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On Groups of Baer Collineations Acting on Cartesian and Translation Planes

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0. INTRODUCTION

Let G be a collineation group of a finite projective plane Π, whose order is n. We shall call G a *B-group* if all its non-trivial elements are Baer collineations of Π and $(|G|, n) = 1$.

All known B-groups G are planar, i.e., their fixed elements form a subplane of Π, which we shall always denote by Π_G. If Π_G is a Baer subplane and Π is a translation plane then Foulser [3] implies G is cyclic. Using this fact Ostrom [10] showed that if S is a B-group of exponent 2, acting on a translation plane of order n, then S is planar and Π_S has order $n^{1/|S|}$.

The object of this paper is to consider the problem of classifying those B-group G that act on the cartesian and translation planes. Our main result, stated below, incorporates the theorems of Foulser and Ostrom mentioned above, and may be regarded as being a partial description of how B-groups act on translation planes. In particular, our result implies that G is planar and, more surprisingly, that if G is not of exponent 2, then Π_G is either a square root or a fourth root subplane of Π.

THEOREM A. *Let Π be a translation plane of order n admitting a B-group G. Then G is a planar group and one of the following cases must occur.*

(a) *G is cyclic and Π_G is a Baer subplane of Π;*

(b) *G is an elementary abelian 2-group and Π_G has order $n^{1/|G|}$;*

(c) *$G \subseteq S_5$ or G is dihedral and in both cases Π_G has order $n^{1/4}$.*

COROLLARY. *An even order translation plane Π admits a B-group G if and only if G is cyclic (and so Π_G must be a Baer subplane).*

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Though we do not have counterexamples to the corollary, even when Π has odd order, it seems quite likely that some presently known translation planes of odd admit dihedral B-groups (and possibly even A_4 and S_4). However, none of the other non-cyclic groups listed in the theorem (viz., A_5, S_5 or groups of type $(2, 2, \ldots, 2)$) are known to act as B-groups on any known projective plane. Such B-groups, if they exist, probably act on entirely new kinds of planes.

We consider now how far we are able to extend Theorem A to arbitary finite cartesian planes, i.e., those planes which are (P, l) transitive for some incident point–line pair. In order to do so it will be convenient to introduce the following non-standard terminology.

DEFINITION. A collineation group of a projective plane will be called locally cyclic if only its cyclic subgroups may fix Baer subplanes elementwise.

(N.B. Cyclic subgroups of B-groups do in fact always fix some Baer subplane elementwise; moreover, as indicated earlier, Foulser [3] implies that B-groups of finite translation planes are always locally cyclic.)

THEOREM A′. *If a finite cartesian plane Π admits a planar locally cyclic B-group G then the conclusions of Theorem* A *apply to* (Π, G).

Remark. The theorem does not tell us whether there are finite groups that are categorically excluded from being B-groups of cartesian planes. In fact, by Section 5, many Frobenius groups are of this type.

We now outline some of the main steps involved in proving Theorems A and A′, in the case when the B-group G is non-solvable. It is easily seen (though we have delayed the proof until Section 6) that by a theorem of Brauer[1] et al. [1] T, a Sylow 2-subgroup of G, must be elementary abelian or dihedral. In the former case $N = N_G(T)$ turns out to be a Frobenius group and so by Section 5 we find $N \cong A_4$ and $|T| = 4$. Thus we need only consider the case when T is dihedral; but in Section 3, by considering elementary abelian B-groups, we find that the odd order Sylow subgroups of G are all cyclic. Hence by a theorem of Suzuki [11] G is closely related to some $LF(2, p)$, where p is a prime >5. So G contains a metacyclic subgroup of order $p(p-1)/2$. But the work on "$\mathbb{Z}$-groups" in Section 4 now shows that $(p-1)/2 = 2$, and so G is "approximately" $LF(2, 5) \cong A_5$. A little more work shows that $G \cong A_5$ or S_5. It remains to analyze the fixed points of G. This is done by considering the Wielandt polynomials for G. These polynomials play an important role throughout the paper, and the ones that we need are discussed in the next section.

[1] All the group theory used in this article predates the odd order paper.

For background on projective planes and spreads the reader should consult [7, Chaps. 5–7] and [9, Chap. 1]. We shall also occasionally use the fact that B-groups of translation planes always lie in some translation complement. This follows immediately from the following simple remark.

LEMMA 1. *Let V be a finite vector space over the prime field $GF(p)$. Suppose G is any p'-subgroup in* $\operatorname{Aut}\mathfrak{A}$, *where $\mathfrak{A}$ denotes the affine space associated with V. Then G must fix a point of V.*

Proof. Consider the action of G on the projective closure of $\mathfrak{A}$. Now by Maschke's theorem G fixes a point outside the hyperplane at infinity. The lemma follows.

1. WIELANDT POLYNOMIALS

Let G be a subgroup of $\operatorname{Aut} H$, where H is any finite group such that $(|G|, |H|) = 1$. Also choose a family $\zeta = (G_i)_1^n$ of n distinct subgroups of G_i and write f_i for $|\mathrm{Fix}(G_i)|$.

Wielandt has given a technique for computing polynomial relationships between the f_i that do not depend on the choice of H; e.g., the well-known 4-group formula of Brauer may be deduced using Wielandt's technique. Our interest in Wielandt's polynomials stems from the fact that if G happens to be a B-group of a cartesian plane then usually some "y-axis" of the plane may be identified with an additive group $(C, +)$, such that G lies in $\operatorname{Aut}(C, +)$.

The object of this section is to compute some Wielandt polynomials that will be needed later on. We begin with a matrix theoretic description of Wielandt's technique.

Let $\gamma_1, \dots, \gamma_m$ be a set of elements in G such that every element of G is conjugate with (at least) one of the γ_i's. Now define $W(\zeta)$, or simply W, to be the matrix $\|w_{ij}\|$ of order $m \times n$ such that

$$w_{ij} = \#\text{ conjugates of } \gamma_i \text{ in } G_j.$$

Now let $(k_1, \dots, k_n)^T$ be any solution for $\mathbf{x}$ in the matrix equation $W\mathbf{x} = \mathbf{0}$.

1. THEOREM (Wielandt [12, Hauptsatz 2.2]). *Let $g_i = |G_i|$. Then*

$$\prod_{i=1}^{n} f_i^{g_i k_i} = 1.$$

We call any polynomial of the above type a *Wielandt polynomial* for G. It will now be convenient to introduce the following more graphic notation for Wielandt polynomials.

Conventions. Suppose A is a subgroup of $G \subseteq \operatorname{Aut} H$. Then f_A denotes $|\operatorname{Fix}(A)|$ and, when $A = \langle \alpha \rangle$, we write f_α for f_A. Also, f_0 denotes $|H|$.

Wielandt has shown that a Wielandt polynomial for dihedral groups coincides with the 4-group formula of Brauer [12, Item 3.1].

Result 2 (Brauer, Wielandt). Let $G = \langle \alpha, \beta \rangle$, where α, β are distinct involutions and write $\gamma = \alpha\beta$. Then a Wielandt polynomial for G is

$$f_0 f_G^2 = f_\alpha f_\beta f_\gamma.$$

Wielandt has also given a Wielandt polynomial for groups of prime exponent [12, Item 3.2]; in particular, the following holds.

Result 3. Let $G = Z_p \oplus Z_p$, where p is prime. Then a Wielandt polynomial for G is

$$f_0 f_G^p = \prod_{i=1}^{p+1} f_i$$

where $(G_i)_1^{p+1}$ is the set of distinct subgroups of G with order p and $f_i = |\operatorname{Fix}(G_i)|$ for all i.

Now we compute a Wielandt polynomial of a Frobenius group G with elementary abelian kernal K and cyclic complement H. We shall restrict ourselves to the (essentially unique) polynomial that arises from the family of subgroups $\zeta = (O_G, K, H, G)$. Now by elementary properties of Frobenius groups [4, Theorems 7.6 and 7.7, Chap. 2], and the fact that H, K are abelian, we find that W can be chosen to be the inner rectangle of the following diagram.

	O_G	K	H	G
O_G	1	1	1	1
$k \in K^*$	0	$\lvert H\rvert$	0	$\lvert H\rvert$
$h \in H^*$	0	0	1	$\lvert K\rvert$

Now Theorem 1 easily gives the following result.

LEMMA 4. *Let $G = KH$ be a Frobenius group of the above type. Then a Wielandt polynomial for G is*

$$f_0 f_G^{|H|} = f_K f_H^{|H|}.$$

We shall also require Wielandt polynomials arising from finite groups whose Sylow subgroups are all cyclic. Such groups have been classified by Zassenhaus [5, Theorem 9.4.3] and are often called **Z**-groups. Basically, if G

is a non-cyclic **Z**-group then G' is cyclic with a cyclic complement H such that $(|H|, |G'|) = 1$. The **Z**-groups that we shall be interested in satisfy the following further condition, which makes their polynomials easier to compute.

DEFINITION 5. A finite **Z**-group G is a *strict* **Z**-group if it is non-cyclic and futhermore:

$$g \in G \Rightarrow |g| \mid |G'| \qquad \text{or} \qquad |g| \mid [G\colon G'].$$

The following lemma is a straightforward consequence of the fact that in a strict **Z**-group no non-trivial element of G' can normalize any subgroup whose order divides $[G\colon G']$.

LEMMA 6. *Let G be a strict* **Z***-group. Then the following conditions all hold.*

(a) $G = \langle a\rangle\langle b\rangle$, *where* $G' = \langle a\rangle$ *and* $(|a|, |b|) = 1$;

(b) $\langle b\rangle^g \cap \langle b\rangle = \{O_G\}$ *whenever* $g \in G\backslash\langle b\rangle$;

(c) *if* $\alpha \in \langle a\rangle\backslash\langle(O_G\}$ *then* α *is conjugate to exactly* $|b|$ *elements of G (and they all lie in* G'*).*

With the aid of the above lemma it is straightforward to check the following result.

LEMMA 7. *Let G be a strict* **Z***-group as described in the lemma above. Then*

$$f_0 f_G^{|b|} = f_a f_b^{|b|}$$

is a Wielandt polynomial for G.

2. *B*-GROUP CONVENTIONS

For the rest of this article we shall restrict ourselves to the study of B-groups acting on finite cartesian planes. Given this objective, the following lemma allows us to make some convenient conventions.

LEMMA 1. *Let* Π *be a cartesian plane that admits a B-group G. Then, up to duality, either*

(a) Π *is a translation plane and G leaves invariant a translation axis (even when* Π *is desarguesian*); *or*

(b) *Π is (P, l) transitive, relative to a unique flag (P, l), which must therefore be G-invariant.*

Proof. For non-desarguesian translation planes part (a) applies because of the uniqueness of the translation axis. If Π is desarguesian we clearly have $|G| \leqslant 2$ and so part (a) continues to be valid. On the other hand, if Π is neither a translation plane, nor its dual, then by the Lenz–Barlotti tables [2, pp. 126–128], and with a little help from the Hering–Kantor theorem [6], Π has a unique flag (P, l) relative to which it is transitive. The result follows.

Because of the lemma, we may (up to duality) assume that G acts on an *affine* cartesian plane Π^l such that l is the translation axis, whenever Π is a translation plane. This assumption, and related conventions, are summarized in (*) below and will be tacitly invoked whenever convenient.

(*) *Conventions.* (a) Π^l is an affine plane of order n which is (Y, l) transitive for some Y on l. Also, if Π is a translation plane, l is its translation axis.

(b) G denotes a B-group in Aut Π such that G fixes Y when Π is not a translation plane; (so we allow G to be fixed point free (fpf) on l, if Π is a translation plane).

Remark. We emphasize that the conventions above have been introduced only to eliminate tedious details from proofs; we do *not* require the assumptions in (*) to be valid in any of the theorems proved. In particular, (*) plays no role in Sections 4 and 5.

We shall frequently need the following result on *autotopism groups* i.e. collineation groups whose fixed sets include at least three non-collinear points.

Result 2. Assume H is an autotopism group of the projective plane Π such that H fixes a proper triangle OXY, where Π is (Y, XY) transitive. Now coordinatize Π so that

$$O = (0, 0), \qquad Y = (\infty), \qquad X = (0)$$

and let $(C, +, \cdot)$ be the resulting cartesian group [7, Chap. 6]. Also let

$$\eta = (\{(0, y) : y \in C\}, +)$$

be the additive group that is canonically isomorphic to $(C, +)$. Then the restriction map $H \to H \mid \eta$ is a group homomorphism from H into Aut$(y, +)$. Also this homomorphism is injective if H contains no perspectivities.

Proof. The fact that H is additive on $(\eta, +)$ has been pointed out in [8, Lemma 2.2]. Everything else is standard [7, Chap. 6].

Remark. There seems no reason to expect H to be additive on the "X-axis"; if this could be shown then many of our results could be considerably improved, e.g., the planarity hypothesis of Theorem A′ could be dropped.

3. Elementary Abelian and Dihedral B-groups

In this section we consider mainly the action of a non-cyclic elementary abelian B-group E, acting on the cartesian plane Π. We shall show that if E is planar, or Π is a translation plane, then E must be a planar 2-group such that Π_E has order $n^{1/|E|}$. Thus we shall be extending Jha [8, Theorem A], which in turn generalizes Ostrom [10].

Hypothesis (H). *E is a subgroup of the B-group G such that $E \cong \mathbf{Z}_p \oplus \mathbf{Z}_p$, where p is a prime.*

Lemma. *Assume Hypothesis* (H). *Then, in the notation of convention* (*), *E fixes the triangle OXY, where O is an affine point of Π^l and $l = XY$.*

Proof. If Π is not a translation plane then convention (*) implies that E fixes Y. Now simple counting gives the lemma. So consider the remaining viz. when Π is a translation plane and l is its translation axis. So, to get a contradiction, we shall assume E is fpf on l. Writing $E = \langle \alpha, \beta \rangle$ we now find that β leaves $\Pi_\alpha \cap l$ invariant and is also fpf on this set. Thus $p \mid \sqrt{n} + 1$. But the semiregularity of β on $l \setminus (l \cap \Pi_\beta)$ shows that $p \mid n - \sqrt{n}$ and so $p \mid \sqrt{n} - 1$. Thus $p = 2$ and $\beta \mid \Pi_\alpha$ is now a homology or a planar element. Both possibilities are consistent with the lemma.

We now prove the main facts about E, when $p > 2$. We continue with the terminology of the lemma above.

Proposition 1. *Assume Hypothesis* (H). *Then for $p > 2$, all the following statements must be valid.*

(a) *E is an autotopism group of Π that fixes exactly $n^{(p-1)/2p}$ affine points of Π^l that lie on the Y-axis OY. Apart from these points the only other points of Π fixed by E are the points X, Y on l.*

(b) *E is not a planar group.*

(c) *Π is not a translation plane.*

Proof. By the lemma above, and Result 2.2, we may regard the affine part of OY as being a group $(C, +)$ such that E acts on OY so as to be a subgroup of $\mathrm{Aut}(C, +)$. Since E is a B-group, the Wielandt polynomial for E on C (Result 1.3) yields

$$nf_E^p = (\sqrt{n})^{p+1}.$$

Thus E fixes exactly $n^{(p-1)/2p}$ affine points of OY. Next suppose, if possible, that E is a planar group. Thus Π_E has order $n^{(p-1)/2p} < n^{1/2}$ and so Π_E is a proper subplane of Π_α, whenever α is a Baer collineation of E. Hence the Baer condition for subplanes shows that

$$(n^{(p-1)/2p})^2 < n^{1/2}$$

and so we have the contradiction $p = 2$. Now part (b) follows and hence part (a) is also valid. Part (c) follows readily from part (a) if we note that we can interchange the roles of OX and OY, where Π is a translation plane. Hence the proposition is proved.

Now recall that a finite p-group with a unique subgroup of order p is either cyclic or (generalized) quaternion [5, Theorem 12.5.2]. Thus we immediately have the following corollaries, applied to a B-group G.

COROLLARY 2. *If Π is a translation plane then the odd order Sylow subgroups of G are cyclic. In particular, if G has odd order then G is a $\mathbf{Z}$-group.*

COROLLARY 3. *Let G be a planar B-group acting on a cartesian plane. Then G has odd order only if it is a $\mathbf{Z}$-group.*

Both corollaries will be strengthened in the next section. The type of conclusion we get will be similar to the case when G is dihedral. The analysis of this case quickly leads to [8, Theorem A].

THEOREM 4. *Let G be a dihedral B-group acting on a cartesian plane of order n. Then G is planar and Π_G has order $n^{1/4}$.*

(N.B. G is allowed to be a Klein group).

Proof. Let $G = \langle \alpha, \beta \rangle$, where α, β are involutions and write $\gamma = \alpha\beta$. Now $\langle \gamma \rangle \lhd G$ and $\bar{\alpha} = \alpha \mid \Pi_\gamma$ is certainly an autotopism, since $|\bar{\alpha}| \leqslant 2$. But $G = \langle \alpha, \gamma \rangle$ now shows that G itself is an autotopism group. So one can assume, in accordance with conventions (*), that G fixes a triangle OXY, where $l = XY$ is such that Π is (Y, l) transitive and O is an affine point of Π^l. Again, looking at the Wielandt polynomial of G (Result 1.2) acting on OY (cf. Result 2.2) we find that

$$nf_G^2 = \sqrt{n}\,\sqrt{n}\,\sqrt{n}$$

and so

$$f_G = n^{1/4}. \tag{a}$$

Thus G fixes precisely $n^{1/4}$ affine points of the line OY. Now let us consider the restriction map $\bar{\alpha} = \alpha \mid \Pi_\gamma$. If $\bar{\alpha}$ is a Baer involution then we are done; if $\bar{\alpha} = 1$ we contradict (a); so assume $\bar{\alpha}$ is a homology of Π_α. Now by (a) the axis of $\bar{\alpha}$ must be the Y-axis and so α, γ both fix precisely the same set of points on OY and hence G fixes $\sqrt{n}$ points of OY, again contradicting (a). Thus the theorem is valid.

Now applying the theorem to the Klein group case and proceeding by induction, as in Ostrom [10], we find that if G is an elementary abelian 2-group which acts as a B-group on the cartesian plane Π, then G is planar and Π_G has order $n^{1/|G|}$ [8, Theorem A]. Combining with the corollaries above we now establish the following.

THEOREM 5. *Let G be a B-group of the cartesian plane Π, whose order is n. Assume that G is elementary abelian. Then*

(a) *if G is a 2-group then it must be planar and Π_G has order $n^{1/|G|}$* [8, *Theorem* A]; *and*

(b) *if Π is a translation or G is planar then either G is cyclic* (*and so fixes a Baer subplane elementwise*) *or G is a 2-group and part* (a) *applies.*

Before leaving this section it will be convenient to record the following simple corollary to the theorem.

LEMMA 6. *Let H be a non-cyclic B-group that acts on the cartesian plane Π. Assume further that*

(i) *H contains a cyclic subgroup $\langle\lambda\rangle$ such that $[H: \langle\lambda\rangle] = 2$; and*

(ii) *H is "locally cyclic" in the sense of the introduction.*

Then H is dihedral and so the conclusions of Theorem 5 hold.

Proof. Suppose $\alpha \in H \backslash \langle\lambda\rangle$. We claim that any such α must be an involution; for otherwise $\Pi_\alpha = \Pi_{\alpha^2} = \Pi_\lambda$ and now condition (ii) yields the contradiction that $H = \langle\alpha, \lambda\rangle$ is a cyclic group. Hence H is generated by the involutions $\alpha, \alpha\lambda$ and so must be dihedral. The lemma follows.

4. STRICT $\mathbf{Z}$-GROUPS

The following lemma shows the importance of *strict* $\mathbf{Z}$-groups (Definition 1.5) in the study of B-groups. The proof is a variation of the previous lemma.

LEMMA 1. *Let H be a B-group acting on an arbitrary finite projective plane Π. Assume also that H is a $\mathbf{Z}$-group. Then either*

(a) *H is planar and Π_H is a Baer subplane of Π; or*

(b) *H is a strict $\mathbf{Z}$-group.*

Proof. Part (a) is obvious for cyclic groups so we shall assume that the **Z**-group H is non-cyclic. Thus by the classification of non-cyclic **Z**-groups (cf. Section 1 and Definition 1.5) the following conditions apply

(i) $|H| = mn$, where $(m, n) = 1$ and m, n are both >1;

(ii) $H' = \langle \alpha \rangle$ and $|\alpha| = m$;

(iii) $\exists \beta \in H \setminus H'$ such that $|\beta| = n$.

We shall now show that (a) occurs when H is not strict, i.e.,

(iv) $\exists h \in H$ such that $|h| = m_1 n_1$, where $m_1 \mid m$, $n_1 \mid n$ and m_1, n_1 are both >1.

First observe that $h^{n_1} \in \langle \alpha \rangle \setminus \{1\}$ and so $\Pi_\alpha = \Pi_{h^{n_1}} = \Pi_h$, since we are only dealing with Baer collineations. Next note that h^{m_1} has a conjugate γ in the Hall subgroup $\langle \beta \rangle$. Now since $\mathrm{Fix}(h^{m_1}) = \mathrm{Fix}(h) = \Pi_\alpha$, and $\langle \alpha \rangle \lhd H$, we must have $\Pi_\gamma = \Pi_\alpha$. However, we also have $\Pi_\gamma = \Pi_\beta$ and so $H = \langle \alpha, \beta \rangle$ fixes Π_α ($=\Pi_\beta$) identically. Thus part (a) and the lemma follow.

THEOREM 2. *Suppose Π is a finite cartesian plane admitting a strict* **Z**-*group H such that H is a planar B-group. Then*

(a) *there is a Baer chain $\Pi \supset \Pi_{H'} \supset \Pi_H$; and*

(b) $[H : H'] = 2$ *and so* $2 \,\|\, |H|$.

Proof. Coordinatize Π, with axes in Π_H, so that the corresponding *PTR* is a cartesian group $(C, +, \cdot)$. Thus H may be identified with a subgroup of $\mathrm{Aut}(C, +)$, e.g., use Result 2.2. The Wielandt polynomial for H on $(C, +)$ (Lemma 1.7) now becomes

$$nf_H^\beta = \sqrt{n}\,(\sqrt{n})^\beta$$

where $\beta = [H : H']$ and n, as usual, denotes the order of Π.
Hence

$$f_H = n^{(\beta-1)/2\beta} < n^{1/2}.$$

But this means Π_H is a proper subplane of all the Baer collineations in H and so the Baer condition yields

$$(n^{(\beta-1)/2\beta})^2 \leqslant n^{1/2}$$
$$\Rightarrow \beta = 2 \qquad \text{and} \qquad f_H = n^{1/4}.$$

The theorem follows.

When Π is a translation plane there is no need to assume that the **Z**-group H is planar, nor that it is strict.

THEOREM 3. *Suppose Π^l is a finite affine translation plane that admits a non-trivial B-group H. Assume that H is a* $\mathbf{Z}$*-group. Then H is a planar group and one of the following cases must occur.*

(a) *H is cyclic and Π_H is a Baer subplane of Π.*

(b) $2 \,\|\, |H|$, $[H\colon H'] = 2$ *and there is a Baer chain of planes* $\Pi \supset \Pi_{H'} \supset \Pi_H$.

Moreover, H is a strict $\mathbf{Z}$*-group.*

Proof. As mentioned in the introduction, part (a) follows from Foulser [3] if Π_H is a Baer subplane. Thus by Lemma 1 we may assume that the $\mathbf{Z}$-group H is strict. Now if H is planar, Theorem 2 applies. So assume H is not planar. Also identify Π^l with a spread (V, Γ) defined on the elementary abelian group $(V, +)$, whose order is n^2; thus by Lemma 0.1, H lies in $\mathrm{Aut}(V, +)$ and permutes the elements of the component set Γ, among themselves. So by Lemma 1.7 the Wielandt polynomial of H on V yields

$$n^2 f_H^{|\beta|} = n n^{|\beta|}$$

where $H = \langle\alpha\rangle\langle\beta\rangle$, $H' = \langle\alpha\rangle$, and $(|\alpha|, |\beta|) = 1$.

Hence we have

$$f_H = n^{(b-1)/b} > 1 \qquad \text{where} \quad b = |\beta|.$$

Since H is assumed non-planar, $\mathrm{Fix}(H)$ lies entirely in some component $W \in \Gamma$. But this means that each member of $H \setminus \{0\}$ fixes exactly $\sqrt{n}$ points of W while H itself fixes exactly $n^{(b-1)/b}$ points of W. Now the Wielandt polynomial of H on W yields the contradiction

$$n(n^{(b-1)/b})^b = \sqrt{n}\,(\sqrt{n})^b.$$

So the theorem is valid.

We now improve Corollaries 3.2 and 3.3.

COROLLARY 4. *Let Π be a cartesian plane admitting a non-trivial B-group H, whose order is odd. Assume further that Π is a translation plane or that H is a planar group.*

Then H is a $\mathbf{Z}$*-group only if it is planar and Π_H is a Baer subplane; also, if Π is a translation plane, then H is cyclic.*

Proof. If Π is a translation plane, use Theorem 3; otherwise use Lemma 1 and Theorem 2.

5. Frobenius B-group

Hypothesis (F). *$\Gamma = \Sigma\langle\lambda\rangle$ is a Frobenius group such that its complement $\langle\lambda\rangle$ has odd order >1 and its kernel Σ is a non-cyclic elementary abelian 2-group.*

As indicated in the introduction, the proofs of Theorems A and A′ require us to consider whether Γ can be a B-group of a cartesian plane Π. Our main conclusion is that this is only possible if $\Gamma = A_4$: in particular, this implies that many families of finite groups can never be B-groups of cartesian planes. We first consider the case when, up to duality, Π is not a translation plane.

Theorem 1. *Assume Γ is a B-group of an affine cartesian plane Π^l of order n, such that Π is (Y, l) transitive relative to exactly one point Y on l. Suppose also that Γ satisfies hypothesis* (F). *Then*

(i) *$\Gamma \cong A_4$; and*

(ii) *Γ fixes precisely $n^{1/4}$ points of an affine line of Π^l.*

Proof. Σ is a planar group such that Π_Σ is a plane of order $n^{1/s}$, where s denotes $|\Sigma|$ (Theorem 3.5). So if we choose $\lambda_1 \in \langle\lambda\rangle$ such that λ_1 has prime order, then clearly λ_1 fixes a proper triangle OXY in Π_Σ such that $XY = l$ and $O \in \Pi^l$. Hence Γ also fixes OXY and so (cf. Result 2.2) OY may be identified with a group $(C, +)$ such that $\Gamma \leqslant \mathrm{Aut}(C, +)$ and $|C| = n$. Hence the Wielandt polynomial for Γ(Lemma 1.4) on $(C, +)$ is

$$f_0 f_\Gamma^{|\lambda|} = f_\Sigma f_\lambda^{|\lambda|}.$$

But we have just noted that $f_\Sigma = n^{1/s}$ and so, writing l for $|\lambda|$, we have

$$\begin{aligned} nf_\Gamma^l &= n^{1/s} n^{l/2} \\ f_\Gamma &= n^{1/2 - (s-1)/sl}. \end{aligned} \tag{i}$$

But since $\Gamma \supset \Sigma$, we also have $f_\Gamma \leqslant f_\Sigma = n^{1/s}$. Hence (i) yields

$$\frac{1}{s} \geqslant \frac{1}{2} - \frac{s-1}{ls}$$

which may be rewritten

$$\frac{1}{s} + \frac{1}{l}\left(1 - \frac{1}{s}\right) \geqslant \frac{1}{2}. \tag{ii}$$

But since $l \geqslant 3$, we have

$$\frac{1}{s}+\frac{1}{3}\left(1-\frac{1}{s}\right) \geqslant \frac{1}{2}$$

and so $s=4$. Hence (ii) forces $l=3$ and therefore $\Gamma = A_4$. Finally, Eq. (i) now shows that Γ fixes exactly $n^{1/4}$ affine points of OY.

A slightly different argument is needed when Π is a translation plane because now Γ may not fix any slopes. Also, this time we get a stronger conclusion, viz., Γ is planar.

THEOREM 2. *Suppose Π^l is an affine translation plane of order n admitting a B-group Γ such that Γ satisfies hypothesis* (F). *Then*

(i) *$\Gamma \cong A_4$; and*

(ii) *Γ is a planar group such that Π_Γ has order $n^{1/4}$.*

Proof. Lemma 0.1 allows us to regard Γ as being in the translation complement of Π^l. Thus, in terms of spreads, we have a vector space $(V, +)$ of order n^2 equipped with a spread $\mathscr{S}$; also $\Gamma \subseteq \mathrm{Aut}(V, +)$ and Γ permutes the members of $\mathscr{S}$ among themselves. Since Γ is also a B-group one of its Wielandt polynomials (Lemma 1.4) on V yields

$$n^2 f_\Gamma^l = f_\Sigma f_\lambda^l \qquad \text{where} \quad l = |\lambda|.$$

But since we are working on the whole affine plane, $f_\lambda = n$ and Theorem 3.5 shows that $f_\Sigma = n^{2/s}$. Hence

$$f_\Gamma = n^{1-(2/l)((s-1)/s)}. \tag{i}$$

But since $n^{2/s} = f_\Sigma \geqslant f_\Gamma$ we again get (cf. Theorem 2) the inequality

$$\frac{1}{s}+\frac{1}{l}\left(1-\frac{1}{s}\right) > \frac{1}{2}. \tag{ii}$$

As in the Theorem 2, we now easily get $\Gamma = A_4$; now Eq. (i) yields $f_\Gamma = n^{1/2}$. Hence if Γ is planar it has the correct order. So consider the alternative, viz. $\mathrm{Fix}(\Gamma) \subset M$, where $M \in \mathscr{S}$. Now every non-trivial member of the B-group Γ fixes precisely $n^{1/2}$ points of M and so these points must coincide with $\mathrm{Fix}(\Gamma)$. But this means that the plane Π_Σ has order $n^{1/2}$, contradicting Theorem 3.5, as $|\Sigma| > 2$. The theorem follows.

6. PROOF OF THEOREMS A AND A′

In addition to conventions (*) of Section 2, we shall from now on assume that

Condition (A). Π is a translation plane (and so the B-group G is "locally cyclic"); or

Condition (B). Π is a cartesian plane and G is a planar locally cyclic B-group.

Note that conditions (A) and (B) correspond respectively to the hypotheses of Theorems A and A′. Thus we need to show that each condition leads to the conclusions of Theorem A. Usually we are able to consider both cases simultaneously.

LEMMA 1. *Maximal cyclic subgroups of G are pairwise disjoint.*

Proof. Suppose $\langle\alpha\rangle$, $\langle\beta\rangle$ are maximal cyclic subgroups of G such that

$$\langle\alpha\rangle \cap \langle\beta\rangle = \langle\gamma\rangle \neq 1.$$

Now we must have $\Pi_\alpha = \Pi_\gamma = \Pi_\beta$. So the B-group $\langle\alpha,\beta\rangle$ fixes Π_γ elementwise. But since G is locally cyclic we see that $\langle\alpha,\beta\rangle$ is cylcic. This contradicts the maximality of $\langle\alpha\rangle$ or $\langle\beta\rangle$ and so the lemma follows.

DEFINITION. A finite group $\mathscr{S}$ is of type-(S) if both the following conditions are satisfied:

(I) $\mathscr{S}$ is of even order;

(II) if $\mathfrak{A}$, $\mathfrak{B}$ are two maximal cyclic subgroups of even order then $\mathfrak{A} = \mathfrak{B}$ or $\mathfrak{A} \cap \mathfrak{B} = \{1\}$.

According to Brauer, Suzuki, and Wall [1, Theorem I.F] the Sylow 2-subgroups of such $\mathscr{S}$ must be cyclic, elementary abelian or dihedral. Thus by Lemma 1 the same holds true for G, when $|G|$ is even. Also, if $|G|$ is odd, Corollary 4.4 and the fact that G is locally cyclic shows that G is cyclic. Hence we have established the following results.

PROPOSITION 2. (a) *If $|G|$ is odd then G is cyclic;*

(b) *If $|G|$ is even then the Sylow 2-subgroups of G are cyclic, elementary abelian or dihedral.*

We shall consider in turn each of the three possibilities for the Sylow 2-subgroups of G. But first we mention a simple argument that will repeatedly be used.

"KLEIN LEMMA". *The centralizer of any Klein group in G is a 2-group.*

Proof. Suppose $\lambda \neq 1$ is an element of odd order in G that centralizes a Klein group $\{\alpha, \beta, \alpha\beta, 1\}$. Then since $\langle\lambda,\alpha\rangle$, $\langle\lambda,\beta\rangle$ are both cyclic we must have $\Pi_\alpha = \Pi_\lambda = \Pi_\beta$. We now contradict Theorem 3.4 and so the lemma is valid.

LEMMA. *Suppose that T, a Sylow 2-subgroup of G, is elementary abelian and that $T \neq G$. Then $|T| \leqslant 4$.*

Proof. To get a contradiction assume $|T| > 4$. Now according to Brauer *et al.* [1, Theorem I.G] the group G, being of type-(S) must satisfy the following condition:

$$C_G(\theta) = T \qquad \forall \theta \in T \setminus \{1\}. \tag{†}$$

Hence $C_G(T) = T$. Next consider $N = N_G(T)$. We break up our argument into the following subcases.

(i) $N = T$;

(ii) $N \supsetneqq T$ and $[\lambda, \alpha] = 1$, where α is some involution in T and λ is some element of odd order in N;

(iii) $N \supsetneqq T$ but $[\lambda, \alpha] \neq 1$ whenever $\lambda \neq 1$ has odd order and α is any involution.

Case (i) $N = T$. Now by a theorem of Burnside [4, Theorem 4.3, p. 252] T has a normal 2-complement in G which must be cyclic (Proposition 2(a)). Hence an index 2 subgroup of T centralizes a group of odd prime order, contrary to the "Klein lemma." Hence this case cannot occur.

Case (ii). Now $\langle \alpha, \lambda \rangle$ is cyclic and so $\Pi_\alpha = \Pi_\lambda$. But if β is any involution in T it certainly leaves Π_α invariant and then λ^β fixes Π_λ identically. So, since G is locally cyclic, $\langle \lambda, \lambda^\beta \rangle$ is a cyclic group and hence $\beta \in N_G(\langle \lambda \rangle)$. But since this is true for all β in T, we have $T\langle \lambda \rangle = T \oplus \langle \lambda \rangle$. But the existence of these abelian groups contradict the Klein lemma. So case (ii) does not occur.

Case (iii). Now $T\langle \lambda \rangle$ is a Frobenius group if λ ($\neq 1$) has odd order. Thus by Theorem 5.1 (for condition B) and Theorem 5.2 (for condition A) we have $|T| \leqslant 4$. The lemma follows.

Next consider the case when the Sylow 2-subgroups of G are cyclic (cf. Proposition 2(b)). Now by a well-known theorem of Burnside [4, Theorem 6.1, p. 257], G still has a normal 2-complement which, for the usual reason, is a cyclic group $\langle \lambda \rangle$. Thus G is a $\mathbf{Z}$-group and so, by Theorem 4.3, when Π is a translation plane, G contains a cyclic subgroup C such that $[G: C] \leqslant 2$. Also, Lemma 4.1 and Theorem 4.2 yield the same conclusion if we assume condition (B) of this section. Hence, combining with the previous lemma, we have now established the following

PROPOSITION 3. *If G has elementary abelian or cyclic Sylow 2-subgroups, then one of the following cases must occur*:

(i) *G is a 2-group*;

(ii) *G contains a cyclic subgroup C such that $|G: C| \leqslant 2$*;

(iii) *the Sylow 2-subgroups of G are Klein groups.*

We now turn to the remaining case of Proposition 2(b), viz., when the Sylow 2-subgroups of G are dihedral. Let us first consider what happens when G is non-solvable. Now finite non-solvable groups with dihedral 2-Sylow subgroups and only cyclic Sylow subgroups of odd order were classified long ago by Suzuki [11]; when applied to G his results yield the following.

LEMMA. *Suppose the 2-Sylow subgroups of G are dihedral and that G is non-solvable. Then G contains a subgroup H such that*

(i) $[G: H] \leqslant 2$; *and*

(ii) *$H = Z \oplus L$, where Z is a $\mathbf{Z}$-group and $L \cong LF(2, p)$ for some prime $p \geqslant 5$.*

Let us further consider the group $L \cong LF(2, p)$. It is easily seen that L contains a (strict) $\mathbf{Z}$-group M of order $p(p-1)/2$ such that $|M'| = p$ and $[M: M'] = (p-1)/2$. Now Theorem 4.3(b) (for condition (A)) and Theorem 4.2(b) (for condition (B)) forces us to conclude $(p-1)/2 = 2$. This means that $L \simeq LF(2, 5) \cong A_5$. But we can go even further.

PROPOSITION 4. *If the B-group G is non-solvable then either $G \cong A_5$ or $G \cong S_5$.*

Proof. By Propositions 2 and 3 the Sylow 2-subgroups of G must be dihedral and so the lemma above applies. Using the terminology of the lemma, we saw that $L \cong A_5$. Now we claim $|Z| = 1$; otherwise we can assume that $L = A_5$ is centralized by a prime order element $\theta \in G$. If $|\theta| = 5$, then G contains $\mathbf{Z}_5 \oplus \mathbf{Z}_5$, contrary to Theorem 3.5. So we may conclude that $\langle \theta, \varphi \rangle$ is cyclic whenever $\varphi \in L$ has order 5. But now we have $\Pi_\theta = \Pi_\varphi$ $\forall \varphi \in L$ whenever $|\varphi| = 5$. This means that A_5 fixes Π_θ elementwise, contradicting many things, e.g., Theorem 3.5. Thus $Z = 1$ and, more generally, $L = A_5$ has trivial centralizer in G. Moreover, returning to the lemma above, we have now shown that $G \cong A_5$ or $[G: A_5] = 2$. But the latter case is only possible if G is S_5 since $|C_G(A_5)| = 1$. Hence the proposition is proved.

We now examine the case when G is solvable and has dihedral Sylow 2-subgroups.

PROPOSITION 5. *Suppose T, a Sylow 2-subgroup of G, is dihedral and that G is solvable. Then one of the following cases must occur:*

(i) $G = T$;

(ii) $G \cong A_4$ *or* S_4;

(iii) *G contains a cyclic subgroup C such that* $[G: C] = 2$.

We now prove the proposition by using a series of lemmas. So for convenience we may assume $G \neq T$; also, by Proposition 2(a), G contains a cyclic Hall $2'$-subgroup $\langle \lambda \rangle$, where $|\lambda| > 1$.

LEMMA. *If T is a Klein group then $G \cong A_4$ or G contains a cyclic subgroup C such that* $[G: C] = 2$.

Proof. Since G is solvable it contains a non-trivial normal elementary abelian subgroup E. We consider separately the three possibilities for E, viz., $|E| = \text{odd}$, $|E| = 2$, $|E| = 4$.

If $|E|$ is odd, then $E = \langle \mu \rangle$, where μ has prime order. Thus μ is centralized by an involution θ and so $\langle \mu, \theta \rangle$ is cyclic. Since $\mu \in \langle \lambda \rangle$ we now have $\Pi_\theta = \Pi_\mu = \Pi_\lambda$; this means, since G is "locally cyclic," that $\langle \theta, \lambda \rangle$ is a cyclic group of the required order. If $|E| = 2$ then G has a central involution and so again G contains an index 2 cyclic subgroup. Finally, consider the case when E is a 4-group, i.e., $T \lhd G$. Now the "Klein lemma" easily shows that $G \cong A_4$ since we are assuming $G \neq T$. Hence the lemma is valid.

So to prove Proposition 5 it is now legitimate for us to assume the following:

HYPOTHESIS (T). $|T| > 4$ *and $\langle \alpha \rangle$ is the unique cyclic stem of T; in particular $|\alpha| \geqslant 4$ and* $[T: \langle \alpha \rangle] = 2$.

Remark. If G normalizes a non-trivial subgroup of $\langle \lambda \rangle$ then $\langle \lambda \rangle \lhd G$.

Proof. Otherwise $\exists g \in G$ such that $\langle \lambda \rangle \neq \langle \lambda^g \rangle$ and $\langle \lambda \rangle \cap \langle \lambda^g \rangle \neq 1$. But this means $\Pi_\lambda = \Pi_\lambda\, g$ and so, since G is locally cyclic, $\langle \lambda, \lambda^g \rangle$ is a cyclic group. The remark follows.

LEMMA. *If G has a cyclic normal subgroup* ($\neq 1$) *then an index 2 subgroup of G is cyclic.*

Proof. Consider first the case when G normalizes a non-trivial cyclic group of odd order. Now by the remark above $\langle \lambda \rangle \lhd G$ and so by hypothesis (T) above we find that G contains a $\mathbf{Z}$-group $\langle \alpha \rangle \langle \lambda \rangle$, where $\langle \alpha \rangle$ is the stem of T. Now by the $\mathbf{Z}$-group section (Theorems 4.2 and 4.3) we find that $\langle \alpha \rangle \langle \lambda \rangle$ is cyclic or $|\alpha| = 2$. However, the latter condition contradicts hypothesis (T).

It remains to consider the case when G normalizes a cyclic subgroup of even order. Now G centralizes an involution θ which must lie in the stem $\langle\alpha\rangle$ of T, since otherwise T would be abelian. Hence $\Pi_\lambda = \Pi_\theta = \Pi_\alpha$ and so $\langle\lambda, \alpha\rangle$ is cyclic of order $|G|/2$. The lemma follows.

LEMMA. *Suppose G has no index 2 cyclic subgroup and that hypothesis* (T) *holds. Then $G \cong A_4$ or S_4.*

Proof. The solvable group G has a normal elementary abelian group $K \neq 1$. By the previous lemma we have a contradiction unless K is a Klein group; also, of course, $K \subset T$.

Consider the representation ψ of G on K via conjugation and let $\Sigma = \operatorname{Ker} \psi$. By the "Klein lemma" Σ is a (clearly non-trivial) 2-group, normal in G. But since $\operatorname{Im} \psi \subset S_3$, it is clear that $\Sigma \subset T$ such that $[T: \Sigma] = 2$. Now let us consider in turn the two possibilities for Σ, viz., $|\Sigma| = 4$ and $|\Sigma| > 4$.

If $|\Sigma| = 4$ then the Klein lemma shows that the $2'$-subgroups of G have order 3 and that $G \supseteq A_4 \supset \Sigma$. But $[T: \Sigma] = 2$ now further implies that $|G| \mid 24$ and hence $G = A_4$, S_4 or $G = \mathbf{Z}_2 \oplus A_4$. But the final case leads to an easy contradiction (e.g., argue as in the proof of Proposition 2.4). It now remains to consider the case when $|\Sigma| > 4$. Now Σ contains a unique cyclic subgroup F of order 4 (which is inside $\Sigma \cap \langle\alpha\rangle$). But now F must be normal in G and the previous lemma leads to a contradiction. This completes the proof of the lemma.

Looking back at the above lemmas one finds that Proposition 5 has now been proved. Moreover, Propositions 2 to 5 and Lemma 3.6 now show that G must be one of the groups listed in Theorems A and A$'$. So all that remains to be done is to verify that G is a planar group such that the plane Π_G has the correct order. This also has already been done in Sections 3 to 5 in all but the following cases:

$$G \cong S_4, A_5, S_5.$$

We now consider each of these cases in turn.

LEMMA. *If $G \cong S_4$ then G is a planar group and Π_G has order $n^{1/4}$.*

Proof. Now $G = \langle\alpha, H\rangle$ where $H \cong A_4$, $|\alpha| = 4$ and $\alpha^2 \in H$. But we have already accepted that H is a planar and Π_H has order $n^{1/4}$. Since $\operatorname{Fix}(\alpha) = \operatorname{Fix}(\alpha^2) \supset \operatorname{Fix}(H)$ the lemma follows immediately.

The same argument prove the following.

LEMMA. *Suppose $G \cong S_5$ and that $H \cong A_5$ is a planar subgroup such that Π_H has order $n^{1/4}$. Then G itself is planar and Π_G also has order $n^{1/4}$.*

So to complete the proofs of Theorems A and A′ it is now sufficient to verify the following.

LEMMA. *Suppose $G \cong A_5$. Then G is a planar group and Π_G has order $n^{1/4}$.*

Proof. On systematically computing all the Wielandt polynomials for A_5 one notices the following polynomial.

$$f_{12} f_{10}^2 = f_4 f_{60}^2 \tag{W}$$

where the subscripts refer to the orders of the subgroups of G involved.

Let us now apply (W) to the case when Π is not a translation plane: this is legitimate since by condition (B), G is planar and Result 2.2 applies. Now f_4, f_{10}, f_{12} are all of order $n^{1/4}$ (when we think of G as acting on the "Y-axis"), cf. Results 3.5, 4.2, and 5.1. Hence f_{60} is also $n^{1/4}$.

It remains to consider the case when Π^l is an affine translation plane. So by Lemma 0.1, $G \subset \mathrm{Aut}(V, +)$ such that G permutes the components of Γ, the spread on V associated with Π^l. This time Eq. (W) shows that $F = \mathrm{Fix}(G)$ is a subgroup of V such that $|F| = n^{1/2}$, as $|V| = n^2$. So either G is a planar group of the required type or $F \subset M$, where M is some component of Γ. But by Theorem 3.5 any 4-group K in G fixes precisely $n^{1/4}$ points of M and so we contradict $|F| = n^{1/2}$. Hence the lemma must be valid.

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