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Transient analysis of collinear cracks under anti-plane dynamic loading

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Abstract

The problem of a homogeneous linear elastic body containing multiple collinear cracks under anti-plane dynamic load is considered in this work. The cracks are simulated by distributions of dislocations and an integral equation relating tractions on the crack planes and the dislocation densities is derived. The integral equation in the Laplace transform domain is solved by Gaussian-Chebyshev integration quadrature. The dynamic stress intensity factor associated with each crack tip is calculated by a numerical inverse Laplace scheme.

The proposed method was applied to calculate the stress intensity factors for $M$ equally spaced cracks of identical length subject to impact loading with $M = 1 \sim 4$. Comparison of the numerical result for a single crack with the analytic solution shows that the present method is highly accurate.

Keywords: collinear cracks; dynamic stress intensity factor.

1. Introduction

When a linear elastic solid containing cracks is subject to applied loads, the stress field near the crack tips exhibits a square-root singularity. The amplitudes of the singular stress field are characterized by the stress intensity factors, which are used as the parameters governing crack propagation. The stress intensity factors depend on applied loading, geometry, and applied loading. Because of the difficulties in satisfying
the boundary conditions, only a limited number of closed-form solutions of stress intensity factor exist. For most problems numerical methods must be employed to determine the stress intensity factors.

One of the popular methods for solving crack problems with static loading is the dislocation method. Analysis of the problem by this method may be broken down into stages: first the tractions arising along the line of the crack in the uncracked body are determined. The crack is then inserted and the unsatisfied tractions cancelled by inserting a continuously varying density of displacement discontinuities, or dislocations, along the line of the crack. This formulation leads to an integral equation which may be discretized using Gaussian-Chebyshev quadrature to the desired degree of refinement [1]. Once the resulting simultaneous linear equations are solved the stress intensity factors can be readily determined from the dislocation densities at the crack tips. The dislocation method, however, has rarely been applied to the problems involving dynamic loading.

For isotropic materials Cochard and Madariaga [2] has derived an integral equation for dynamic anti-plane shear (mode III) loading. The integral equation contains a space-time convolution integral, which is Cauchy singular in space as in the static case. Additionally there is another term which is only present in the dynamic case and is associated with radiation damping by wave emission. The integral equation has been numerically implemented by Morrissey and Geubelle [3] with a spectral scheme, in which an integral in time obtained by applying the Fourier transform in space is solved. In the spectral formulation the Cauchy singularity of the integral is removed and the Gauss integration quadrature cannot be utilized. Chen and Tang [4] obtained an integral equation as that in [2] but did not correctly include the radiation term. Nevertheless they showed that if the Laplace transform in time is applied to the integral, the Cauchy singularity is preserved and the same integration quadrature for static loading may be employed.

It is the main objective of this work to extend the method to study collinear cracks under dynamic loading. The integral equations in [2] will be derived based on the fundamental solution of a dislocation. The equation will be solved first in the Laplace transform domain using Gauss integration quadrature and then inverted to calculate the stress intensity factors in the time domain. The proposed method was applied to calculate the stress intensity factors for \( M \) equally spaced cracks of identical length subject to impact loading with \( M = 1 \sim 4 \).

2. Basic Equations

The equation of motion in anti-plane shear deformation is

\[
\tau_{1,1} + \tau_{2,2} = \rho \ddot{u},
\]

where \( \tau_{\alpha} = \sigma_{3\alpha}, \quad \alpha = 1, 2, \quad u \) is the displacement in the \( x_j \) direction, \( t \) is time and \( \rho \) is the mass density, a comma in the subscript denotes partial differentiation, and an overhead dot stands for time derivative. The stress-strain relations for isotropic materials are

\[
\tau_{\alpha} = \mu u_{,\alpha}, \quad \alpha = 1, 2,
\]

where \( \mu \) is the shear modulus. The equation of motion in terms of the displacement is given by substituting (2) into (1) as

\[
\mu \left( u_{,11} + u_{,22} \right) = \rho \ddot{u}.
\]

The general solution of (3) for \( x_2 > 0 \) can be represented as [5,6]
where $\text{Re}$ stands for real part, $H$ is the Heaviside step function, $c = \sqrt{\mu/\rho}$ is the shear wave speed, and $w$ is

$$w = \frac{y_1 + y_2 \sqrt{1 - (y/c)^2}}{1 - (y_2/c)^2}.$$  

(5)

Here $y_\alpha = x_\alpha / t, \alpha = 1, 2$ and $y = \sqrt{y_1^2 + y_2^2}$. From Eq. (2), $\tau_2$ can be expressed as

$$\tau_2 = 2\mu \text{Re} \left[ f'(\omega) \frac{\partial w}{\partial x_2} - 2\mu \text{Re} [f(\omega)] \delta(ct - x_2) \right],$$  

(6)

where $\delta$ is the Dirac delta function.

3. Fundamental Solution

Consider a screw dislocation of Burgers vector $\beta$, which suddenly appears at $t = 0$ at the origin in an infinite body that is initially at rest and stress-free. The slip plane is assumed to coincide with the negative $x_1$-axis. From the jump condition for $u$ and continuity condition for $\tau_2$ at $x_2 = 0$, (4) and (6) yield, for $t > 0$,

$$f'' - f'' + (f'' - f'') - (f'' - f'') = 0,$$

(7)

(8)

where the superscripts $\pm$ denote the limits as $x_2 \to 0^\pm$. The solution of $f(w)$ may be shown to be

$$f(w) = \frac{\beta}{4\pi t} \ln w.$$  

(9)

Substitution of (9) into (4) and (6) gives

$$u = \frac{\beta}{2\pi} \text{Im} \ln w H(t - x_2/c).$$  

(10)

From (6) $\tau_2$ at $x_2 = 0$ is given as

$$\tau_2 = \frac{L(y_1)\beta}{2\pi x_1} - \frac{\mu\beta}{2c} H(-x_1) \delta(t),$$  

(11)

where

$$L(y_1) = \mu \sqrt{1 - (y_1/c)^2} H(1 - (y_1/c)^2).$$  

(12)

4. Collinear Cracks

Consider $N$ collinear cracks located at $|x_i - b_i| \leq a_i, i = 1 \sim M$ and $x_2 = 0$ in an infinite body. The cracks may be simulated as a distribution of dislocation $\delta(x_i, t)$ as
\[ \beta(x,t) = \int_{0}^{c} \hat{\alpha}(\xi, \tau) d\tau d\xi. \]  

(13)

From (11) and (13) the stress \( \tau \) at \( x_2 = 0 \) may be expressed as

\[ \tau(x_1, t) = -\frac{1}{2} \frac{\partial \beta}{\partial t} + \frac{1}{2\pi} \sum_{n=-a}^{b} \int_{x_1 - \xi}^{x_1 + \xi} \frac{1}{(x_1 - \xi)} L\left( \frac{x_1 - \xi}{t - \tau} \right) \hat{\alpha}(\xi, \tau) d\tau d\xi. \]  

(14)

Equation (14) recovers the result obtained in [2] using a different approach. Taking Laplace transform of (14) yields the following Cauchy type singular integral equation:

\[ \tilde{\tau}(x_1, s) = -\frac{1}{2} \frac{\partial \tilde{\beta}}{\partial t} + \frac{1}{2\pi} \sum_{n=-a}^{b} \int_{x_1 - \xi}^{x_1 + \xi} \frac{1}{(x_1 - \xi)} \tilde{U}\left( \frac{x_1 - \xi}{s / c} \right) \hat{\alpha}(\xi, s) d\xi, \]  

(15)

where \( \tilde{f} \) denotes Laplace transform of \( f \),

\[ U_{III}(z) = z \left( K_1(z) - \int_{0}^{\infty} K_0(\eta) d\eta \right). \]  

(16)

and \( K_n \) is the modified Bessel function of order \( n \). Equation (15) may be solved for \( \hat{\alpha}(x_1, s) \) and subsequently, by inverse Laplace transform, \( \alpha(x_1, t) \).

5. Numerical Methods

Let \( \xi = b_i + a_i \eta, -1 \leq \eta \leq 1 \), the integral in (15) may be expressed as

\[ \int_{x_1 - \xi}^{x_1 + \xi} \frac{1}{\xi} \tilde{U}\left( \frac{x_1 - \xi}{s / c} \right) \hat{\alpha}(\xi, s) d\xi = \int_{-1}^{1} \frac{1}{\xi} \tilde{U}\left( \frac{x_1 - \xi}{s / c} \right) \hat{\alpha}(\xi, s) a_i d\eta. \]  

(17)

Moreover to incorporate the square-root singularity of dislocation density at the crack tips, let

\[ \hat{\alpha}(\xi, s) = \frac{\hat{g}(\xi, s)}{a_i \sqrt{1 - \eta^2}}, \]  

(18)

where \( \hat{g}(\xi) \) is a regular function, and (17) becomes

\[ \int_{x_1 - \xi}^{x_1 + \xi} \frac{1}{\xi} \tilde{U}\left( \frac{x_1 - \xi}{s / c} \right) \hat{\alpha}(\xi, s) d\xi = \int_{-1}^{1} \frac{1}{\xi} \tilde{U}\left( \frac{x_1 - \xi}{s / c} \right) \frac{\hat{g}(\xi, s)}{\sqrt{1 - \eta^2}} d\eta. \]  

(19)

Since \( x \rightarrow \xi, U \rightarrow 1 \), (19) is an integral of Cauchy's type and can be evaluated using the following Gauss-Chebyshev integration formula [1]

\[ \int_{x_1 - \xi}^{x_1 + \xi} \frac{1}{\xi} \tilde{U}\left( \frac{x_1 - \xi}{s / c} \right) \hat{\alpha}(\xi, s) d\xi = \frac{\pi}{N} \sum_{j=1}^{N} \frac{1}{x_1 - \xi_j} \tilde{U}\left( \frac{x_1 - \xi_j}{s / c} \right) \hat{g}(\xi_j, s), \]  

(20)

where \( \xi_j = b_i + a_j \cos \left( j \frac{1}{2} \frac{\pi}{N} \right), \quad j = 1, \ldots, N. \)

Consider next \( \tilde{\beta}(x_1, s) \) in (16), which is related to \( \hat{\alpha}(\xi, s) \) by

\[ \tilde{\beta}_i(x_1, s) = -\int_{-1}^{1} \hat{\alpha}_i(\xi, s) d\xi, \]  

(21)

for \( x_1 = b_i + a_i \eta, -1 \leq \eta \leq 1 \). By substituting (18) into (21) and approximating \( \hat{g}(\xi) \) by Chebyshev polynomials, \( \tilde{\beta}(x_1, s) \) may be expressed as
\[
\hat{\beta}_i(x_i) = -\frac{1}{N} \sum_{j=1}^{N} \left( \pi - \theta - 2 \sum_{k=1}^{N} \frac{T_k(\eta)}{k} \sin k\theta \right) \hat{g}(\xi_{ij}),
\]
where \( \theta = \cos^{-1}(\eta) \).

With (20) and (22), (15) can be discretized as
\[
\hat{\varepsilon}_i(x_i,s) = \sum_{i=1}^{M} \sum_{j=1}^{N} C_{ij}(x_i,\xi_{ij}) \hat{g}(\xi_{ij},s).
\]
Equation (23) can be solved for \( \hat{g}(\xi_{ij},s) \) by setting \( x_i = b_i + a_i \cos \left( \frac{k\pi}{N} \right), \ k = 1, \ldots, N-1 \).

Additional equations are obtained by the crack closure conditions
\[
\beta(b_i + a_i, s) = \sum_{j=1}^{N} \hat{g}(\xi_{ij},s) = 0, i = 1, \ldots, M.
\]

The stress intensity factors in the Laplace transform domain is determined by
\[
\hat{K}_{III}(s) = \mu \sqrt{\frac{\pi r}{2}} \lim_{s \to 0} \hat{\alpha}(r,s),
\]
where \( r \) is the distance from the crack tip. Using (18), (25) yields
\[
\hat{K}_{III}(b_i + a_i, s) = \frac{\mu}{2N} \sqrt{\pi / a_i} \sum_{j=1}^{N} \left( -1 \right)^{j+1} \cot \left( \frac{2j-1}{2N} \pi \right) \hat{g}(\xi_{ij}),
\]
\[
\hat{K}_{III}(b_i - a_i, s) = \frac{\mu}{2N} \sqrt{\pi / a_i} \sum_{j=1}^{N} \left( -1 \right)^{j+1} \tan \left( \frac{2j-1}{2N} \pi \right) \hat{g}(\xi_{ij}).
\]

The stress intensity factors in the time domain may be calculated using several numerical schemes for Laplace inversion. In this work, the method proposed by Miller and Guy [7] is adopted.

6. Examples

Consider \( M \) equally spaced cracks of length \( 2a \) subject to \( \tau_2 = -\tau_\theta H(t) \), where \( \tau_\theta \) is a constant. The distance between the centers of the cracks is assumed to be \( 3a \). The stress intensity factors for \( M = 1 \sim 4 \), were computed and the numerical results are shown, respectively, in Fig1. \sim Fig4.
For $M = 1$, the computed maximum normalized $K_{III}$ was 1.24, which agrees closely with the analytic value $4/\pi \approx 1.27$ [8]. For $M = 2$, the maximum normalized $K_{III}$ occurred at tip B and reached a higher value 1.33 than that for a single crack. For $M = 3, 4$, the maximum normalized $K_{III}$ occurred at tip C and D, respectively, but no further enhancement was observed. It appears that as far as the dynamic overshoot of the stress intensity factors is concerned, it is sufficient to consider only two cracks. It was also checked that the corresponding static values were approached as the time increased in all cases considered.

References