma)$. Since $y^{(n-2)}(\gamma) = \psi^{(n-2)}(\gamma)$, it is clear that $y^{(n-2)}(t) > \psi^{(n-2)}(t)$ in some interval (γ, γ'), $\gamma' > \gamma$. Then, for any $\tau > \gamma$, $E^+(P) \cap (\{(\tau, x): x \in \mathbf{R}^n\} \cup (\partial W \cap \{(t, x): t \leqslant \tau, x \in \mathbf{R}^n\}))$ consists of the single point P and thus is connected.

The other possibility is $\alpha^{n-1} = \psi^{(n-1)}(\gamma)$. Choose $\tau - \gamma > 0$ sufficiently small that all solutions of (6) through P exist on $[\gamma, \tau]$ and do not intersect B_2. Let

$$\Sigma_P = \{Y(t) = (y(t),..., y^{(n-1)}(t)) \colon y \text{ is a solution of (6) on } [\gamma, \tau] \text{ through } P\}.$$

By the generalized Kneser Theorem for ordinary differential equations (see [5]), Σ_P is a compact connected subset of the Banach space of continuous functions on $[\gamma, \tau]$.

Define $K = E^+(P) \cap (\{(\tau, x): x \in \mathbf{R}^n\} \cup (\partial W \cap \{(t, x): t \leqslant \tau, x \in \mathbf{R}^n\}))$. K is closed and bounded, so it is compact. If K is not connected, then $K = K_1 \cup K_2$, where $K_1 \cap K_2 = \varnothing$ and K_1 and K_2 are compact and nonempty. For $i = 1, 2$, let

$$\Sigma_i = \{Y \in \Sigma_P \colon (t_0, Y(t_0)) \in K_i, \text{ where } t_0 = \sup\{\gamma \leqslant t \leqslant \tau \colon (t, Y(t)) \in W\}\}.$$

Clearly, $\Sigma_1 \cap \Sigma_2 = \varnothing$. By Lemma 3.3, if $Y \in \Sigma_P$ and $(t_0, Y(t_0)) \in W$ for some $t_0 \geqslant \gamma$, then $(t, Y(t)) \in W$ for $\gamma \leqslant t \leqslant t_0$. Thus $\Sigma_1 \cup \Sigma_2 = \Sigma_P$. Next, we prove Σ_1 is closed.

Let $\langle Y_k(t) \rangle_{k=1}^{\infty}$ be a sequence in Σ_1 such that $Y_k(t) \to Y(t) \in \Sigma_P$ uniformly for $t \in [\gamma, \tau]$. For each k, there is a $t_k \in [\gamma, \tau]$ such that $(t_k, Y_k(t_k)) \in K_1$ and either $t_k = \tau$ or $(t, Y_k(t)) \notin W$ for $t > t_k$. Taking a subsequence, if necessary, suppose $t_k \to t_0$. Then $(t_k, Y_k(t_k)) \to (t_0, Y(t_0)) \in K_1$. If $t_0 = \tau$, then $(\tau, Y(\tau)) \in K_1$ and $Y \in \Sigma_1$. If $t_0 < \tau$, then for k sufficiently large $t_k < \tau$ and $(t, Y_k(t)) \notin W$ for $t \in (t_k, \tau]$. Thus $(t, Y(t))$ cannot belong to the interior of W for any $t \in [t_0, \tau]$. Suppose $Y \in \Sigma_2$. Then $\exists t_1 \in (t_0, \tau]$ so that $(t_1, Y(t_1)) \in K_2$. Now $(t_0, Y(t_0)) \in K_1$ and $(t, Y(t)) \in K_1 \cup K_2$ for $t_0 \leqslant t \leqslant t_1$. This is clearly impossible since the distance between K_1 and K_2 is positive. Since $Y \notin \Sigma_2$, $Y \in \Sigma_1$. It follows that Σ_1 is closed in Σ_P. Similarly, Σ_2 is closed.

Finally, we show that Σ_1 and Σ_2 are nonempty. Suppose Σ_1 is empty. Then no point in K_1 has t component τ. Let $Q = (\lambda, \beta^0,..., \beta^{n-1})$ be a point in K_1 with the largest possible t component λ. Now $Q \in B_1$, so $\beta^{n-1} \geqslant \psi^{(n-1)}(\lambda)$. If $\beta^{n-1} > \psi^{(n-1)}(\lambda)$, then any $Y = (y,..., y^{(n-1)}) \in \Sigma_P$ with $(\lambda, Y(\lambda)) = Q$ satisfies $y^{(n-1)}(\lambda) > \psi^{(n-1)}(\lambda)$ and $y^{(n-2)}(\lambda) = \psi^{(n-2)}(\lambda)$, so $y^{(n-2)}(t) > \psi^{(n-2)}(t)$ for t in some open interval with left endpoint λ. By Lemma 3.3, $y^{(n-2)}(t) > \psi^{(n-2)}(t)$ for $t \in (\lambda, \tau]$. Thus $Y \in \Sigma_1$, so in this case $\Sigma_1 \neq \varnothing$, and we can assume that $\beta^{n-1} = \psi^{(n-1)}(\lambda)$. Also, $\beta^{n-2} = \psi^{(n-2)}(\lambda)$. By definition of $E^+(P)$, $Q \in K$ implies Q is reached by some $Y = (y,..., y^{(n-1)}) \in \Sigma_P$ such that $(t, Y(t)) \in W$ for $\gamma \leqslant t \leqslant \lambda$. Thus $y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ for $t \in [\gamma, \lambda]$. Since $\alpha^i \leqslant \psi^{(i)}(\gamma)$ for $i = 0,..., n-3$, successive integrations yield $\beta^i \leqslant \psi^{(i)}(\lambda)$ for $i = 0,..., n-3$.

Let y be a solution of (6) which emanates from Q. If there is no $\lambda' \in (\lambda, \tau]$ such that $y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ on $[\lambda, \lambda']$, then $y^{(n-2)}(t) > \psi^{(n-2)}(t)$ on $(\lambda, \tau]$ by Lemma 3.3, contradicting $\Sigma_1 = \varnothing$. Thus $y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ on some interval $[\lambda, \lambda']$. Fix an $\epsilon > 0$ which is smaller than ϵ_1, $\lambda' - \lambda$ and the distance from K_1 to K_2.

For $t \in [\lambda, \lambda + \epsilon]$, $\psi(t)$ satisfies (4) and $y(t)$ satisfies (5) because of the monotony of f and $\psi^{(i)}(t) \geqslant y^{(i)}(t)$ for $i = 0,..., n-2$. Therefore, by Theorem 3.1, there is a solution $z(t)$ of (2) such that $z^{(i)}(\lambda) = \beta^i$ for $i = 0,..., n-2$, $z^{(n-2)}(\lambda + \epsilon) = \psi^{(n-2)}(\lambda + \epsilon)$ and $y^{(i)}(t) \leqslant z^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [\lambda, \lambda + \epsilon]$ and $i = 0,..., n-2$. Since $y^{(n-1)}(\lambda) = \psi^{(n-1)}(\lambda)$, it must be true that $z^{(n-1)}(\lambda) = \beta^{n-1}$, so $(\lambda, z(\lambda),..., z^{(n-1)}(\lambda)) = Q$. Now the point $(\lambda + \epsilon,..., z^{(n-1)}(\lambda + \epsilon))$ must belong to either K_1 or K_2, but it cannot be in K_1 by the assumption on Q and it cannot be in K_2 by the choice of ϵ.

This contradiction implies that $\Sigma_1 \neq \varnothing$, and a similar argument proves the assertion for Σ_2. Thus $\Sigma_P = \Sigma_1 \cup \Sigma_2$ is a separation, but we know Σ_P is connected. It must be true that K is connected. Q.E.D.

We can use the preceding lemma to obtain an additional result for the left-hand boundary set considered in Section 3.

THEOREM 4.1. *Let $\beta^0,..., \beta^{n-2}$ be real numbers satisfying $\phi^{(i)}(a) \leqslant \beta^{(i)} \leqslant \psi^{(i)}(a)$ for $i = 0,..., n-2$ and some $a \in \mathbf{R}^1$. Let $Z = \{(a, \beta^0,..., \beta^{n-2}, y^{(n-1)}): y^{(n-1)} \in \mathbf{R}^1\}$ and assume $H([c, d])$ is satisfied for all $a \leqslant c < d$. Then there is a solution $y(t)$ of (2) emanating from Z which exists for $t \in [a, \infty)$ and satisfies $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [a, \infty)$ and $i = 0,..., n-2$.*

Proof. Let $b > a$. By Theorem 3.1, there are solutions y_1 and y_2 of (2) emanating from Z such that $y_1^{(n-2)}(b) = \psi^{(n-2)}(b)$ and $y_2^{(n-2)}(b) = \phi^{(n-2)}(b)$ and $\phi^{(n-2)}(t) \leqslant y_i^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ for $t \in [a, b]$ and $i = 1, 2$. Let $Z^1 = \{(a, \beta^0,..., \beta^{n-2}, y^{(n-1)}): y^{(n-1)}$ belongs to the closed interval with endpoints $y_1^{(n-1)}(a)$ and $y_2^{(n-1)}(a)\}$. Now Z^1 is compact and connected, $T^+(Z^1)$ is not connected and $T^+(Z^1) \subset S$ by Lemma 4.1. Thus by Lemmas 2.1 and 2.2 there is a solution $y(t)$ emanating from Z^1 with $\phi^{(n-2)}(t) \leqslant y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ on its right maximal interval of existence. Since $H([a, d])$ is satisfied for all $d > a$, $y(t)$ exists for all $t \geqslant a$, and $y(t)$ satisfies $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \geqslant a$ and $i = 0,\dots, n-2$. Q.E.D.

We now consider boundary value problems with very general boundary sets. Let Z_1 be a compact connected subset of $\{(a, y,\dots, y^{(n-1)}: \phi^{(i)}(a) \leqslant y^{(i)} \leqslant \psi^{(i)}(a)$ $(i = 0,\dots, n-2)$ and $y^{(n-1)} \in \mathbf{R}^1\}$ which intersects the boundary $\{y^{(n-2)} = \psi^{(n-2)}(t)\}$ in a nonempty subset of B_1 and intersects $\{y^{(n-2)} = \phi^{(n-2)}(t)\}$ in a nonempty subset of B_2. The proof of the following theorem is like that of Theorem 4.1, except that Theorem 3.1 is not needed.

THEOREM 4.2. *Suppose $H([c, d])$ is satisfied for all $a \leqslant c < d$. Then there is a solution $y(t)$ of (2) emanating from Z_1 which exists for $t \in [a, \infty)$ and satisfies $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [a, \infty)$ and $i = 0,\dots, n-2$.*

Let $b > a$ be fixed. We shall now take W to be the set

$$W = \{(t, y,\dots, y^{(n-1)}): t \in (-\infty, b], \phi^{(n-2)}(t) \leqslant y^{(n-2)} \leqslant \psi^{(n-2)}(t) \text{ and } y,\dots, y^{(n-3)}, y^{(n-1)} \text{ any real numbers}\}.$$

If the sets B_1 and B_2 are adjusted in the obvious way, Lemma 4.1 remains true. Let $E = W \cap \{(b, y,\dots, y^{(n-1)})\}$ and let Z_2 be a subset of E such that there is a separation $E \sim Z_2 = E_1 \cup E_2$ with $B_1 \cap E \subset E_1$ and $B_2 \cap E \subset E_2$.

THEOREM 4.3. *Assuming hypothesis $H([c, d])$ is satisfied for $a \leqslant c < d \leqslant b$, there exists a solution $y(t)$ of (2) emanating from Z_1 and terminating in Z_2 which satisfies $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [a, b]$ and $i = 0,\dots, n-2$.*

Proof. Suppose there is no solution $y(t)$ of (2) from Z_1 to Z_2 which satisfies $\phi^{(n-2)}(t) \leqslant y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ for $t \in [a, b]$. Then $T^+(Z_1) \cap Z_2 = \varnothing$, and we have

$$T^+(Z_1) = [(T^+(Z_1) \cap B_1) \cup (T^+(Z_1) \cap E_1)] \cup [(T^+(Z_1) \cap B_2) \cup (T^+(Z_1) \cap E_2)]$$

is a separation of $T^+(Z_1)$.

Now $T^+(Z_1) \subset S$ by Lemma 4.1 and the fact that $E \subset S$. By Lemmas 2.1 and 2.2 there is a solution $y(t)$ of (2) emanating from Z_1 with $\phi^{(n-2)}(t) \leqslant y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ on its right maximal interval of existence with respect to $\mathbf{R}^1 \times \mathbf{R}^n$. By hypothesis $H([a, b])$, this situation is impossible.

Then there is a solution $y(t)$ of (2) from Z_1 to Z_2 with $\phi^{(n-2)}(t) \leqslant y^{(n-2)}(t) \leqslant \psi^{(n-2)}(t)$ for $t \in [a, b]$. It follows that $\phi^{(i)}(t) \leqslant y^{(i)}(t) \leqslant \psi^{(i)}(t)$ for $t \in [a, b]$ and $i = 0,\dots, n-2$. Q.E.D.

5. An Example

To illustrate how the above theorems can be applied, we consider the following equation from boundary-layer theory:

$$y''' = -yy'' + \lambda(y'^2 - 1) \qquad (\lambda \geqslant 0). \tag{10}$$

First, let us specify the boundary conditions $y(a) = \beta^0$, $y'(a) = \beta^1$, $y'(b) = \delta^1$, where $a, b, \beta^0, \beta^1, \delta^1$ are real numbers with $a < b$, $\beta^1 \geqslant -1$, and $\delta^1 \geqslant -1$. Let $C_1 \geqslant \max\{1, \beta^1, \delta^1\}$ and $-1 \leqslant C_2 \leqslant \min\{1, \beta^1, \delta^1\}$. Define $\psi(t) = C_1 t + \beta^0 - C_1 a$ and $\phi(t) = C_2 t + \beta^0 - C_2 a$. Then

$$\psi'''(t) = 0 \leqslant \lambda(C_1{}^2 - 1) = y\psi''(t) + \lambda(\psi'^2(t) - 1)$$

and

$$\phi'''(t) = 0 \geqslant \lambda(C_2{}^2 - 1) = -y\phi''(t) + \lambda(\phi'^2(t) - 1),$$

so ψ and ϕ satisfy (4) and (5), respectively, for

$$f(t, y, y', y'') = -yy'' + \lambda(y'^2 - 1).$$

Also, $\psi(a) = \phi(a) = \beta^0$, $\psi'(a) \geqslant \beta^1 \geqslant \phi'(a)$, and $\psi'(b) \geqslant \delta^1 \geqslant \phi'(b)$.

Next, we show that $H([a, b])$ is satisfied for (10). Let $y(t)$ be a solution of (10) with maximal interval of existence J with respect to $[a, b] \times \mathbf{R}^3$.

Suppose that $|y'(t)| \leqslant R$ for $t \in J$. Note that

$$|y'''(t)| \leqslant |y|\,|y''| + \lambda\,|y'^2 - 1| \leqslant Q\,|y''| + \lambda(R^2 + 1) \equiv \Phi(|y''|),$$

where Q is a positive number such that $Q > R(b - a) + |y(t_0)|$, where $t_0 \in J$. Now

$$\int^{\infty} \frac{s\,ds}{\Phi(s)} = \int^{\infty} \frac{s\,ds}{Qs + \lambda(R^2 + 1)} = \infty,$$

so by Lemma 5.1 of [4, p. 428], there is an M such that $|y''(t)| \leqslant M$ for $t \in J$.

By Theorem 3.1, there is a solution $y(t)$ of (10) such that $y(a) = \beta^0$, $y'(a) = \beta^1$, $y'(b) = \delta^1$, $C_2 t + \beta^0 - C_2 a \leqslant y(t) \leqslant C_1 t + \beta^0 - C_1 a$, and $C_2 \leqslant y'(t) \leqslant C_1$ for $t \in [a, b]$.

Now consider the following modification of (10):

$$y''' = -y\,|y''| + \lambda(y'^2 - 1) \qquad (\lambda \geqslant 0). \tag{11}$$

Let a, $\beta^0 \in \mathbf{R}^1$ and define $\psi(t) = t + \beta^0 - a$, $\phi(t) = \beta^0$. Then ψ and ϕ satisfy (8) and (9), respectively, and the right-hand side of (11) is non-increasing in y for fixed values of y'', y', and λ. We can apply Theorems 4.1, 4.2, and 4.3 to obtain the following results (a), (b), and (c), respectively.

(a) Suppose $0 \leqslant \beta^1 \leqslant 1$. There is a solution $y(t)$ of (11) which exists for all $t \geqslant a$ and satisfies $y(a) = \beta^0$, $y'(a) = \beta^1$, $\beta^0 \leqslant y(t) \leqslant t + \beta^0 - a$ and $0 \leqslant y'(t) \leqslant 1$ for all $t \geqslant a$.

(b) Suppose $C_3 > 0$ and $0 \leqslant C_4 \leqslant 1$. There is a solution $y(t)$ of (11) which exists for all $t \geqslant a$ and satisfies $y(a) = \beta^0$, $y'(a) - C_3 y''(a) = C_4$, $\beta^0 \leqslant y(t) \leqslant t + \beta^0 - a$ and $0 \leqslant y'(t) \leqslant 1$ for $t \geqslant a$.

(c) Suppose C_3 and C_4 are as above, $0 < \delta^1 < 1$ and $b > a$. There is a solution $y(t)$ of (11) such that $y(a) = \beta^0$, $y'(a) - C_3 y''(a) = C_4$, $y'(b) = \delta^1$, $\beta^0 \leqslant y(t) \leqslant t + \beta^0 - a$ and $0 \leqslant y'(t) \leqslant 1$ for $t \in [a, b]$.

Suppose $y(t)$ is a solution of (11) with $y(a) \geqslant 0$ and $0 \leqslant y'(t) \leqslant 1$ for all $t \geqslant a$. Assume $y''(t_0) < 0$ for some $t_0 \geqslant a$. Now $y(t) \geqslant 0$ for $t \geqslant a$, so $y'''(t) \leqslant 0$ for $t \geqslant a$. For $t \geqslant t_0$, $y''(t) \leqslant y''(t_0) < 0$, so $y'(t) \geqslant 0$ for all $t > a$ is impossible. Hence, $y''(t) \geqslant 0$ for all $t \geqslant a$, and $y(t)$ is a solution of (10). Thus, if $\beta^0 \geqslant 0$, we may replace (11) by (10) in (a) and (b) above.

If, in addition, $\lambda > 0$, we can reason as follows: Since $y''(t) \geqslant 0$, $y'(t)$ is nondecreasing, so $y'(t)$ approaches a finite limit as $t \to \infty$. Now $y'''(t) \leqslant \lambda(y'^2(t) - 1)$ and $y''(t)$ is bounded; hence, it follows that $y'(t) \to 1$ as $t \to \infty$. Thus, for $\lambda > 0, \beta^0 \geqslant 0$, we can assert in (a) and (b) that $y'(t) \to 1$ as $t \to \infty$. Then result (a) contains the classical boundary conditions for (10): $y(0) = 0$, $y'(0) = 0$, and $y'(\infty) = 1$ (see [3, p. 23]).

Remark. There are a number of interesting possibilities for the boundary sets Z_1 and Z_2 in Theorems 4.2 and 4.3. In (b) and (c) above, we have made the simplest choices. One could substitute, for example, $y'(b) + C_5 y''(b) = C_6$ for $y'(b) = \delta^1$ in (c), where $C_5 > 0$ and $0 \leqslant C_6 \leqslant 1$.

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