The neighborhood union of independent sets and hamiltonicity of graphs

Guantao Chen\textsuperscript{a, b, 1}, Xuechao Li\textsuperscript{c}, Zhengsheng Wu\textsuperscript{d}, Xingping Xu\textsuperscript{e}

\textsuperscript{a}Georgia State University, Atlanta, GA 30303, USA
\textsuperscript{b}Central China Normal University, Wuhan, China
\textsuperscript{c}University of Georgia, Athens, GA 30609, USA
\textsuperscript{d}Nanjing Normal University, Nanjing, China
\textsuperscript{e}Jiangsu Institute of Education, Nanjing, China

Received 21 February 2002; received in revised form 25 September 2006; accepted 25 October 2006
Available online 31 December 2006

Abstract

Let $G$ be a graph, $N(u)$ the neighborhood of $u$ for each $u \in V(G)$, and $N(U) = \bigcup_{u \in U} N(u)$ for each $U \subseteq V(G)$. For any two positive integers $s$ and $t$, we prove that there exists a least positive integer $N(s, t)$ such that every $(s + t)$-connected graph $G$ of order $n \geq N(s, t)$ is hamiltonian if $|N(S)| + |N(T)| \geq n$ for every two disjoint independent vertex sets $S, T$ with $|S| = s$ and $|T| = t$.

© 2007 Published by Elsevier B.V.

Keywords: Hamiltonian; Vertex insertion; The neighborhood union

1. Introduction

All graphs considered in this paper are finite simple graphs. We will generally follow the terminology and notation of Bondy and Murty in [3]. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, let $N(v) := \{w : vw \in E(G)\}$ and $d(v) := |N(v)|$, where $N(v)$ is called the neighborhood of $v$ and $d(v)$ is called the degree of $v$, respectively. More generally, for any $U \subseteq V(G)$, let $N(U) = \bigcup_{u \in U} N(u)$ and $d(U) = |N(U)|$. Further, $N(U)$ is called the neighborhood of $U$ while $d(U)$ is the degree of $U$. Let $\delta(G)$ be the minimum degree of $G$ and $\sigma_2 := \min\{d(u) + d(v) | uv \notin E(G)\}$. Two classic sufficient degree conditions for hamiltonian graphs are obtained by Dirac and Ore, respectively.

**Theorem 1.1 (Dirac [8]).** Let $G$ be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$ then $G$ is hamiltonian.

**Theorem 1.2 (Ore [14]).** Let $G$ be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$ then $G$ is hamiltonian.

\textsuperscript{1}Partially supported by NSA Grant H98230-04-1-0300 and NSF Grant DMS-0500951.
A natural generalization of the above results is to replace the degree of each vertex by the degree of a set of vertices. Let $G$ be a graph and $k$, $s$, and $t$ be three positive integers. We define

$$
\delta_k(G) := \min\{d(U) : U \subseteq V(G) \text{ is an independent set of } k \text{ vertices}\},
$$

$$
\delta^*_k(G) := \min\{d(U) : U \subseteq V(G) \text{ and } |U| = k\},
$$

$$
\sigma_{s,t} := \min\{d(S) + d(T) : |S| = s, |T| = t, S \cap T = \emptyset, \text{ and } S \cup T \text{ is an independent set of } G\}.
$$

Clearly, $\delta_k^*(G) \leq \delta_k(G) \leq \sigma_{s,t}(G)$ if $s + t = k$, $\delta(G) = \delta_1(G)$, and $\sigma_{1,1}(G) = \sigma_2(G)$. A natural question is that under what circumstances we can replace $\delta(G)$ in Theorem 1.1 by $\delta_k(G)$ and $\sigma_2(G)$ by $\sigma_{s,t}(G)$. For $k = 2$, Faudree et al. obtained the following result.

**Theorem 1.3** (Faudree et al. [10]). Let $G$ be a 2-connected graph of order $n \geq 3$. If $\delta_2(G) \geq (2n - 1)/3$ then $G$ is hamiltonian.

The graph $K_2 + 3K_p$ illustrates that the lower bound $(2n - 1)/3$ in Theorem 1.3 is best possible. However, the following three theorems show that the $(2n - 1)/3$ can be reduced considerably under some circumstances.

**Theorem 1.4** (Faudree et al. [9]). If $G$ is a 2-connected graph of sufficiently large order $n$ such that $\delta^*_2(G) \geq n/2$ then $G$ is hamiltonian.

**Theorem 1.5** (Jackson [12]). Let $G$ be a 3-connected graph of order $n$. If $\delta_2(G) \geq (n + 1)/2$ then $G$ is hamiltonian.

**Theorem 1.6** (Broersma et al. [4]). Let $G$ be a 3-connected graph of order $n$. If $\delta_2(G) \geq n/2$ then $G$ is either hamiltonian or the Petersen graph.

In general, Fraisse obtained the following result.

**Theorem 1.7** (Fraisse [11]). Let $G$ be a $k$-connected graph of order $n$. If $\delta_k(G) > k(n - 1)/(k + 1)$ then $G$ is hamiltonian.

The graph $K_k + (k + 1)K_p$ illustrates that the above result is best possible. However, $k(n - 1)/(k + 1)$ is much bigger than $n/2$ when $n$ is large. Let $G = (V, E)$ be a $k$-connected graph of order $n$. For $S \subseteq V$, let $J(S) = \{u \notin S | N(u) \supseteq S\}$ if $|S| \geq 2$ and $J(S) = \emptyset$ otherwise. Ainouche generalized Fraisse’s result as follows.

**Theorem 1.8** (Ainouche [1]). Let $G$ be a $k$-connected graph of order $n$. Suppose there exists some $s$, $1 < s < k$, such that for every independent set $X \subseteq V$ of cardinality $s + 1$ there is a vertex $x \in X$ such that

$$
d(X \setminus \{x\}) + |N(x) \cup J(X \setminus \{x\})| \geq n.
$$

Then $G$ has a hamiltonian cycle.

By increasing the connectivity, Chen and Liu obtained the following result.

**Theorem 1.9** (Chen and Liu [5]). Let $k$ be a positive integer and $G$ be a $4(k - 1)$-connected graph of order $n$. If $\delta_k(G) \geq n/2$ then $G$ is hamiltonian.

Note that $4(k - 1) = 0$ when $k = 1$. Thus, connectivity $4(k - 1)$ imposes no constraints for the case $k = 1$. The well-known Petersen graph shows that $4(k - 1) = 4$ is best possible in some sense for the case $k = 2$. However, when $k \geq 3$, the lower bound $4(k - 1)$ may not be the best possible. The following result improves Theorem 1.9 in terms of connectivity although it requires that $n$ is much larger than $k$.

**Theorem 1.10** (Chen et al. [6]). Let $k$ be a positive integer and let $G$ be a $(2k - 1)$-connected graph of order $n \geq 16k^3$. If $\delta_k \geq n/2$ then $G$ is hamiltonian.
In the same paper, the following conjecture was posted.

**Conjecture 1.11 (Chen et al. [6]).** Let \( k \) be a positive integer and let \( G \) be a \((2k - 1)\)-connected graph of order \( n \). If \( \delta_k \geq n/2 \) then \( G \) is a hamiltonian graph except \( G \) is the Petersen graph.

The purpose of this article is to generalize Theorem 1.2 in terms of \( d(S) + d(T) \) for any two disjoint sets \( S \) and \( T \) such that \( S \cup T \) is an independent set. When \( |S| = |T| \), the following result, stronger than Theorem 1.9 is obtained.

**Theorem 1.12 (Chen and Liu [5]).** Let \( k \) be a positive integer and \( G \) be a \((4k - 1)\)-connected graph of order \( n \geq 3 \). If \( \sigma_{k,k} \geq n \) then \( G \) is hamiltonian.

Only case \( d(S) + d(T) \) with \( |S| = |T| \) are considered in the above results. In this paper, we generalize the results to include the case \( |S| \neq |T| \) as follows.

**Theorem 1.13.** Let \( s \) and \( t \) be two positive integers and let \( G \) be a \((2s + t)\)-connected graph of order \( n \). If \( \sigma_{s,t}(G) \geq n \) then \( G \) is hamiltonian.

We strongly believe that the connectivity \( s + t \) can be reduced to \( s + t - 1 \) with some exceptions.

**Conjecture 1.15.** Let \( s \) and \( t \) be two positive integers and let \( G \) be a \((s + t - 1)\)-connected graph of order \( n \). If \( \sigma_{s,t}(G) \geq n \) then \( G \) is hamiltonian unless \( G \) is isomorphic to the Petersen graph.

Clearly, Ore’s theorem is the case when \( s = t = 1 \). In general, the case \( s = 1 \) is a corollary of Theorem 1.8. The graph \( G = K_{s+t-2} + \cup_{i=1}^{s+t-1} G_i \), where each \( G_i \) is a complete graph with \((n - (s + t - 2))/(s + t - 1)\) vertices, is \((s + t - 2)\)-connected with independence number \( \alpha(G) = s + t - 1 \). In addition, \( G \) does not contain disjoint vertex sets \( S \) and \( T \) such that \( |S| = s, |T| = t, \) and \( S \cup T \) is independent. Thus, \( \sigma_{s,t}(G) \geq n \). It is not difficult to check that \( G \) is not hamiltonian. So the connectivity condition \( s + t - 1 \) is best possible in Conjecture 1.15.

Let \( \alpha(G) \) denote the independent number of \( G \) and \( \kappa(G) \) denote the connectivity of \( G \).

**Theorem 1.16 (Chvátal and Erdös [7]).** If \( \alpha(G) \leq \kappa(G) \), then \( G \) is hamiltonian.

So \( \alpha(G) > s + t - 1 \) for each \((s + t - 1)\)-connected non-hamiltonian graph (see Fig. 1). Consequently, there exist \( S \subseteq V(G) \) and \( T \subseteq V(G) \) such that \( |S| = s, |T| = t, \) and \( S \cup T \) is independent.

![Fig. 1. An \((s + t - 2)\)-connected non-hamiltonian graph.](image)
Let \( Y \) be a vertex set of \( G \). For each \( i \in \{0, 1, 2, \ldots, |Y|\} \), let \( V_i(Y) := \{v \in V(G) : |N(v) \cap Y| = i\} \), i.e. each vertex in \( V_i(Y) \) is adjacent to exactly \( i \) vertices in \( Y \). Slightly abusing notation, we use \( H \subseteq G \) for a subgraph of \( G \) as well as for a vertex set \( H \) provided no ambiguity. For any \( A \subseteq G \) and \( B \subseteq G \), let \( N_B(A) := N(A) \cap B \).

2. Basic lemmas

In this section, we will state some lemmas regarding insertible vertices of a maximal cycle in a non-hamiltonian graph.

Let \( G \) be a graph. We assume that all cycles and paths of \( G \) are given with a fixed orientation. For a cycle (or a path) \( C \) of \( G \), we let \( \bar{C} \) denote \( C \) with the reverse orientation. For \( u, v \in V(C) \), let \( C[u, v] \) denote the subpath of \( C \) from \( u \) to \( v \). Let \( C(u, v) = C[u, v] - \{u\} \) and define \( C[u, v) \) and \( C(u, v) \) similarly. Let \( u^+ \) denote the successor of \( u \) along \( C \) and \( u^- \) denote the predecessor of \( u \) along \( C \). A \( uv \)-path of \( G \) is a path of \( G \) connecting \( u \) and \( v \) with the fixed orientation from \( u \) to \( v \). Let \( H \) be a connected subgraph of \( G \) and let \( u \) and \( v \) be two vertices of \( H \). Then \( uHv \) will denote a longest \( uv \)-path in \( H \). A bridge \( B[x_i, x_j] \) of \( H \) is a path such that all internal vertices are in \( G - V(H) \) except the two endvertices \( x_i \) and \( x_j \) which are in \( H \). A maximal cycle \( C \) of \( G \) is a cycle such that no other cycle in \( G \) contains all of vertices of \( C \) as a proper subset of vertices.

Let \( G \) be a non-hamiltonian graph of order \( n \), \( C \) be a maximal cycle of \( G \) with an orientation, \( H \) be an arbitrary component of \( G - V(C) \), and \( v_1, v_2, \ldots, v_h \) be \( h \) distinct vertices in \( N_C(H) \). We assume that \( u_i v_i \in E(G) \) where \( u_i \in H \) for \( 1 \leq i \leq h \) and that \( v_1, v_2, \ldots, v_h \) are labeled in the order along the orientation of \( C \).

The vertices \( v_1, v_2, \ldots, v_h \) divide the cycle \( C \) into \( h \) segments,

\[
Q_i := C(v_i, v_{i+1}) = w_{i1}w_{i2} \cdots w_{iq_i}v_{i+1} \quad \text{for} \quad 1 \leq i \leq h,
\]

where \( v_{i+1} := v_1 \). A vertex \( w_i \in Q_i \) is called an insertible vertex if there are a pair of consecutive vertices \( I(w_i) \) and \( I(w_i)^+ \in C - Q_i \) such that \( w_i I(w_i), w_i I(w_i)^+ \in E(G) \) (see Fig. 2).

Suppose that \( w_{i1}, w_{i2}, \ldots, w_{i\beta_i} \) are insertible vertices. Let \( \beta_1 \) be the largest integer in \( [1, x] \) such that \( I(w_{i1}) = I(w_{i\beta_1}) \), and \( \beta_2 \) be the largest integer in \( [\beta_1 + 1, x] \) such that \( I(w_{i\beta_1+1}) = I(w_{i\beta_2}) \). Then we insert the segment \( C[w_{i1}, w_{i\beta_1}] \) between \( I(w_{i1}) \) and \( I(w_{i1})^+ \), the segment \( C[w_{i\beta_1+1}, w_{i\beta_2}] \) between \( I(w_{i\beta_1+1}) \) and \( I(w_{i\beta_1+1})^+ \), \ldots, the segment \( C[w_{i\beta_2+1}, w_{i\beta_3}] \) between \( I(w_{i\beta_2+1}) \) and \( I(w_{i\beta_2+1})^+ \), to obtain a path \( P \) from \( w_{i\beta_2}^+ \) to \( v_i \) such that \( V(P) = V(C) \), as shown in Fig. 3. We name such an insertion the segment insertion and denote it as \( SI(C[w_{i1}, w_{i\beta}]) \).

The following lemmas are obtained in \([2,5,13,15]\).

**Lemma 2.1.** For each \( Q_i \) there is a non-insertible vertex in \( Q_i - \{v_{i+1}\} \).

For each \( 1 \leq i \leq h \), let \( t_i \) be the smallest integer such that \( w_{it_i} \) is not an insertible vertex in \( Q_i \) and let \( S_i = \{w_{i1}, w_{i2}, \ldots, w_{it_i}\} \). Notice that from Lemma 2.1, \( S_i \cap N_C(H) = \emptyset \). Moreover, it is not difficult to verify the following lemmas hold.

![Fig. 2. An insertible vertex.](image-url)
Lemma 2.2. For each $i \neq j$, each $1 \leq s_i \leq t_i$, and each $1 \leq s_j \leq t_j$, the following two properties hold.

(i) There does not exist a bridge $B[w_{i_s}, w_{j_s}]$ of $C$.
(ii) For every $w \in C[w_{i_s}, w_{j_s}]$, if $ww_{i_s} \in E(G)$, then $w^-w_{j_s} \notin E(G)$. Similarly, for any $w \in C[w_{j_s}, w_{i_s}]$, if $ww_{j_s} \in E(G)$, then $w^-w_{i_s} \notin E(G)$.

Without confusion, let $w_i := w_{i_s}$ for $1 \leq i \leq h$ and let $W := \{w_1, w_2, \ldots, w_h\}$. By Lemma 2.2, $W$ is an independent vertex set. Let $J_H := \bigcup_{q=1}^{h} C[w_q, v_{q+1}]$ and $K_H := V(G) \setminus J_H$. That is, $K_H$ is the union of $\bigcup_{i=1}^{h} C(v_i, w_i)$ and all components of $G - V(C)$. The following lemma holds.

Lemma 2.3. $K_H \subseteq V_0(W) \cup V_1(W)$.

So, for any $S, T \subset W$, if $S \cap T = \emptyset$, then $N(S) \cap N(T) \cap K_H = \emptyset$.

Lemma 2.4. For any $S, T \subset W$, if $S \cap T = \emptyset$, then

$$|N(S) \cap K_H| + |N(T) \cap K_H| \leq |K_H| - |V(H)|.$$

For any $i = 1, 2, \ldots, h$, a segment $C[z_1, z_2] \subseteq C[w_i, w_{i+1}]$ is called an NE-segment if $C(z_1, z_2) \subseteq N(W)$, and $z_1 \notin N(W)$ and $z_2 \notin N(W)$. An NE-segment $C[z_1, z_2]$ is said to be trivial if $C[z_1, z_2] = \{z_1\}$. Clearly, each path $C[w_i, w_{i+1}]$ is divided into disjoint NE-segments. The following lemma is a direct consequence of Lemma 2.2 and the definition of insertible vertices.

Lemma 2.5. For each $i = 1, 2, \ldots, h$ and each NE-segment $C[z_1, z_2] \subseteq C[w_i, w_{i+1}]$, let $L_j = N(w_j) \cap C(z_1, z_2)$ ($j \in \{1, 2, \ldots, h\}$). Then

$L_i, L_{i-1}, \ldots, L_1, L_{h-1}, \ldots, L_{i+1}$

(some of them may be empty) form consecutive subpaths of $C[z_1, z_2]$ which can have only their endvertices in common. Moreover, $|L_j| \leq 1$ for all $j \neq i$.

3. Proof of Theorem 1.13

We prove the following result which is slightly stronger than Theorem 1.13.

Theorem 3.1. Let $G$ be a non-hamiltonian graph of order $n$ such that $\sigma_{s,t} \geq n$. Then $|N_C(H)| < 2(s + t)$ for any maximal cycle $C$ and any component $H$ of $G - V(C)$.
Proof. Suppose, to the contrary, there is a maximal cycle $C$ of $G$ and a component $H$ of $G - V(C)$ such that $h = |N_C(H)| \geq 2(s + t)$. Following the notation of Section 2, let $N_C(H) = \{v_1, v_2, \ldots, v_h\}$, where $v_1, v_2, \ldots, v_h$ are listed in the order along the orientation of $C$. Let $w_i$ be the first non-insertible vertex of $C(v_i, v_{i+1})$ along the orientation of $C$.

Let

$$S_1 = \{w_1, w_2, \ldots, w_s\},$$

$$S_2 = \{w_{s+1}, w_{s+2}, \ldots, w_{2s}\},$$

$$T_1 = \{w_{2s+1}, w_{2s+2}, \ldots, w_{2s+t}\},$$

$$T_2 = \{w_{2s+t+1}, w_{2s+t+2}, \ldots, w_{2s+2t}\}.$$

Claim 3.1. For each $i = 1, 2, \ldots, h$ and each NE-segment $I = C[z_1, z_2] \subseteq C[w_i, w_{i+1}]$,

$$|N_I(S_1)| + |N_I(S_2)| + |N_I(T_1)| + |N_I(T_2)| \leq 2|I|.$$

Proof. If $|I| = 1$, i.e. $I = \{z_1\}$, then, by the definition of NE-segments,

$$N_I(S_1) = N_I(S_2) = N_I(T_1) = N_I(T_2) = N_I(W) = \emptyset.$$

Thus, $|N_I(S_1)| + |N_I(S_2)| + |N_I(T_1)| + |N_I(T_2)| = 0 \leq 2$.

If $|I| = 2$, let $I = \{z_1, z_2\}$. In this case, we have that

$$N_I(S_1) \cup N_I(S_2) \cup N_I(T_1) \cup N_I(T_2) \subseteq N_I(W) \subseteq \{z_1, z_2\}.$$

Therefore,

$$|N_I(S_1)| + |N_I(S_2)| + |N_I(T_1)| + |N_I(T_2)| \leq 4 = 2|I|.$$

Suppose that $|I| \geq 3$. Without loss of generality, we assume that $1 \leq i \leq s$. By Lemma 2.5, $C(z_1, z_2)$ is divided into five internal disjoint subpaths which are neighbors of $S_1, T_2, T_1, S_2$, and $S_1$, respectively. Thus,

$$|N_I(S_1)| + |N_I(S_2)| + |N_I(T_1)| + |N_I(T_2)| \leq |C(z_1, z_2)| + 4 = (|I| - 1) + 4 \leq 2|I|. \quad \square$$

Claim 3.2. The inequality $|N(S_1)| + |N(S_2)| + |N(T_1)| + |N(T_2)| \leq 2(n - |V(H)|)$ holds.

Proof. Since $C$ is union of disjoint NE-segments, applying Claim 3.1 to all NE-segments, we obtain

$$|N_C(S_1)| + |N_C(S_2)| + |N_C(T_1)| + |N_C(T_2)| \leq 2|V(C)|. \quad (1)$$

By Lemma 2.3, $N_{G-V(C)}(S_1), N_{G-V(C)}(S_2), N_{G-V(C)}(T_1), N_{G-V(C)}(T_2)$ are pairwise disjoint. Since $N_C(H) \cap W = \emptyset, V(H) \cap (N(S_1) \cup N(S_2) \cup N(T_1) \cup N(T_2)) = \emptyset$. Combining these two statements together, we have the following:

$$|N_{G-V(C)}(S_1)| + |N_{G-V(C)}(S_2)| + |N_{G-V(C)}(T_1)| + |N_{G-V(C)}(T_2)| \leq n - |V(C)| - |V(H)|. \quad (2)$$

Combining inequalities (1) and (2), we have

$$d(S_1) + d(S_2) + d(T_1) + d(T_2) \leq n + |V(C)| - |V(H)| \leq 2(n - |V(H)|). \quad (3)$$

On the other hand,

$$d(S_1) + d(T_1) + d(S_2) + d(T_2) + 2s_{s,t} \geq 2n,$$

which is a contradiction to (3). \quad \square
4. Proof of Theorem 1.14

By contradiction, suppose that $G$ is an $(s + t)$-connected non-hamiltonian graph of order $n$ satisfying $\sigma_{s+t} \geq n$. Let $L = \{v : d(v) < n/(s + t)\}$, and $(L)$ denotes the subgraph induced by $L$. Since $G$ is an $(s + t)$-connected graph and $s + t \geq 2$, $G$ contains a cycle. Let $C$ be a cycle of $G$ satisfying

1. $|V(C) \cap L|$ is maximum, and
2. subject to above, $|V(C)|$ is maximum.

Claim 4.1. $\alpha((L)) < s + t$ where $\alpha((L))$ is the independence number of $(L)$.

Proof. Suppose, to the contrary, $\alpha((L)) \geq s + t$. Let $X \subseteq L$ be an independent set of $s + t$ vertices and let $S \cup T$ be an arbitrary partition of $X$ with $|S| = s$, $|T| = t$. Then,

$$d(S) + d(T) \leq \sum_{x \in X} d(x) < \frac{n}{s + t} (s + t) = n,$$

a contradiction. □

Claim 4.2. $L \subset V(C)$.

Proof. Suppose, to the contrary, that there exists an $v_0 \in L - V(C)$. Since $G$ is $(s+t)$-connected, there are $(s+t)$ vertex-disjoint paths (except $v_0$) $P_1(v_0, v_1)$, $P_2(v_0, v_2)$, ..., $P_{s+t}(v_0, v_{s+t})$ from $v_0$ to $C$. We assume that $v_1, v_2, ..., v_{s+t}$ occur on $C$ in the order of along the orientation of $C$. For each $i \in \{1, 2, ..., s + t\}$, we claim that $C(v_i, v_{i+1}) \cap L \neq \emptyset$. Otherwise, the cycle $C' = P_{i+1}(v_0, v_{i+1})C[v_{i+1}, v_i]P_i(v_i, v_0)$ contains more vertices of $L$ than $C$ does, a contradiction.

Let $w_i$ be the first vertex of $L$ along the segment $C(v_i, v_{i+1})$, for each $i \in \{1, 2, ..., s + t\}$. If $w_iw_j \in E(G)$, for some $i \neq j \in \{1, 2, ..., s + t\}$, then the cycle $C'' = P_i[v_0, v_i]C[v_i, v_j]C[w_i, v_j]P_j(v_j, v_0)$ contains more vertices of $L$ than $C$ does, a contradiction. Thus, $w_1, w_2, ..., w_{s+t}$ is an independent set, a contradiction to Claim 4.1. □

Claim 4.3. For any component $H$ of $G - V(C)$, we have

$$|V(H)| \geq \frac{n}{s + t} - (2s + 2t - 1) > (s + t - 1)(s + t).$$

Proof. By Claim 4.2, we have $d(v_0) \geq n/(s + t)$ for every vertex $v_0 \in V(H)$. By Theorem 3.1, $|N_C(v_0)| \leq |N_C(H)| \leq 2(s + t) - 1$. Thus,

$$|V(H)| \geq d_H(v_0) \geq \frac{n}{s + t} - 2(s + t) + 1 > (s + t - 1)(s + t).$$

The second inequality comes from the fact that $n \geq (s + t)^2(s + t + 1)$. □

Since $G$ is $(s + t)$-connected and $|V(H)| > (s + t - 1)(s + t) \geq s + t$, there are $s + t$ independent edges $u_1v_1, u_2v_2, ..., u_{s+t}v_{s+t}$ such that $u_i \in V(H)$ and $v_i \in V(C)$ for each $i = 1, 2, ..., s + t$. Let $S = \{w_1, w_2, ..., w_s\}$ and $T = \{w_{s+1}, w_{s+2}, ..., w_{s+t}\}$. For convenience, we let $w_{s+t+1} = v_1$ and $w_{s+t+1} = w_1$. For each $i \in \{1, 2, ..., s + t\}$, let $w_i$ be the first non-insertible vertex in $C(v_i, v_{i+1})$. By Lemma 2.1, such $w_i$ exists for each $i$. Let $W = \{w_1, w_2, ..., w_{s+t}\}$. Then $C = \bigcup_{i=1}^{s+t} C[w_i, w_{i+1}]$ and each segment $C[w_i, w_{i+1}]$ is the union of disjoint $NE$-segments with respect to $W$. A vertex $v \in V(C)$ is called a connector if there are two distinct non-insertible vertices $w_i$ and $w_j$ such that $v \in C(w_i, w_j)$ and $v^-w_j \in E(G)$ and $v^+w_i \in E(G)$, as shown in Fig. 4.

Claim 4.4. If $v$ is a connector, then $v \in L$.

Proof. Assume $v \notin L$. Let $w_i$ and $w_j \in W$ such that $v \in C(w_i, w_j)$ and $w_jv^- \in E(G)$ and $w_iv^+ \in E(G)$. By Lemma 2.2(ii), no vertices of $C(v_i, w_i)$ and $C(v_j, w_j)$ are adjacent to $v$. 

Therefore, without loss of generality, assume that 1. Proof.

Suppose, to the contrary, there are two bad connectors $P$ and $Q$.

Claim 4.5. For each $q = 1, 2, \ldots, s + t$ and each NE-segment $I = C[z_1, z_2] \subseteq C[w_q, w_{q+1}]$,

$$|N_I(S) \cap N_I(T)| \leq \begin{cases} 2 & \text{if } q \in \{1, 2, \ldots, s + t\} \setminus \{s, s + t\}; \\ 1 & \text{if } q = s \text{ or } s + t. \end{cases}$$

Therefore,

$$|N_I(S)| + |N_I(T)| \leq |I| - 1 + |N_I(S) \cap N_I(T)| \leq \begin{cases} |I| + 1 & \text{if } |N_I(S) \cap N_I(T)| = 2, \\ |I| & \text{if } |N_I(S) \cap N_I(T)| \leq 1. \end{cases}$$

Furthermore, if $1 \leq q \leq s - 1$ and $|N_I(S)| + |N_I(T)| = |I| + 1$, then $z_2 \hat{w}_i \in E(G)$ for some $i = q, q + 1, \ldots, s - 1$; if $s + 1 \leq q \leq s + t - 1$ and $|N_I(S)| + |N_I(T)| = |I| + 1$, then $z_2 \hat{w}_i \in E(G)$ for some $i = q, q + 1, \ldots, s + t - 1$.

Proof. Without loss of generality, assume that $1 \leq q \leq s$. If $q = s$, then $C(z_1, z_2)$ is a union of two segments $P_1$ and $P_2$ such that $P_1 \subseteq N(S)$ and $P_2 \subseteq N(T)$. If $1 \leq q \leq s - 1$, then $C(z_1, z_2)$ is a union of three disjoint segments $P_1, P_2,$ and $P_3$ such that $P_1 \subseteq N((w_q, w_{q-1}, \ldots, w_1)), P_2 \subseteq N(T), P_3 \subseteq N((w_s, w_{s-1}, \ldots, w_{q+1}))$. Moreover, $P_1, P_2,$ and $P_3$ are listed in the order along the orientation of $C$. So Claim 4.5 follows. \(\Box\)

Claim 4.6. Let $q = 1, 2, \ldots, s$ and let $I = C[z_1, z]$ and $J = C[z, z_2] \subseteq C[w_q, w_{q+1}]$ be two consecutive NE-segments in $C[w_q, w_{q+1}]$. Suppose $|N_I(S) \cap N_I(T)| = 2$.

(i) If $N_J(S) \cap N_J(T) = \emptyset$, then

$$|N_{I \cup J}(S)| + |N_{I \cup J}(T)| \leq (I - 1) + 2 + (J - 1) = |I| + |J|.$$  

(ii) If $|N_J(S) \cap N_J(T)| \neq 0$, then there exists $i \in \{q, q - 1, \ldots, 1, s + t, s + t - 1, \ldots, s + 1\}$, such that $w_i z^+ \in E(G)$.

Similar result holds for $s \leq q \leq s + t - 1$.

If (ii) of Claim 4.6 happens, we call $z$ a bad connector. It is obvious a bad connector is a connector. Let $B$ denote the set of all bad connectors and $\eta = |B|$. Let $B_q = B \cap C(w_q, w_{q+1})$ for $q \in \{1, 2, \ldots, s + t\} \setminus \{s, s + t\}$. Clearly, $B = \bigcup_{q=1}^{s+t+1} B_q$ where $B_s = B_{s+t} = \emptyset$.

Claim 4.7. $B_q$ is an independent vertex set for each $q \in \{1, 2, \ldots, s + t\} \setminus \{s, s + t\}$.

Proof. Suppose, to the contrary, there are two bad connectors $y$ and $z$ in $C(w_q, w_{q+1})$ such that $yz \in E(G)$. Without loss of generality, we assume that $z \in C(y, v_{q+1})$. Note that $q \neq s, s + t$. Without loss of generality, assume that
Thus, Claim 4.8 holds by combining (4) and (5).

Proof. Note

Claim 4.8. \( d(S) + d(T) \leq n - |V(H)| + \eta + s + t - 2. \)

Proof. Note C is a union of disjoint NE-segments I. By Claim 4.5, we have that

\[ |N_I(S)| + |N_I(T)| \leq |I| + 1 \]

for each NE-segment \( I = C[z_1, z_2] \) where \( I \subseteq C[w_q, w_{q+1}] \). Moreover, if the equality holds, we have that \( q \neq s, s + t \). By Claim 4.6, we have that

\[ |N_C(S)| + |N_C(T)| \leq |V(C)| + \eta + (s + t) - 2, \]

where \( s + t - 2 \) comes from the NE-segments \( C[z_1, z_2] \) with \( z_1 = w_i \) for some \( i \in \{1, 2, \ldots, s + t\} \setminus [s, s + t] \). By Lemma 2.2(i),

\[ |N_{G-V(C)}(S)| + |N_{G-V(C)}(T)| \leq n - |V(C)| - |V(H)|. \]

Thus, Claim 4.8 holds by combining (4) and (5). □

By Claim 4.8 and \( d(S) + d(T) \geq n \), we have that

\[ \eta \geq |V(H)| - (s + t) + 2 \geq (s + t - 1)(s + t) - (s + t) + 2 > (s + t)(s + t - 2). \]

Since \( B = \bigcup_{q=1}^{s+t} B_q \) where \( B_s = B_{s+t} = \emptyset \). By the Pigeonhole Principle, there exists an integer \( q \) such that \( |B_q| \geq s + t \). By Claim 4.7, \( B_q \) is an independent set. By Claim 4.4, \( B_q \subseteq L \). Thus, \( \alpha(L) \geq s + t \), which contradicts that \( \alpha(L) < s + t \). □

References