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Orthogonality of the Jacobi polynomials with negative integer parameters

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Abstract

It is well known that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal with respect to a quasi-definite linear functional whenever α, β, and $\alpha+\beta+1$ are not negative integer numbers. Recently, Sobolev orthogonality for these polynomials has been obtained for α a negative integer and β not a negative integer and also for the case $\alpha=\beta$ negative integer numbers.

In this paper, we give a Sobolev orthogonality for the Jacobi polynomials in the remainder cases.

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1. Introduction

Sobolev orthogonality has played an important role in the study of orthogonality for some families of classical orthogonal polynomials with non-classical parameters. For Laguerre polynomials this problem has been completely solved. In [4], Kwon and Littlejohn established the orthogonality of the generalized Laguerre polynomials $\{L_n^{(-k)}\}_{n\geqslant 0}$, $k\geqslant 1$, with respect to a Sobolev inner product and later Pérez and Piñar gave an unified approach to the orthogonality of the generalized Laguerre polynomials $\{L_n^{(\alpha)}\}_{n\geqslant 0}$, for any real value of the parameter α, by proving their orthogonality with

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respect to a Sobolev non-diagonal inner product, see [6]. However, this is not the situation for Jacobi polynomials.

It is well known (see [3]) that the monic Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_n$ satisfy, for any real value of α and β, the three-term recurrence relation

$$\begin{aligned} &xP_n^{(\alpha,\beta)}(x) = P_{n+1}^{(\alpha,\beta)}(x) + c_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x) + \lambda_n^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(x), \quad n \geqslant 0, \\ &P_{-1}^{(\alpha,\beta)}(x) = 0, \quad P_0^{(\alpha,\beta)}(x) = 1 \end{aligned} \tag{1.1}$$

where

$$c_n^{(\alpha,\beta)} = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$\lambda_n^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)}.$$

This formula holds for every n, except at most for two values depending on α and β.

Whenever α, β and $\alpha+\beta+1$ are not negative integers, we have $\lambda_n^{(\alpha,\beta)} \neq 0$ for all $n \geqslant 1$ and Favard's theorem (see [3]) ensures that the sequence $\{P_n^{(\alpha,\beta)}\}_n$ is orthogonal with respect to a quasi-definite linear functional. Besides, if $\alpha, \beta > -1$, the functional is positive definite and the polynomials are orthogonal with respect to the weight $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$. Observe that standard orthogonality results cannot be deduced from Favard's theorem when either α, or β, or $\alpha+\beta+1$ are negative integer numbers, since $\lambda_n^{(\alpha,\beta)}$ vanishes for some values of n.

However, in the cases when α is a negative integer, β being not a negative integer and when $\alpha=\beta$ are negative integer numbers, the Sobolev orthogonality for these polynomials has been obtained in [1] and [2], respectively. More precisely, in [1] it has been established the orthogonality of the monic Jacobi polynomials $\{P_n^{(-N,\beta)}\}_{n\geqslant 0}$, for N a positive integer number and β not a negative integer, with respect to a Sobolev bilinear form defined as follows:

$$(f,g)_S = F(1)\mathbf{A}G(1)^{\mathrm{T}} + \int_{-1}^{1} f^{(N)}(x)g^{(N)}(x)(1+x)^{\beta+N}\,\mathrm{d}x,$$

where $F(1) = (f(1), f'(1), \ldots, f^{(N-1)}(1))$ and $\mathbf{A}$ is a symmetric $N \times N$ real matrix such that $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^{\mathrm{T}}$, with $\mathbf{D}$ any regular diagonal matrix and $\mathbf{Q}$ the matrix whose entries are the derivatives of the Jacobi polynomials $P_n^{(-N,\beta)}$ at the point 1.

Otherwise, in [2] it is shown that the generalized Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_n$, $N \geqslant 1$, are orthogonal with respect to the inner product

$$(f,g)_S = (F(1)|F(-1))\mathbf{A}(G(1)|G(-1))^{\mathrm{T}} + \int_{-1}^{1} f^{(2N)}(x)g^{(2N)}(x)(1-x^2)^N\,\mathrm{d}x,$$

where $\mathbf{A}$ is a symmetric positive definite matrix defined by $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^{\mathrm{T}}$, with $\mathbf{D}$ any diagonal positive definite matrix of order $2N$ and $\mathbf{Q}$ the regular matrix, $\mathbf{Q} = (\mathbf{Q}(1)|\mathbf{Q}(-1)) \in \mathscr{M}_{2N}(\mathbb{R})$, with

$$\mathbf{Q}(1) = (D^k C_n^{(-N+1/2)}(1))_{\substack{n=0,\ldots,2N-1,\\ k=0,\ldots,N-1}} \quad \mathbf{Q}(-1) = (D^k C_n^{(-N+1/2)}(-1))_{\substack{n=0,\ldots,2N-1,\\ k=0,\ldots,N-1}}.$$

The particular case of the monic Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n\geqslant 0}$ had been previously considered in [5].

For the remainder cases, nothing is known about orthogonality of Jacobi polynomials. The aim of this paper is to fill up this gap in the literature.

Now, we describe the structure of the paper. Since the Jacobi polynomials $\mathscr{P}_n^{(\alpha,\beta)}$ (see [7]) with negative integer parameters vanish identically for some values of n, it is necessary to redefine these polynomials in order to have a sequence of monic orthogonal polynomials. This is done in Section 2, where some of their properties are also deduced. In Section 3, the Sobolev orthogonality for the monic Jacobi polynomials $\{P_n^{(-k,-l)}\}_{n\geqslant 0}$, introduced in the above section, is given.

2. The monic Jacobi polynomials $P_n^{(-k,-l)}$

For α and β arbitrary real numbers, the Jacobi polynomials $\mathscr{P}_n^{(\alpha,\beta)}$ can be defined (see [7, p. 62]) by means of their explicit representation

$$\mathscr{P}_n^{(\alpha,\beta)}(x)=\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}\left(\frac{x-1}{2}\right)^{n-m}\left(\frac{x+1}{2}\right)^{m},\quad n\geqslant 0.$$

When $\alpha=-k$ and $\beta=-l$ with k,l positive integers, $\mathscr{P}_n^{(-k,-l)}$ vanishes identically for $\max\{k,l\}\leqslant n<k+l$ and reduces its degree for $(k+l)/2\leqslant n<\max\{k,l\}$.

In the case $\alpha=-k$, with k a positive integer, and β not a negative integer, the nth monic Jacobi polynomial is given by (see [1])

$$P_n^{(-k,\beta)}(x)=\binom{2n-k+\beta}{n}^{-1}\sum_{m=0}^{n}\binom{n-k}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^{m}, \tag{2.1}$$

with the convention $\binom{0}{0}=1$.

Observe that if we take $\beta=-l$ with l a positive integer in formula (2.1), the binomial coefficient $\binom{2n-k-l}{n}$ vanishes for some values of n. That is the reason for which it is necessary to give a new definition of the Jacobi polynomials.

Obviously, the expression of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ in the case $\beta=-l$, for l a positive integer, and α not a negative integer, is similar to (2.1).

We define the nth monic generalized Jacobi polynomial $P_n^{(-k,-l)}$, for k and l positive integers, by means of the following formula:

$$P_n^{(-k,-l)}(x)=\frac{1}{2}\left[\lim_{\alpha\to -k}P_h^{(\alpha,-l)}(x)+\lim_{\beta\to -l}P_h^{(-k,\beta)}(x)\right], \tag{2.2}$$

with $h=k+l-n-1$ if $(k+l)/2\leqslant n<\max\{k,l\}$ and $h=n$ otherwise, where the polynomials $P_h^{(\alpha,-l)}$ and $P_h^{(-k,\beta)}$ are defined as in (2.1). Notice that $\deg P_n^{(-k,-l)}=h$.

Therefore, it can be written:

$$P_n^{(-k,-l)}(x) = \frac{1}{2}\left[\lim_{\alpha\to -k}\left(\kappa_h^{(\alpha,-l)}\sum_{m=0}^{h}\binom{h+\alpha}{m}\binom{h-l}{h-m}(x-1)^{h-m}(x+1)^m\right)\right.$$
$$\left. + \lim_{\beta\to -l}\left(\kappa_h^{(-k,\beta)}\sum_{m=0}^{h}\binom{h-k}{m}\binom{h+\beta}{h-m}(x-1)^{h-m}(x+1)^m\right)\right], \tag{2.3}$$

with $h=k+l-n-1$ if $(k+l)/2 \leqslant n < \max\{k,l\}$ and $h=n$ otherwise. Here and in the sequel we will put

$$\kappa_h^{(\alpha,\beta)} = \binom{2h+\alpha+\beta}{h}^{-1}.$$

We want to point out that for $0 \leqslant n < \max\{k,l\}$ and for $n \geqslant k+l$, the polynomials $P_n^{(-k,-l)}$ defined by (2.2) are the monic polynomials corresponding to those described in [7] (see Section 4.22) and that for $\max\{k,l\} \leqslant n < k+l$, taking into account (2.2), they satisfy $\deg P_n^{(-k,-l)} = n$.

It is easy to proof, as a first consequence of (2.2), that the symmetry property of the classical Jacobi polynomials $\mathscr{P}_n^{(\alpha,\beta)}$ also holds for the polynomials $P_n^{(-k,-l)}$, that is

$$P_n^{(-k,-l)}(-x) = (-1)^h P_n^{(-l,-k)}(x), \quad n \geqslant 0 \tag{2.4}$$

where $h = \deg P_n^{(-k,-l)}$

From now on, without loss of generality, we suppose that $l \leqslant k$.

Remark. It is worthy to observe that the conservation of the symmetry property justify the definition given in (2.2).

Next, we show the explicit representation for the monic generalized Jacobi polynomials $P_n^{(-k,-l)}$ according to the different values of n:

(a) For $0 \leqslant n < l$, all the binomial coefficients in (2.3) are nonzero. So, we have

$$P_n^{(-k,-l)}(x) = \kappa_n^{(-k,-l)}\sum_{m=0}^{n}\binom{n-k}{m}\binom{n-l}{n-m}(x-1)^{n-m}(x+1)^m. \tag{2.5}$$

(b) When $l \leqslant n < (k+l)/2$, from [7, Eq. (4.22.2)] and (2.4), we obtain that the polynomial $P_n^{(-k,-l)}$ can be expressed in terms of the polynomial $P_{n-l}^{(-k,l)}$ and then

$$\begin{aligned} P_n^{(-k,-l)}(x) &= (x+1)^l P_{n-l}^{(-k,l)}(x) \\ &= (x+1)^l \kappa_{n-l}^{(-k,l)}\sum_{m=0}^{n-l}\binom{n-l-k}{m}\binom{n}{n-l-m}(x-1)^{n-l-m}(x+1)^m. \end{aligned} \tag{2.6}$$

(c) If $(k+l)/2 \leqslant n < k$, from [7, Eq. (4.22.3)], the Jacobi polynomial $\mathscr{P}_n^{(-k,-l)}$ reduces its degree which is precisely $k+l-n-1$. For n in this range, by definition (2.2), $P_n^{(-k,-l)} = P_{k+l-n-1}^{(-k,-l)}$ and then since $P_{k+l-n-1}^{(-k,-l)}$ corresponds to the case (b), it can be concluded:

$$\begin{aligned}P_n^{(-k,-l)}(x) &= (x+1)^l P_{k-n-1}^{(-k,l)}(x)\\ &= (x+1)^l \kappa_{k-n-1}^{(-k,l)} \sum_{m=0}^{k-n-1} \binom{-n-1}{m}\binom{k+l-n-1}{k-n-1-m}(x-1)^{k-n-1-m}(x+1)^m.\end{aligned} \tag{2.7}$$

(d) If $k \leqslant n < k+l$, from [7, Eq. (4.22.4)], the polynomial $\mathscr{P}_n^{(-k,-l)}$ vanishes identically. After some computations, it can be deduced that the limits in (2.3) exist and then

$$\begin{aligned}P_n^{(-k,-l)}(x) = A_n^{(k,l)} \Bigg[&(-1)^{n-k}(n-l)! \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{(-1)^m (l-m-1)!}{(n-m)!}(x-1)^{n-m}(x+1)^m \\ &+ (n-k)! \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{(-1)^m (k+l+m-n-1)!}{(m+l)!}(x-1)^{n-m-l}(x+1)^{m+l}\Bigg],\end{aligned} \tag{2.8}$$

where

$$A_n^{(k,l)} = \frac{n!}{2(k+l-n-1)!(2n-k-l)!}.$$

(e) In the case $n \geqslant k+l$, all the binomial coefficients in (2.3) are nonzero, and we have

$$\begin{aligned}P_n^{(-k,-l)}(x) &= (x+1)^l (x-1)^k \kappa_n^{(-k,-l)} \sum_{m=l}^{n-k} \binom{n-k}{m}\binom{n-l}{n-m}(x-1)^{n-k-m}(x+1)^{m-l}\\ &= (x+1)^l (x-1)^k P_{n-k-l}^{(k,l)}(x).\end{aligned} \tag{2.9}$$

Using either formula (2.2) or the above explicit representations of the polynomials $P_n^{(-k,-l)}$ it can be shown that some of the properties of the classical Jacobi polynomials hold for the generalized Jacobi polynomials. More precisely:

Proposition 2.1. *Let k and l be arbitrary positive integers. Then, the monic generalized Jacobi polynomials $P_n^{(-k,-l)}$ satisfy the following properties*:

(i) *The three-term recurrence relation* (1.1) *with* $\alpha = -k$, $\beta = -l$ *and* $n \geqslant \max\{k,l\}+1$.

(ii) *The differentiation formula*

$$D^i P_n^{(-k,-l)}(x) = \frac{h!}{(h-i)!} P_{n-i}^{(-k+i,-l+i)}(x), \quad n \geqslant 0, \quad 0 \leqslant i \leqslant h = \deg P_n^{(-k,-l)}, \tag{2.10}$$

where $D = \mathrm{d}/\mathrm{d}x$.

(iii) *The second order differential equation*

$$(1-x^2)y'' + [k-l+(k+l-2)x]y' + n(n-k-l+1)y = 0, \quad n \geqslant 0.$$

(iv) *The generalized Gegenbauer polynomials* (*see* [2]) *are a particular case of the generalized Jacobi polynomials*, *that is*

$$C_n^{(-k+\frac{1}{2})}(x) = P_n^{(-k,-k)}(x), \quad n \geqslant 0.$$

Next, using differentiation property and formulas (2.5)–(2.9), the values of these polynomials and their derivatives at the points 1 and −1 that will be used in Section 3, can be computed. The results obtained are resumed in the following:

Lemma 2.2. (a) *If* $0 \leqslant n < l$,

$$D^i P_n^{(-k,-l)}(1) = \frac{n!}{(n-i)!} 2^{n-i} \kappa_{n-i}^{(-k+i,-l+i)} \binom{n-k}{n-i}, \quad 0 \leqslant i < n,$$

$$D^j P_n^{(-k,-l)}(-1) = \frac{n!}{(n-j)!} (-2)^{n-j} \kappa_{n-j}^{(-k+j,-l+j)} \binom{n-l}{n-j}, \quad 0 \leqslant j < n.$$

(b) *If* $1 \leqslant l \leqslant n < (k+l)/2$,

$$D^i P_n^{(-k,-l)}(1) = \frac{n!}{(n-i)!} 2^{n-i} \kappa_{n-l}^{(-k,l)} \binom{n-k-l+i}{n-l}, \quad 0 \leqslant i < n.$$

$$D^j P_n^{(-k,-l)}(-1) = 0, \quad 0 \leqslant j < l.$$

(c) *In the case* $(k+l)/2 \leqslant n < k$

$$D^i P_n^{(-k,-l)}(1) = \frac{h!}{(h-i)!} 2^{k+l-n-1-i} \kappa_{k-n-1}^{(-k,l)} \binom{i-n-1}{k-n-1}, \quad 0 \leqslant i < h = \deg P_n^{(-k,-l)},$$

$$D^j P_n^{(-k,-l)}(-1) = 0, \quad 0 \leqslant j < l.$$

(d) *When* $k \leqslant n < k+l$,

$$D^i P_n^{(-k,-l)}(1) = \frac{n!}{(n-i)!} 2^{n-i-1} (-1)^{n-l} \kappa_{n-k}^{(k,-l)} \binom{k-i-1}{n-l}, \quad 0 \leqslant i < k,$$

$$D^j P_n^{(-k,-l)}(-1) = \frac{n!}{(n-j)!} 2^{n-j-1} (-1)^{k-j} \kappa_{n-l}^{(-k,l)} \binom{l-j-1}{n-k}, \quad 0 \leqslant j < l.$$

(e) *For* $n \geqslant k+l$,

$$D^i P_n^{(-k,-l)}(1) = 0, \quad 0 \leqslant i < k.$$

$$D^j P_n^{(-k,-l)}(-1) = 0, \quad 0 \leqslant j < l.$$

3. Sobolev orthogonality for $\{P_n^{(-k,-l)}\}_{n \geqslant 0}$

Next, we shall give orthogonality for the sequence $\{P_n^{(-k,-l)}\}_{n \geqslant 0}$. Given k and l positive integers with $l \leqslant k$, if there exists n such that $(k+l)/2 \leqslant n < k$ then $\deg P_n^{(-k,-l)} < n$ and the polynomials $\{P_n^{(-k,-l)}\}_{n \geqslant 0}$ are not orthogonal with respect to any quasi-definite bilinear form.

So, a more general kind of orthogonality will be considered:

Given a bilinear form $(.,.)$, by Q_n, $n \geqslant 0$, we will denote the nth monic polynomial of least degree, not identically equal to zero, such that

$$(Q_n, p) = 0 \quad p \in \mathbf{P}_{n-1}$$

where $\mathbf{P}_{n-1}$ denotes the linear space of all polynomials of degree less than or equal $n-1$.

Such a polynomial does exist and it is unique by minimality of degree for the polynomial solution. If the bilinear form is positive definite then $\deg Q_n = n$ and thus all the Q_n's are distinct. In general, this is not so and for different values of n the same polynomial Q_n can appear.

Theorem 3.1. *Let k, l be positive integer numbers. There exists a symmetric $(k+l)\times(k+l)$ matrix* $\mathbf{A}$ *such that the sequence $\{P_n^{(-k,-l)}\}_{n\geqslant 0}$ is orthogonal with respect to the Sobolev bilinear form in the space of the real polynomials*

$$(f,g)_S = (f,g)_D + \int_{-1}^{1} f^{(k+l)}(x) g^{(k+l)}(x)(1-x)^l (1+x)^k \,\mathrm{d}x, \tag{3.1}$$

where $(f,g)_D = (F(1)|\tilde{F}(-1))\mathbf{A}(G(1)|\tilde{G}(-1))^{\mathrm{T}}$ *and*

$$(F(1)|\tilde{F}(-1)) = (f(1), f'(1), \ldots, f^{(k-1)}(1), f(-1), f'(-1), \ldots, f^{(l-1)}(-1)).$$

Proof. From Lemma 2.2 (e) and formula (2.10) it follows that the polynomials $P_n^{(-k,-l)}$, for $n \geqslant k+l$, are orthogonal to the linear space $\mathbf{P}_{n-1}$ with respect to (3.1), for any symmetric $(k+l)\times(k+l)$ matrix $\mathbf{A}$.

In order to have the orthogonality of the sequence $\{P_n^{(-k,-l)}\}_{n\geqslant 0}$ it suffices to prove that there exists a symmetric $(k+l)\times(k+l)$ matrix $\mathbf{A}$ such that $(P_n, P_m)_D = 0$ for $0 \leqslant m < n < k+l$.

The existence of such a matrix follows solving the homogeneous linear system

$$(P_n(1)|\tilde{P}_n(-1))\mathbf{A}(P_m(1)|\tilde{P}_m(-1))^{\mathrm{T}} = 0, \quad 0 \leqslant m < n < k+l$$

with $\binom{k+l+1}{2}$ unknowns and, at most, $\binom{k+l}{2}$ equations. □

Further information about the matrix $\mathbf{A}$ can be derived analyzing the structure of the sequence $\{P_n^{(-k,-l)}\}_{n\geqslant 0}$. In general, the matrix $\mathbf{A}$ is not regular because of for every n satisfying $(k+l)/2 \leqslant n < k$, the polynomial $P_n^{(-k,-l)}$ reduces its degree. However, if such a positive integer n does not exist, then for any diagonal positive definite matrix $\mathbf{D}$ of order $(k+l)$, a symmetric positive definite matrix $\mathbf{A}$ can be explicitely constructed by means of $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^{\mathrm{T}}$, where $\mathbf{Q}$ is the regular matrix $\mathbf{Q} = (\mathbf{Q}(1)|\tilde{\mathbf{Q}}(-1)) \in \mathscr{M}_{(k+l)\times(k+l)}(\mathbb{R})$, with

$$\mathbf{Q}(1) = (D^i P_n^{(-k,-l)}(1))_{\substack{n=0,\ldots,k+l-1,\\ i=0,\ldots,k-1}} \quad \text{and}$$

$$\tilde{\mathbf{Q}}(-1) = (D^j P_n^{(-k,-l)}(-1))_{\substack{n=0,\ldots,k+l-1,\\ j=0,\ldots,l-1}}.$$

Observe that this situation can only occur when either $k = l$ (generalized Gegenbauer polynomials, see [2]) or $k = l+1$.

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