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On the dynamics of a class of nonclassical parabolic equations ☆

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Abstract

We consider the first initial and boundary value problem of nonclassical parabolic equations $u_t - \mu\Delta u_t - \Delta u + g(u) = f(x)$ on a bounded domain Ω, where $\mu \in [0, 1]$. First, we establish some uniform decay estimates for the solutions of the problem which are independent of the parameter μ. Then we prove the continuity of solutions as $\mu \to 0$. Finally we show that the problem has a unique global attractor $\mathcal{A}_\mu$ in $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$ in the topology of $H^2(\Omega)$; moreover, $\mathcal{A}_\mu \to \mathcal{A}_0$ in the sense of Hausdorff semidistance in $H_0^1(\Omega)$ as μ goes to 0.

Keywords: Nonclassical parabolic equation; Singular point; Uniform decay estimate; Global attractor; Upper-continuity

1. Introduction

In this paper we are mainly concerned with the dynamical behavior of the following nonclassical parabolic equation:

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$$u_t - \mu\Delta u_t - \Delta u + g(u) = f(x), \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$u(t,x) = 0, \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

$$u(0,x) = u_0(x), \quad sx \in \Omega, \tag{1.3}$$

where Ω is an open bounded set of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$, $f(x)$ is a given function, $\mu \in [0,1]$. This consideration is motivated by an increasing interest in such types of equations in recent years [5,9,16,17].

Nonclassical parabolic equations arise as models to describe physical phenomena such as non-Newtonian flow, soil mechanics and heat conduction, etc.; see [1,3,10,14,15] and references therein. Aifantis [1] provides a quite general approach for obtaining these equations.

As we will see in Section 2, Eq. (1.1) can be transformed into the following abstract equation in appropriate spaces:

$$u_t + \mathcal{L}(\mu)u + \tilde{g}(u) = \tilde{f},$$

where $\mathcal{L}(\mu) = (I + \mu A)^{-1}A$, and A is an operator corresponding to $-\Delta$ with respect to the homogeneous Dirichlet boundary condition. Note that in case $\mu = 0$, $\mathcal{L}(\mu) = A$ is an unbounded operator, while in case $\mu > 0$, it is a bounded one. Thus from some point of view, $\mu = 0$ can be seen as a singular limit for the equation. It is therefore of great interest to understand both the dynamics of the nonclassical equation itself and the influence of the term "$-\mu\Delta u_t$" to the dynamics of the classical equation as μ varies in [0, 1], in particular, as $\mu \to 0$.

The main aim of this paper is as follows.

First, we are interested in the uniform dissipativity of the equation, where the uniformity is with respect to the parameter μ. Roughly speaking, we will establish some uniform decay estimates for (1.1)–(1.3) which are independent of $\mu \in [0,1]$. These estimates are particularly useful in understanding the effects of the term $\mu\Delta u_t$ to the dynamics of the equation as $\mu \to 0$.

Secondly, we consider the continuous dependence of solutions of (1.1)–(1.3) on μ as $\mu \to 0$. Let $R, T > 0$. Then we will show that for some constant $C_T(R) > 0$,

$$\left\| S_\mu(t)u_0 - S_0(t)u_0 \right\|_1 \leqslant C_T(R)\sqrt{\mu}, \quad \forall t \in [0,T],$$

for any $u_0 \in V_2 = H^2(\Omega) \cap H_0^1(\Omega)$ with $\|u_0\|_2 \leqslant R$, where $S_\mu(t)$ is the solution semigroup of (1.1)–(1.3).

Finally, we establish the existence of the global attractor $\mathcal{A}_\mu$ for the system and prove the upper semicontinuity of $\mathcal{A}_\mu$ at $\mu = 0$.

In case $\mu = 0$ (i.e., for the classical parabolic equation), if the initial data u_0 belongs, say, for instance, to $H_0^1(\Omega)$, then one can usually establish a $H^2(\Omega)$ decay estimate, which guarantees the asymptotic compactness of the solution semigroup in $H_0^1(\Omega)$. Unfortunately such an estimate can be hardly obtained for the nonclassical equation. This brings us some difficulty in establishing the existence of the attractors. In the present work we will try to overcome this difficulty by developing some techniques in [8], etc., which are based on the noncompactness measure theory and show that the solution semigroup $S_\mu(t)$ of the system (1.1)–(1.3) has a global attractor $\mathcal{A}_\mu$ in the topology of $H^2(\Omega)$. We also show the upper semicontinuity of the attractors $\mathcal{A}_\mu$ as $\mu \to 0$. More precisely, we will prove that $d(\mathcal{A}_\mu, \mathcal{A}_0) \to 0$ as $\mu \to 0$, where $d(\cdot,\cdot)$ is the semi-Hausdorff distance in $H_0^1(\Omega)$.

We need the following assumptions on g:

g is C^2 function from $\mathbb{R}^1$ to $\mathbb{R}^1$, and

(G1) there exists $l > 0$ such that

$$g'(s) \geqslant -l, \quad \forall s \in \mathbb{R};$$

(G2) there exists $\kappa_1 > 0$ such that

$$\left|g'(s)\right| \leqslant \kappa_1\left(1 + |s|^\gamma\right), \quad \forall s \in \mathbb{R},$$

with $0 \leqslant \gamma < \infty$ when $n = 1, 2$, and $0 \leqslant \gamma \leqslant \frac{2}{n-2}$ when $n \geqslant 3$;

(G3) we denote by $G(s)$ the primitive of $g(s)$,

$$G(s) = \int_0^s g(r)\,\mathrm{d}r.$$

Then

$$\liminf_{|s|\to\infty} G(s)/s^2 \geqslant 0;$$

(G4) there exists $\kappa_2 > 0$ such that

$$\liminf_{|s|\to\infty} \frac{sg(s) - \kappa_2 G(s)}{s^2} \geqslant 0.$$

A typical function in applications is $g(u) = au^3 - bu$ with $a, b > 0$. It is easy to check that this function satisfies all the conditions (G1)–(G4). We infer from (G3) and (G4) that for any $\delta > 0$ there exist positive constants C_δ, C'_δ such that

$$G(s) + \delta s^2 \geqslant -C_\delta, \quad \forall s \in \mathbb{R}, \tag{1.4}$$

$$sg(s) - \kappa_2 G(s) + \delta s^2 \geqslant C'_\delta, \quad \forall s \in \mathbb{R}. \tag{1.5}$$

This paper is organized as follows. In Section 2 we discuss existence of solutions for (1.1)–(1.3). In Section 3 we establish some uniform decay estimates for the solutions of the system. Section 4 is concerned with the continuity of the solutions as $\mu \to 0$. In Section 5 we establish the existence of the global attractor $\mathcal{A}_\mu$ and show the upper-continuity of $\mathcal{A}_\mu$ at $\mu = 0$.

2. Mathematical setting and the existence of solutions

In this section we give an abstract form and state the existence results for the problem (1.1)–(1.3).

Let $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, and $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and norm of H, respectively. Denote by $((\cdot,\cdot))$ and $[\cdot,\cdot]$ the inner products in V_1 and V_2, respectively,

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall u, v \in V_1,$$

$$[u,v] = \int_\Omega \Delta u \Delta v \, \mathrm{d}x, \quad \forall u, v \in V_2.$$

Let $\|\cdot\|_s$ be the corresponding norm of V_s $(s = 1, 2)$. It is well known that the norm $\|\cdot\|_s$ is equivalent to the usual one of V_s.

Define the operator A_1 on V_1 as follows: for any $u \in V_1$, $A_1 u \in V_1'$, and

$$\langle A_1 u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall v \in V_1,$$

where $V_1' = H_0^{-1}(\Omega)$ is the dual space of V_1.

The operator A_2 on V_2 is defined simply by setting $A_2 = -\Delta$.

For $u \in V_s$, we also define $g(u) \in V_s'$ by

$$\langle g(u), v\rangle = \int_\Omega g(u) v \, \mathrm{d}x, \quad \forall v \in V_s,$$

where $\langle\cdot,\cdot\rangle$ is the dual between V_s and V_s'. Then the problem (1.1)–(1.3) can be formulated into an abstract equation in V_s:

$$u_t + \mu A_s u_t + A_s u + g(u) = f, \quad u(0) = u_0. \tag{2.1}$$

Consider the case $\mu > 0$. By the basic theory of second-order PDEs we know that the operator $I + \mu A_s$, where I is the identity operator, is an isomorphism from V_s to V_s'. (Note that $V_2' = H$.) Now we can reformulate Eq. (2.1) as

$$u_t + \mathcal{L}_s(\mu) u + \tilde{g}(u) = \tilde{f}, \quad u(0) = u_0, \tag{2.2}$$

where

$$\mathcal{L}_s(\mu) = (I + \mu A_s)^{-1} A_s, \qquad \tilde{g} = (I + \mu A_s)^{-1} g, \quad \text{and} \quad \tilde{f} = (I + \mu A_s)^{-1} f.$$

Clearly $\mathcal{L}_s(\mu)$ maps V_s into itself. We observe that

$$\mathcal{L}_s(\mu) = \frac{1}{\mu}\big(I - (I + \mu A_s)^{-1}\big).$$

Therefore $\mathcal{L}_s(\mu)$ is a bounded linear operator on V_s.

Concerning the operator g, we have

Lemma 2.1. [13] *The operator g is locally Lipschitz from V_1 to H. More precisely, there exists a constant C_g such that*

$$\big|g(u) - g(v)\big| \leqslant C_g\big(1 + \|u\|_1^\gamma + \|v\|_1^\gamma\big)\|u - v\|_1, \quad \forall u, v \in V_1. \tag{2.3}$$

As a direct consequence of the above lemma, one concludes immediately that $g : V_s \to V_s'$ is locally Lipshitz. It then follows that $\tilde{g} := (I + \mu A_s)^{-1} g : V_s \to V_s$ is locally Lipshitz.

Note also that $\tilde{f} := (I + \mu A_s)^{-1} f \in V_s$ if $f \in V_s'$. Thanks to the basic theory of abstract ordinary differential equations in Banach spaces, we conclude immediately that the following existence result holds.

Theorem 2.2. *Let $\mu > 0$. Assume that $f \in V_s'$ $(s = 1, 2)$. Then for each $u_0 \in V_s$ the system (1.1)–(1.3) has on some interval $[0, \tau)$ a unique solution $u = u(t) = u(t; u_0)$ with*

$$u \in C^1\big([0, \tau), V_s\big),$$

and for each t fixed, u is continuous in u_0.

In case $\mu = 0$, as far as the existence of solutions for (1.1)–(1.3) is concerned, we have

Theorem 2.3. *Let $\mu = 0$, and assume that $f \in H$. Then for each $u_0 \in V_1$, there exists a unique global solution $u = u(t) = u(t; u_0)$ of the system (1.1)–(1.3) which satisfies*

$$u \in C\big([0, T]; V_1\big) \cap L^2(0, T; V_2), \quad \forall T > 0. \tag{2.4}$$

If $u_0 \in V_2$, then

$$u \in C\big([0, T]; V_2\big) \cap C^1\big([0, T]; V_1'\big), \quad \forall T > 0. \tag{2.5}$$

Proof. In case $u_0 \in V_1$, the existence of a global solution for (1.1)–(1.3) is well known; see [13], etc. Here we give a proof in case $u_0 \in V_2$ for the reader's convenience.

Assume $u_0 \in V_2$. Then there is a global solution u satisfying (2.4). Using the Sobolev embeddings and the structure condition (G2), one can easily check that $g(u) \in C([0, T]; H)$ for any $T > 0$. Since $u \in L^2(0, T; V_2)$ and

$$u_t = \Delta u - g(u) + f,$$

it is clear that $u_t \in L^2(0, T; H)$ for any $T > 0$. Now let $T > 0$. Then for any $v \in V_1$, we have

$$\begin{aligned}\left| \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\big(g(u)\big) v \,\mathrm{d}x \right| &= \left| \int_\Omega g'(u) u_t v \,\mathrm{d}x \right| \\ &\leqslant \left(\int_\Omega \big|g'(u)\big|^n \,\mathrm{d}x \right)^{1/n} |u_t| \|v\|_{L^{2n/(n-2)}(\Omega)} \\ &\leqslant C\left(\int_\Omega \big(1 + |u|^\gamma\big)^n \,\mathrm{d}x \right)^{1/n} |u_t| \|v\|_1.\end{aligned}$$

Since $\gamma n = 2n/(n-2)$ and $u \in C([0, T]; V_1)$, by the Sobolev embedding $V_1 \subset L^{2n/(n-2)}(\Omega)$ we conclude immediately that for some $C_T > 0$,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} g(u) \right\|_{V_1'} \leqslant C_T |u_t|, \quad \forall t \in [0, T],$$

which implies that $\frac{\mathrm{d}}{\mathrm{d}t} g(u) \in L^2(0, T; V_1')$. Thanks to the classical regularity results (see [11, Theorem 7.9]), we deduce that $u \in C([0, T]; V_2) \cap C^1([0, T]; V_1')$. The proof is complete. □

Remark 2.4. In case $\mu = 0$, we can show that for each $u_0 \in V_2$, the system (1.1)–(1.3) has a unique strong solution $u \in C([0, T]; V_2)$. Unfortunately, we cannot obtain the continuity of the mapping $u_0 \to u(t)$ for fixed t. In spite of this difficulty, we will still establish the existence of the global attractor in V_2 for the system.

3. Uniform decay estimates

In this section we establish some (a priori) uniform decay estimates for the solution u of (1.1)–(1.3). These estimates in turn imply the local solution u we obtained for the system in Section 2 globally exists.

It should be pointed out that some computations in the following argument are not reasonable, as the solution u of the system (1.1)–(1.3) may not possess sufficient regularities, especially in the case $\mu = 0$. However, they can be justified by considering the Galerkin approximations u_m of u, which usually take the form

$$u_m(t) = \sum_{k=1}^{m} g_{m,k}(t)\omega_k$$

and solve some ordinary differential equations, where w_k is the kth eigenvector of the Laplace operator $-\Delta$ with respect to the homogeneous Dirichlet boundary condition. Since u_m is sufficiently regular, all the computations can be performed on u_m rigorously, and hence we know that the estimates for u in the following lemmas hold for all u_m with the constants in the estimates being independent of m. Finally, we obtain the estimates for u by passing to the limit in the estimates for u_m.

Let C and R be two positive constants. If C depends on R, then we will point out this dependence explicitly by writing C as $C(R)$. Otherwise, C and R are independent. This convention will be used throughout the following argument. We also note that all the constants appearing in the following argument are independent of $\mu \in [0, 1]$.

Theorem 3.1. *Assume $f \in H$. Then for any $R > 0$, there exist positive constants $E_1(R)$, ρ_1 and $t_1(R)$ such that for any solution u of problem* (1.1)–(1.3)

$$\|u\|_1 \leqslant E_1(R), \quad t \geqslant 0, \tag{3.1}$$

$$\|u\|_1 \leqslant \rho_1, \quad t \geqslant t_1(R) \tag{3.2}$$

provided $\|u_0\|_1 \leqslant R$, where $E_1(R)$, ρ_1 and $t_1(R)$ are independent of μ.

Proof. Let $R > 0$ and $\|u_0\| \leqslant R$. Let u be the solution of (1.1)–(1.3) with $u(0) = u_0$. Multiply (1.1) by $u_t + u$ and integrate over Ω,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|u|^2 + (\mu + 1)\|u\|_1^2\big) + \|u\|_1^2 + |u_t|^2 + \mu\|u_t\|_1^2 + \big(g(u), u + u_t\big)$$
$$= (f, u + u_t). \tag{3.3}$$

By (1.4) we see that for any $\delta > 0$, there is $C_\delta > 0$ such that

$$\int_\Omega G(v)\,\mathrm{d}x + \delta|v|_1^2 + C_\delta \geqslant 0, \quad \forall v \in V_1.$$

Since $\lambda_1|v|^2 \leqslant \|v\|_1^2$ ($\forall v \in V_1$), we have

$$\int_\Omega G(v)\,\mathrm{d}x + \frac{\delta}{\lambda_1}\|v\|_1^2 + C_\delta \geqslant 0, \quad \forall v \in V_1,$$

where λ_1 is the first eigenvalue of the Laplace operator Δ with domain V_2. Taking $\delta = \lambda_1/4$, one finds that

$$\int_\Omega G(v)\,\mathrm{d}x + \frac{1}{4}\|v\|_1^2 + k_1 \geqslant 0, \quad \forall v \in V_1 \tag{3.4}$$

for some $k_1 > 0$.

Similarly by (1.5), we deduce that there is a constant $k_2 > 0$ such that

$$\big(g(v), v\big) - \kappa_2 \int_\Omega G(v)\,\mathrm{d}x + \frac{1}{2}\|v\|_1^2 + k_2 \geqslant 0, \quad \forall v \in V_1. \tag{3.5}$$

Now we observe that

$$\big(g(u), u + u_t\big) = \big(g(u), u\big) + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u)\,\mathrm{d}x,$$

therefore by (3.5) we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2 + (\mu+1)\|u\|_1^2 + 2\int_\Omega G(u)\,\mathrm{d}x\right) + \|u\|_1^2 + |u_t|^2 + \mu\|u_t\|_1^2 \\
&\qquad + \kappa_2 \int_\Omega G(u)\,\mathrm{d}x \\
&\quad \leqslant \frac{1}{2}\|u\|_1^2 + k_2 + (f, u + u_t) \\
&\quad \leqslant \frac{1}{2}\|u\|_1^2 + k_2 + \frac{1}{2}|u_t|^2 + \frac{1}{2}|f|^2 + (f, u).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2 + (\mu+1)\|u\|_1^2 + 2\int_\Omega G(u)\,\mathrm{d}x\right) + \|u\|_1^2 + |u_t|^2 + 2\kappa_2\int_\Omega G(u)\,\mathrm{d}x \\
&\quad \leqslant 2k_2 + |f|^2 + 2(f, u).
\end{aligned} \tag{3.6}$$

Recalling that $\mu \in [0, 1]$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2 + (\mu+1)\|u\|_1^2 + 2\int_\Omega G(u)\,\mathrm{d}x + 2k_1\right) + \frac{1}{2}\lambda_1|u|^2 + \frac{1}{4}(\mu+1)\|u\|_1^2$$
$$+ 2\kappa_2\int_\Omega G(u)\,\mathrm{d}x + 2k_1$$
$$\leqslant 2k_2 + 2k_1 + |f|^2 + \frac{4}{\lambda_1}|f|^2 + \frac{1}{4}\lambda_1|u|^2.$$

Setting $\alpha_1 = \min\{\lambda_1/4, 1/4, \kappa_2\}$, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \alpha_1 y \leqslant C_1, \tag{3.7}$$

where

$$y = |u|^2 + (\mu+1)\|u\|_1^2 + 2\int_\Omega G(u)\,\mathrm{d}x + 2k_1$$

and

$$C_1 = 2k_1 + 2k_2 + |f|^2 + \frac{4}{\lambda_1}|f|^2.$$

By (3.4) it is easy to see that

$$y \geqslant \frac{1}{2}\|u\|_1^2.$$

Thanks to the classical Gronwall lemma, we infer from (3.7) that

$$y(t) \leqslant y(0)\exp(-\alpha_1 t) + \frac{C_1}{\alpha_1}, \quad \forall t \geqslant 0. \tag{3.8}$$

We assume that $\|u_0\|_1 \leqslant R$. It then follows from (3.8) that for some $E_1(R) > 0$ and $t_1(R) > 0$,

$$\|u\|_1 \leqslant E_1(R), \quad \forall t \geqslant 0, \tag{3.9}$$

$$\|u\|_1 \leqslant \left(\frac{2C_1}{\alpha_1}\right)^{1/2} + 1 := \rho_1, \quad \forall t \geqslant t_1(R). \tag{3.10}$$

The proof is complete. □

Theorem 3.2. *Assume that $f \in V_1$. Then for any $R > 0$, there exist positive constants $E_2(R)$, ρ_2 and $t_2(R)$ independent of μ such that for any solution u of problem* (1.1)–(1.3),

$$\|u\|_2 \leqslant E_2(R), \quad t \geqslant 0, \tag{3.11}$$

$$\|u\|_2 \leqslant \rho_2, \quad t \geqslant t_2(R), \tag{3.12}$$

provided $\|u_0\|_2 \leqslant R$.

Proof. Multiplying (1.1) by $-\Delta u$ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_1^2 + \mu\|u\|_2^2\big) + \|u\|_2^2 + \big(g(u), -\Delta u\big) = (f, -\Delta u). \tag{3.13}$$

By the structure condition (G1), we get

$$\big(g(u), -\Delta u\big) = \int_\Omega g'(u)\nabla u \cdot \nabla u \,\mathrm{d}x \geqslant -l\|u\|_1^2. \tag{3.14}$$

Noting that

$$(f, -\Delta u) \leqslant |f|^2 + \frac{1}{4}\|u\|_2^2,$$

we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_1^2 + \mu\|u\|_2^2\big) + \frac{1}{2}\lambda_1\|u\|_1^2 + \frac{1}{4}\|u\|_2^2 \leqslant l\|u\|_1^2 + |f|^2. \tag{3.15}$$

Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_1^2 + \mu\|u\|_2^2\big) + \frac{1}{2}\lambda_1\|u\|_1^2 + \frac{1}{4}\mu\|u\|_2^2 \leqslant lE_1^2(R) + |f|^2, \quad \forall t \geqslant 0, \tag{3.16}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_1^2 + \mu\|u\|_2^2\big) + \frac{1}{2}\lambda_1\|u\|_1^2 + \frac{1}{4}\mu\|u\|_2^2 \leqslant l\rho_1^2 + |f|^2, \quad \forall t \geqslant t_1(R). \tag{3.17}$$

Setting $\alpha_2 = \min\{\lambda_1, 1/2\}$ and using the classical Gronwall lemma, we obtain

$$\|u\|_1^2 + \mu\|u\|_2^2 \leqslant \big(\|u_0\|_1^2 + \mu\|u_0\|_2^2\big)e^{-\alpha_2 t} + \frac{2lE_1^2(R) + 2|f|^2}{\alpha_2}, \quad \forall t \geqslant 0, \tag{3.18}$$

$$\|u\|_1^2 + \mu\|u\|_2^2 \leqslant \big(\big\|u(t_1)\big\|_1^2 + \mu\big\|u(t_1)\big\|_2^2\big)e^{-\alpha_2(t-t_1)} + \frac{2l\rho_1^2 + 2|f|^2}{\alpha_2}, \quad \forall t \geqslant t_1(R). \tag{3.19}$$

Integrating (3.15) in t from t to $t+1$, one concludes that for some $C_2(R) > 0$ and $\rho_2^* > 0$ independent of μ,

$$\int_t^{t+1} \big\|u(s)\big\|_2^2 \,\mathrm{d}s \leqslant C_2(R), \quad \forall t \geqslant 0, \tag{3.20}$$

$$\int_t^{t+1} \big\|u(s)\big\|_2^2 \,\mathrm{d}s \leqslant \rho_2^*, \quad \forall t \geqslant t_1(R). \tag{3.21}$$

We now prove the results in Lemma 3.2. It can be obtained by multiplying (1.1) with $-\Delta u_t$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + \|u_t\|_1^2 + \mu\|u_t\|_2^2 + \big(g(u), -\Delta u_t\big) = (f, -\Delta u_t). \tag{3.22}$$

By (G2) we get

$$
\begin{aligned}
&\big|\big(g(u), -\Delta u_t\big)\big| \\
&\quad\leqslant \int_\Omega \big|g'(u)\big| |\nabla u||\nabla u_t| \,\mathrm{d}x \\
&\quad\leqslant \kappa_1 \int_\Omega |u|^\gamma |\nabla u||\nabla u_t| \,\mathrm{d}x + \kappa_1 \int_\Omega |\nabla u||\nabla u_t| \,\mathrm{d}x \\
&\quad\leqslant \kappa_1 \bigg(\int_\Omega |u|^{2\gamma} |\nabla u|^2 \,\mathrm{d}x \bigg)^{1/2} \|u_t\|_1 + \kappa_1 \|u\|_1 \|u_t\| \\
&\quad\leqslant \begin{cases} \kappa_1 \big(\int_\Omega |u|^{2\gamma q} \,\mathrm{d}x\big)^{1/(2q)} \big(\int_\Omega |\nabla u|^{2p} \,\mathrm{d}x\big)^{1/(2p)} \|u_t\|_1 + \kappa_1 \|u\|_1 \|u_t\|_1, \\ \quad n = 1, 2, \\ \kappa_1 \big(\int_\Omega |u|^{\gamma n} \,\mathrm{d}x\big)^{1/n} \big(\int_\Omega |\nabla u|^{2n/(n-2)} \,\mathrm{d}x\big)^{(n-2)/(2n)} \|u_t\|_1 + \kappa_1 \|u\|_1 \|u_t\|_1, \\ \quad n \geqslant 3, \end{cases} \\
&\quad\leqslant \kappa_1 C_2^2 \|u\|_1^\gamma \|u\|_2 \|u_t\|_1 + \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 \|u\|_1^2, \qquad (3.23)
\end{aligned}
$$

where $p, q > 0$ are conjugate ($1/p + 1/q = 1$), and C_2 is a positive constant satisfying

$$
C_2 \|u\|_1^\gamma \geqslant \begin{cases} \big(\int_\Omega |u|^{2\gamma q} \,\mathrm{d}x\big)^{1/(2q)}, & n = 1, 2, \\ \big(\int_\Omega |u|^{\gamma n} \,\mathrm{d}x\big)^{1/n}, & n \geqslant 3, \end{cases}
$$

$$
C_2 \|u\|_2 \geqslant \begin{cases} \big(\int_\Omega |\nabla u|^{2p} \,\mathrm{d}x\big)^{1/(2p)}, & n = 1, 2, \\ \big(\int_\Omega |\nabla u|^{2n/(n-2)} \,\mathrm{d}x\big)^{(n-2)/(2n)}, & n \geqslant 3. \end{cases}
$$

Thus we have

$$
\begin{aligned}
\big|\big(g(u), -\Delta u_t\big)\big| &\leqslant \kappa_1 C_2^2 E_1^\gamma(R) \|u\|_2 \|u_t\|_1 + \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 E_1^2(R) \\
&\leqslant \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 C_2^4 E_1^{2\gamma}(R) \|u\|_2^2 + \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 E_1^2(R) \\
&\leqslant \frac{1}{4} \|u_t\|_1^2 + 2\kappa_1^2 C_2^4 E_1^{2\gamma}(R) \|u\|_2^2 + 2\kappa_1^2 E_1^2(R), \quad \forall t \geqslant 0, \qquad (3.24)
\end{aligned}
$$

and

$$
\begin{aligned}
\big|\big(g(u), -\Delta u_t\big)\big| &\leqslant \kappa_1 C_2^2 \rho_1^\gamma \|u\|_2 \|u_t\|_1 + \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 \rho_1^2 \\
&\leqslant \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 C_2^4 \rho_1^{2\gamma} \|u\|_2^2 + \frac{1}{8} \|u_t\|_1^2 + 2\kappa_1^2 \rho_1^2 \\
&\leqslant \frac{1}{4} \|u_t\|_1^2 + 2\kappa_1^2 C_2^4 \rho_1^{2\gamma} \|u\|_2^2 + 2\kappa_1^2 \rho_1^2, \quad \forall t \geqslant t_1(R). \qquad (3.25)
\end{aligned}
$$

Since

$$
(f, -\Delta u_t) \leqslant \|f\|_1^2 + \frac{1}{4} \|u_t\|_1^2,
$$

by (3.24) and (3.25) we easily deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2+\|u_t\|_1^2+2\mu\|u_t\|_2^2\leqslant C_3(R)\|u\|_2^2+C_4(R),\quad \forall t\geqslant 0, \tag{3.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2+\|u_t\|_1^2+2\mu\|u_t\|_2^2\leqslant C_5\|u\|_2^2+C_6,\quad \forall t\geqslant t_1(R), \tag{3.27}$$

where

$$C_3(R)=4\kappa_1^2C_2^4E_1^{2\gamma}(R),\qquad C_4(R)=4\kappa_1^2E_1^2(R)+2\|f\|_1^2,$$
$$C_5=4\kappa_1^2C_2^4\rho_1^{2\gamma},\qquad C_6=4\kappa_1^2\rho_1^2+2\|f\|_1^2.$$

Now thanks to the classical Gronwall lemma and the uniform Gronwall lemma (see [13]), we conclude that

$$\|u\|_2^2\leqslant\|u_0\|_2^2\exp\bigl(C_3(R)t\bigr)+\frac{C_4(R)}{C_3(R)},\quad \forall t\geqslant 0,$$
$$\bigl\|u(t)\bigr\|_2^2\leqslant\bigl(4l\rho_1^2+2\rho_1^2+4|f|^2+C_6\bigr)\exp(C_5),\quad t\geqslant t_1(R)+1,$$

which complete the proof of the desired results. □

4. Continuity of solutions as $\mu\to 0$

To understand the dynamics of the system as $\mu\to 0$, one of the most important steps is to discuss the continuous convergence of the solutions as $\mu\to 0$. Of course, this is also of independent interest. We have

Theorem 4.1. *Assume $f\in V_1$. Denote by $S_\mu(t)$ the solution semigroup of problem* (1.1)–(1.3). *Let $R,T>0$ be given arbitrary. Then there exists a positive constant $C=C(R,T)$ such that for any $u_0\in V_2$ with $\|u_0\|_2\leqslant R$,*

$$\bigl\|S_\mu(t)u_0-S_0(t)u_0\bigr\|_1\leqslant C\sqrt{\mu},\quad \forall t\in[0,T].$$

Proof. Let $u_0\in V_2$ with $\|u_0\|_2\leqslant R$, $u=S_\mu(t)u_0$.

Integrating (3.26) from 0 to T, one finds that

$$\int_0^T\|u\|_1^2\,\mathrm{d}t<C_T, \tag{4.1}$$

where $C_T>0$ is a constant depending only on T and R, etc.

For simplicity we write $v=S_0(t)u_0$. Let $w=u-v$. Then $w(0)=0$ and

$$w_t-\mu\Delta u_t-\Delta w+g(u)-g(v)=0. \tag{4.2}$$

Taking the scalar product of (4.2) with w_t in H, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_1^2+|w_t|^2-\mu(\Delta u_t,w_t)+\bigl(g(u)-g(v),w_t\bigr)=0. \tag{4.3}$$

By (2.3) we see that

$$\begin{aligned}\big|\big(g(u)-g(v),w_t\big)\big| &\leqslant \big|g(u)-g(v)\big||w_t| \leqslant C_g\big(1+\|u\|_1^{\gamma}+\|v\|_1^{\gamma}\big)\|w\|_1|w_t| \\ &\leqslant \frac{1}{2}|w_t|^2+\frac{1}{2}C_g^2\big(1+2E_1^{\gamma}(R)\big)^2\|w\|_1^2, \end{aligned} \tag{4.4}$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_1^2 \leqslant \lambda\|w\|_1^2+\mu\big(\|u_t\|_1^2+\|w_t\|_1^2\big), \tag{4.5}$$

where $\lambda = C_g^2(1+2E_1^{\gamma}(R))^2$. Using the classical Gronwall lemma, we conclude that

$$\begin{aligned}\big\|w(t)\big\|_1^2 &\leqslant \big\|w(0)\big\|_1^2 e^{\lambda t}+\mu\int_0^t e^{\lambda(t-s)}\big(\big\|u_t(s)\big\|_1^2+\big\|w_t(s)\big\|_1^2\big)\,\mathrm{d}s \\ &= \mu\int_0^t e^{\lambda(t-s)}\big(\big\|u_t(s)\big\|_1^2+\big\|w_t(s)\big\|_1^2\big)\,\mathrm{d}s \\ &\leqslant \big(\text{by (4.1)}\big) \leqslant C_T'\mu, \quad \forall t\in[0,T], \end{aligned} \tag{4.6}$$

which completes the proof of the theorem. □

5. Global attractor in V_2

Now let us try to establish the existence of the global attractor in V_2 for the system (1.1)–(1.3). The main obstacle is that it is difficult to obtain a higher regularity estimate to guarantee the asymptotic compactness of the semigroup. We will overcome this difficulty by employing some techniques in [8], etc.

For convenience, we will use $\mathcal{B}_{V_s}(r)$ to denote the ball in V_s centered at 0 with radius r. $d_{V_s}(\cdot,\cdot)$ denotes the semi-Hausdorff distance in V_s,

$$d_{V_s}(X,Y)=\sup_{u\in X}\inf_{v\in Y}\|u-v\|_s$$

for any subsets X, Y of V_s.

First, by summarizing the results in [8], we have

Theorem 5.1. [8] *Let $S(t)$ $(t\geqslant 0)$ be a semigroup on a Hilbert space H. Assume that $S(t)$ satisfies the following dissipativity and compactness conditions*:

(1) *$S(t)$ has a bounded absorbing set $\mathcal{U}$*;
(2) *for any $\varepsilon>0$ and bounded subset B of H, there exist $t(B)>0$ and a finite-dimensional subspace H_1 of H such that $\{\|PS(t)B\|\}_{t\geqslant 0}$ is bounded, and*

$$\big\|(I-P)S(t)x\big\|\leqslant\varepsilon, \quad \textit{for } t\geqslant t(B), x\in B,$$

where $P: H\to H_1$ is the orthogonal projection.

Then $S(t)$ has a global attractor $\mathcal{A}$.

Now let us state and prove the following result concerning the compactness of g.

Lemma 5.2. *Assume $g \in C^2(\mathbf{R}^1; \mathbf{R}^1)$ satisfies* (G2)*; moreover, $g(0) = 0$. Then the corresponding mapping $g : V_2 \to V_1$ is continuously compact, i.e., g is continuous and maps a bounded subset of V_2 into a precompact subset of V_1.*

Proof. In case $n \leqslant 3$, the embedding $H^2(\Omega) \Subset L^\infty(\Omega)$ is compact, from which one can easily check the validity of the conclusion in the lemma.

Thus we only consider the case $n \geqslant 4$. We know that the following embeddings are compact:

$$H^2(\Omega) \Subset L^p(\Omega), \quad \forall p < \frac{2n}{n-4}, \tag{5.1}$$

$$H^2(\Omega) \Subset W^{1,p}(\Omega), \quad \forall p < \frac{2n}{n-2}. \tag{5.2}$$

Assume that $u \in V_2$. We first show that $g(u) \in H^1(\Omega)$. Indeed, we know that $g(u) \in L^2(\Omega)$. On the other hand,

$$\begin{aligned}
\left|\nabla g(u)\right| &= \left(\int_\Omega \left|g'(u)\right|^2 |\nabla u|^2 \,\mathrm{d}x\right)^{1/2} \\
&\leqslant \kappa_1 \left(\int_\Omega \left(1+|u|^\gamma\right)^2 |\nabla u|^2 \,\mathrm{d}x\right)^{1/2} \\
&\leqslant \kappa_1 \left(\int_\Omega \left(1+|u|^\gamma\right)^n \,\mathrm{d}x\right)^{1/n} \left(\int_\Omega |\nabla u|^{2n/(n-2)} \,\mathrm{d}x\right)^{(n-2)/2n}.
\end{aligned}$$

Thanks to (5.1) and (5.2), we deduce immediately from the above estimates that $g(u) \in H^1(\Omega)$.

Let $B_R = \mathcal{B}_{V_2}(R)$. We fix p with

$$\frac{n}{2} < p < \frac{n}{2} \cdot \frac{n-2}{n-4}.$$

Let q be the conjugate of p (i.e., $1/p + 1/q = 1$). Then

$$p' := 2p\gamma < \frac{2n}{n-4}, \quad q < \frac{n}{n-2}.$$

By (5.1) and (5.2), there is a sequence $u_k \in B_R$ such that u_k converges in both the spaces $L^{p'}(\Omega)$ and $W^{1,2q}(\Omega)$.

For simplicity we rewrite u_m and u_k as u and v, respectively. Then

$$\begin{aligned}
&\left(\int_\Omega \left|\nabla\left(g(u) - g(v)\right)\right|^2 \,\mathrm{d}x\right)^{1/2} \\
&\quad \leqslant \left(\int_\Omega \left|g'(u) - g'(v)\right|^2 |\nabla u|^2 \,\mathrm{d}x\right)^{1/2} + \left(\int_\Omega \left|g'(v)\right|^2 |\nabla u - \nabla v|^2 \,\mathrm{d}x\right)^{1/2} \\
&\quad := I_{mk}^1 + I_{mk}^2.
\end{aligned}$$

For I^1_{mk}, we have

$$I^1_{mk} \leqslant \left(\int_\Omega \left|g'(u) - g'(v)\right|^{2p}\,\mathrm{d}x\right)^{1/2p}\left(\int_\Omega |\nabla u|^{2q}\,\mathrm{d}x\right)^{1/2q}$$
$$\leqslant \big(\text{by (5.2)}\big) \leqslant C(R)\left\|g'(u) - g'(v)\right\|_{L^{2p}(\Omega)},$$

where $C(R) > 0$ is a constant depending only R and the embedding constants. Since

$$\left|g'(s)\right| \leqslant \kappa_1\big(1 + |s|^\gamma\big) = \kappa_1\big(1 + |s|^{p'/2p}\big),$$

thanks to a classical continuity result (see Chang [4, Chapter 1, Theorem 1.1]), we know that $g' : L^{p'}(\Omega) \to L^{2p}(\Omega)$ is continuous. It follows that $I^1_{mk} \to 0$ as $m, k \to +\infty$.

As for I^2_{mk}, we have

$$I^2_{mk} \leqslant \left(\int_\Omega \left|g'(v)\right|^{2p}\right)^{1/2p}\left(\int_\Omega |\nabla u - \nabla v|^{2q}\right)^{1/2q}$$
$$\leqslant c_1\left(\int_\Omega \big(1 + |v|^{2p\gamma}\big)\right)^{1/2p}\left(\int_\Omega |\nabla u - \nabla v|^{2q}\right)^{1/2q}, \tag{5.3}$$

where $c_1 > 0$ is a constant depending only on κ_1, from which one concludes immediately that $I^2_{mk} \to 0$ as $m, k \to +\infty$.

Hence $\nabla g(u_k)$ is convergent in $L^2(\Omega)$. Similarly one can check that $g(u_k)$ converges in $L^2(\Omega)$. Therefore we conclude that $g(u_k)$ converges in $H^1(\Omega)$. □

Lemma 5.3. *Assume $g \in C^2(\mathbf{R}^1; \mathbf{R}^1)$ satisfies* (G2); *moreover, $g(0) = 0$. Let λ_m and ω_m be the mth eigenvalue and eigenvector, respectively, of the operator $-\Delta$ with respect to the homogeneous Dirichlet boundary condition, and $\lambda_m \to +\infty$. Let $H_m = \operatorname{span}\{\omega_1, \dots, \omega_m\}$.*

Let B be a bounded subset of V_2. Then for any $\varepsilon > 0$, there exists some m_0 such that when $m > m_0$,

$$\left\|(I - P_m)g(u)\right\|_1 < \frac{\varepsilon}{2}, \quad \forall u \in B, \tag{5.4}$$

where $P_m : H \to H_m$ is the orthogonal projection.

Proof. Note that $g(u) \in V_1$ for $u \in V_2$. By Lemma 5.2, we see that g maps bounded subsets of V_2 into precompact subsets of V_1.

Let B be a bounded subset of V_2, and $\varepsilon > 0$ be given arbitrary. Since $g(B)$ is precompact in V_1, there is a finite number of elements $v_1, v_2, \dots, v_k \in g(B)$ such that

$$g(B) \subset \bigcup_{1 \leqslant i \leqslant k} \mathcal{B}_{V_1}(v_i, \varepsilon/2). \tag{5.5}$$

We take $m_0 > 0$ sufficiently large so that when $m > m_0$,

$$\left\|(I - P_m)v_i\right\|_1 < \frac{\varepsilon}{2}, \quad \text{for all } 1 \leqslant i \leqslant k.$$

Then by (5.5) one concludes immediately that (5.4) holds. □

Theorem 5.4. *Assume that $g \in C^2(\mathbf{R}^1; \mathbf{R}^1)$ and satisfies the structure conditions* (G1)–(G4) *with $g(0) = 0$, $f \in V_1$. Then the semigroup $S_\mu(t)$ generated by the problem* (1.1)–(1.3) *possesses in V_2 a global attractor $\mathcal{A}_\mu$, which attracts all bounded subsets of V_2 in the topology of V_2.*

Proof. We first show that the semigroup $S_\mu(t)$ satisfies condition (2) in Theorem 5.1.

Let λ_m, ω_m, P_m and H_m be as in Lemma 5.3, and B be a bounded subset of V_2. Let $\varepsilon > 0$ be given arbitrary.

Set $u = u(t) = S_\mu(t)u_0$. By Theorem 3.2, we see that there is $t(B) > 0$ independent of μ such that

$$\|u\|_2 \leqslant \rho_2, \quad \forall t \geqslant t(B),\ u_0 \in B. \tag{5.6}$$

By Lemma 5.3 and (5.6), we can fix m sufficiently large (m is independent of μ) so that

$$\big\|(I - P_m)g(u)\big\| < \frac{\varepsilon}{2}, \quad \forall t \geqslant t(B),\ u_0 \in B. \tag{5.7}$$

Since $f \in V_1$, it can be also assumed that m is sufficiently large such that

$$\big|(I - P_m)f\big| + \big\|(I - P_m)f\big\|_1 < \frac{\varepsilon}{2}. \tag{5.8}$$

We decompose u into two parts,

$$u = P_m u + (I - P_m)u := u^1 + u^2. \tag{5.9}$$

For the sake of clarity, we write u^2 as v. Multiplying (1.1) by $-\Delta v - \Delta v_t$ and integrating over Ω, one finds that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v\|_1^2 + \mu\|v\|_2^2 + \|v\|_2^2\big) + \|v_t\|_1^2 + \mu\|v_t\|_2^2 + \|v\|_2^2 \\
&\quad = \big(g(u), \Delta v + \Delta v_t\big) + (f_2, -\Delta v - \Delta v_t) \\
&\quad = -\big(\big((I - P_m)g(u), v + v_t\big)\big) + (f_2, -\Delta v) + ((f_2, v_t)),
\end{aligned}$$

where $f_2 = (I - P_m)f$. In view of (5.7) and (5.8), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v\|_1^2 + (\mu + 1)\|v\|_2^2\big) + \|v_t\|_1^2 + \mu\|v_t\|_2^2 + \|v\|_2^2 \\
&\quad \leqslant \frac{\varepsilon}{2}\|v\|_1 + \frac{\varepsilon}{2}\|v_t\|_1 + \frac{\varepsilon}{2}\|v\|_2 + \frac{\varepsilon}{2}\|v_t\|_1 \\
&\quad \leqslant \frac{\varepsilon}{2}\rho_1 + \frac{1}{2}\|v_t\|_1^2 + \frac{1}{2}\|v\|_2^2 + \frac{5}{8}\varepsilon^2, \quad \forall t \geqslant t(B),
\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v\|_1^2 + (\mu + 1)\|v\|_2^2\big) + \|v\|_2^2 \leqslant \left(\rho_1 + \frac{5}{4}\varepsilon\right)\varepsilon, \quad \forall t \geqslant t(B).$$

We assume that $\|v\|_1 \leqslant c_2\|v\|_2$ for $v \in V_2$. Then

$$\|v\|_2 = \frac{1}{2}\|v\|_2 + \|v\|_2 \geqslant \frac{1}{4}(\mu + 1)\|v\|_2 + \frac{1}{2c_0}\|v\|_1 \geqslant \alpha\big(\|v\|_1^2 + (\mu + 1)\|v\|_2^2\big),$$

where $\alpha = \min\{1/4, 1/2c_0\}$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v\|_1^2 + (\mu+1)\|v\|_2^2\big) + \alpha\big(\|v\|_1^2 + (\mu+1)\|v\|_2^2\big) \leqslant \left(\rho_1 + \frac{5}{4}\varepsilon\right)\varepsilon, \quad \forall t \geqslant t(B).$$

By the classical Gronwall lemma we then get

$$\begin{aligned}
&\big\|v(t)\big\|_1^2 + (\mu+1)\big\|v(t)\big\|_2^2 \\
&\quad \leqslant \big(\big\|v\big(t(B)\big)\big\|_1^2 + (\mu+1)\big\|v\big(t(B)\big)\big\|_2^2\big)e^{-\alpha(t-t(B))} + \frac{(\rho_1 + 5\varepsilon/4)\varepsilon}{\alpha} \\
&\quad \leqslant \big(\rho_1^2 + 2\rho_2^2\big)e^{-\alpha(t-t(B))} + \frac{\rho_1 + 5\varepsilon/4}{\alpha}\varepsilon, \quad \text{for } t \geqslant t(B).
\end{aligned}$$

Hence

$$\big\|v(t)\big\|_1^2 + (\mu+1)\big\|v(t)\big\|_2^2 \leqslant \left(1 + \frac{(\rho_1+1)}{\alpha}\right)\varepsilon, \quad \text{as } t \geqslant t(B) + \frac{1}{\alpha}\log\frac{(\rho_1^2 + 2\rho_2^2)}{\varepsilon}, \tag{5.10}$$

where we have assumed that $5\varepsilon/4 \leqslant 1$.

Now we divide the argument into two cases.

Case 1. $\mu > 0$. In this case we know that $S_\mu(t): V_2 \to V_2$ is continuous, so the existence of the global attractor $\mathcal{A}_\mu$ follows immediately from Theorem 5.1.

Case 2. $\mu = 0$. In this case the problem reduces to a classical one which has been extensively studied in the literature. However, since the continuity in V_2 of the semigroup under our consideration remains unknown, we still need to give a proof for the reader's convenience.

Let α be the noncompactness measure in V_2, which is defined by

$$\alpha(B) = \inf\{\delta\colon\ B \text{ admits a finite cover by subsets of } V_2 \text{ whose diameter} < \delta\}.$$

Then since $S_0(t)$ satisfies the condition (2) in Theorem 5.1, we infer from [8] that for any bounded subset B of V_2 that $\alpha(\bigcup_{t\geqslant\tau} S_0(t)B) \to 0$ as $s \to +\infty$. Now set

$$\mathcal{A}_0 = \bigcap_{\tau\geqslant 0} \mathrm{Cl}_{V_2}\left(\bigcup_{t\geqslant\tau} S_0(t)\mathcal{B}_{V_2}(\rho_2)\right),$$

where we use $\mathrm{Cl}_{V_s}(K)$ to denote the closure of K in V_s, and ρ_2 is the constant in (3.12). Then by [8, Lemma 2.5], we know that $\mathcal{A}_0$ is a nonempty compact subset of V_2. By a very standard argument (see [7,8], etc.), we can show that $\mathcal{A}_0$ attracts each bounded subset of V_2. To show that $\mathcal{A}_0$ is the global attractor of the system, there remains to check that it is invariant under the semigroup $S_0(t)$. This can be done by verifying that $\mathcal{A}_0$ is precisely the global attractor of $S_0(t)$ in less regular spaces.

First, it is easy to verify that $S_0(t): V_1 \to V_1$ is continuous. Secondly, using the uniform Gronwall lemma we can show that there is $\rho_2' > 0$ such that for any bounded subset B of V_1, there exists $t_2 = t_2(B) > 0$ so that

$$\big\|S_0(t)u\big\|_2 < \rho_2', \quad \forall t \geqslant t_2,\ u \in B. \tag{5.11}$$

(See also [13].) Note that (5.11) implies the asymptotic compactness of $S_0(t)$. Hence, by the basic theory of the existence of global attractors (see [2,6,11,12], etc.), we know that $S_0(t)$ possesses a unique global attractor $\mathcal{A}_0^*$ in V_1. Clearly $\mathcal{A}_0^* \subset \mathcal{B}_{V_2}(\rho_2')$. We want to show that $\mathcal{A}_0 = \mathcal{A}_0^*$, and hence $\mathcal{A}_0$ is invariant under $S_0(t)$.

Since $\mathcal{A}_0$ attracts $\mathcal{A}_0^*$, by the invariance of $\mathcal{A}_0^*$ we deduce that $\mathcal{A}_0^* \subset \mathcal{A}_0$. On the other hand, by the definition of $\mathcal{A}_0$ we clearly have

$$\mathcal{A}_0 = \bigcap_{\tau \geqslant 0} \mathrm{Cl}_{V_2}\left(\bigcup_{t \geqslant \tau} S_0(t)\mathcal{U}\right) \subset \bigcap_{\tau \geqslant 0} \mathrm{Cl}_{V_1}\left(\bigcup_{t \geqslant \tau} S_0(t)\mathcal{U}\right) \subset \mathcal{A}_0^*,$$

which completes the proof of what we expected.

The proof is completed. □

Remark 5.5. We point out that some of the computations in the above argument are not reasonable, as v may not possess sufficient regularities. However, they can be justified by considering the Galerkin approximations u_k $(k = 1, 2, \ldots)$ of u.

First, we know that all the estimates for u obtained in Section 3 hold true if u therein is replaced by any u_k. Then corresponding to the decomposition in (5.9), we consider the Galerkin approximations u_{m+k} of u, for which we have

$$u_{m+k}^2 = (I - P_m)u_{m+k} = \sum_{i=m+1}^{m+k} g_{m+k}^i(t)\omega_i.$$

Clearly all the computations for $v = u^2 = (I - P_m)u$ can be performed on u_{m+k}^2 rigorously. Therefore the estimates in (5.10) holds true if u^2 is replaced by u_{m+k}^2; moreover, the constants in the estimates do not depend on k. Finally, since $\|u_{m+k}^2\|_2 \leqslant \|u_{m+k}\|_2$, we see that all the estimates in Section 3 for u remain valid for u_{m+k}^2. This enable us to pass to the limit to find that u_{m+k}^2 converges in suitable spaces with corresponding topologies to $\tilde{u}^2$ as $k \to \infty$. On the other hand, since $u_{m+k} \to u$ as $k \to \infty$ in the same topologies, and $u_{m+k} = u_{m+k}^1 + u_{m+k}^2$, where $u_{m+k}^1 = P_m u_{m+k} \in H_m$, one easily understands that $\tilde{u}^2$ is precisely u^2. Hence, the estimate (5.10) holds for u^2.

In the remaining part of this section, we discuss the upper semicontinuity of the attractors $\mathcal{A}_\mu$ at $\mu = 0$ in the topology of V_1. The main result is contained in the following theorem.

Theorem 5.6. *The global attractor $\mathcal{A}_\mu$ of the problem* (1.1)–(1.3) *is upper semicontinuous in μ at $\mu = 0$ in the topology of V_1, i.e.,*

$$d_{V_1}(\mathcal{A}_\mu, \mathcal{A}_0) = 0, \quad \text{as } \mu \to 0.$$

Proof. Let $\varepsilon > 0$ be given arbitrary. Since $\mathcal{A}_0$ attracts $B = \bar{\mathcal{B}}_{V_2}(\rho_2)$ in the topology of V_2, there exists $T > 0$ such that

$$d_{V_1}\big(S_0(T)B, \mathcal{A}_0\big) < \varepsilon/2.$$

Note that $\mathcal{A}_\mu \subset B$ for any μ. Therefore

$$d_{V_1}\big(S_0(T)\mathcal{A}_\mu, \mathcal{A}_0\big) < \varepsilon/2, \quad \forall \mu \in [0, 1].$$

Now we have

$$\begin{aligned} d_{V_1}(\mathcal{A}_\mu, \mathcal{A}_0) &\leqslant d_{V_1}\big(\mathcal{A}_\mu, S_0(T)\mathcal{A}_\mu\big) + d_{V_1}\big(S_0(T)\mathcal{A}_\mu, \mathcal{A}_0\big) \\ &\leqslant d_{V_1}\big(\mathcal{A}_\mu, S_0(T)\mathcal{A}_\mu\big) + \varepsilon/2 \\ &= d_{V_1}\big(S_\mu(T)\mathcal{A}_\mu, S_0(T)\mathcal{A}_\mu\big) + \varepsilon/2 \\ &\leqslant \text{(by Theorem 4.1)} \leqslant C_T\sqrt{\mu} + \varepsilon/2, \end{aligned}$$

where C_T is a constant depending only on T and ρ_2, etc. Take $\mu_0 = (\varepsilon/2C_T)^2$. Then

$$d_{V_1}(\mathcal{A}_\mu, \mathcal{A}_0) < \varepsilon$$

provided $\mu < \mu_0$. The proof is complete. □

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References

[1] E.C. Aifantis, On the problem of diffusion in solids, Acta Mech. 37 (1980) 265–296.
[2] A.V. Babin, M.I. Vishik, Attractors of Evolution Equation, North-Holland, Amsterdam, 1992.
[3] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1665.
[4] K.C. Chang, Critical Point Theory and Its Applications, Shanghai Science and Technology Press, Shanghai, 1986 (in Chinese).
[5] C.M. Elliott, A.M. Stuart, Viscous Cahn–Hilliard equation, J. Differential Equations 128 (1996) 387–415.
[6] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., Providence, RI, 1998.
[7] A. Haraux, Attractors of asymptotically compact processes and applications to nonlinear partial differential equations, Comm. Partial Differential Equations 13 (1988) 1383–1414.
[8] Q.F. Ma, S.H. Wang, C.K. Zhong, Necessary and sufficient conditions for the existence of global attractors for semigroups and applications, Indiana Univ. Math. J. 51 (2002) 1541–1559.
[9] D.S. Li, Z.X. Wang, Z.L. Wang, Global existence, uniqueness and long-time behaviour for a nonclassical diffusion equations, Acta Math. Appl. Sinica 21 (1998) 267–276 (in Chinese).
[10] J.C. Peter, M.E. Gurtin, On a theory of heat conduction involving two temperatures, Z. Angew. Math. Phys. 19 (1968) 614–627.
[11] J.C. Robinson, Infinite-Dimensional Dynamical Systems, Cambridge Univ. Press, Cambridge, 2001.
[12] G.R. Sell, Y. Yuncheng, Dynamics of Evolutionary Equations, Springer, New York, 2002.
[13] R. Teman, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, second ed., Springer, New York, 1997.
[14] T.W. Ting, Certain nonsteady flows of second-order fluids, Arch. Ration. Mech. Anal. 14 (1963) 1–26.
[15] C. Truesdell, W. Noll, The Nonlinear Field Theories of Mechanics, Encyclopedia of Physics, Springer, Berlin, 1995.
[16] Z.Y. Yin, Well-posedness, blow-up, and global existence for an integrable shallow water equation, Discrete Contin. Dynam. Systems 11 (2004) 393–411.
[17] L.H. Zhang, Long-time asymptotic behaviors of solutions of N-dimensional dissipative partial differential equations, Discrete Contin. Dynam. Systems 8 (2002) 1025–1042.