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n-Star modules and *n*-tilting modules

Wei Jiaqun

Department of Mathematics, Nanjing Normal University, Nanjing 210097, PR China Received 14 January 2004 Available online 7 December 2004 Communicated by Kent R. Fuller

Abstract

We give some characterizations of (not necessarily selfsmall) *n*-star modules and prove that (not necessarily finitely generated) *n*-tilting modules are precisely (not necessarily selfsmall) *n*-star modules *n*-presenting all the injectives. © 2004 Elsevier Inc. All rights reserved.

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1. Introduction

The classical tilting modules (simply, finitely 1-tilting modules) were first considered in the early eighties by Brenner–Butler [4], Bongartz [3] and Happel and Ringel [9] etc. Beginning with Miyashita [10], the defining conditions for a classical tilting module were extended to arbitrary rings by many authors, Wakamatsu [13], Colby and Fuller [5], Colpi and Trlifaj [8], and recently, Angeleri Hügel and Coelho [1], Bazzoni [2] and Wei [14]. Among them, Miyashita [10] considered tilting modules of finitely generated projective dimension $\leq n$ (simply, finitely *n*-tilting modules), Colpi and Trlifaj [8] investigated (not necessarily finitely generated) tilting modules of projective dimension ≤ 1 (simply, 1-tilting modules) and then, Angeleri Hügel and Coelho [1] and Bazzoni [2] considered (not

E-mail address: weijiaqun@njnu.edu.cn.

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necessarily finitely generated) tilting modules of projective dimension $\leq n$ (simply, *n*-tilting modules).

One important result in the theory of finitely tilting modules is the famous Brenner– Butler Theorem which shows that a finitely tilting module induces some equivalences between certain subcategories. In this sense, *-modules (i.e., selfsmall 1-star modules, see Section 3 for the detailed definition) investigated by Menini and Orsatti [11] and Colpi [6] etc., as well as $*^n$ -modules (i.e., selfsmall *n*-star modules) considered in [16], are also generalizations of the classical tilting modules. In fact, the classical tilting modules are just *-modules which generate all the injectives [7] and finitely *n*-tilting modules are just $*^n$ -modules which admit a finitely generated projective resolution and which *n*-present all the injectives [2,16].

Note that *-modules and $*^n$ -modules considered above are selfsmall. In particular, *-modules are always finitely generated [12] (while it is still an open question whether $*^n$ -modules are finitely generated). So (not necessarily finitely generated) *n*-tilting modules cannot be characterized as a subclass of these selfsmall *n*-star modules. From this point of view, it is natural to consider *n*-star modules which are not necessarily selfsmall and to study the relations between them and (not necessarily finitely generated) *n*-tilting modules.

This rises the problem of characterizing n-star modules in the general setting. The techniques used in the literature to study selfsmall n-star modules do not work here because they heavily depended on the property 'selfsmall' (see [6,16]). Hence we adopt a new method in this paper, and successfully, obtain the desired results.

We now state the main result of this paper.

Theorem. Let T be an R-module. Then T is an n-tilting module if and only if T is an n-star module which n-presents all the injectives.

Throughout this paper, *R* will be an associative ring with nonzero identity and *T* will be a left *R*-module. Let *R*-Mod be the class of left *R*-modules. If $f: X \to Y$ and $g: Y \to Z$ are homomorphisms, we denote by fg the composition of f and g.

Given an *R*-module *T*, we denote by $T^{\perp_{1 \leq i \leq n}} := \{M \in R \text{-Mod} \mid \text{Ext}_{R}^{i}(T, M) = 0 \text{ for all } 1 \leq i \leq n\}$. $T^{\perp_{i \geq 1}}$ and $T^{\perp_{1}}$ are defined similarly.

For every *R*-module *T*, we denote by Add *T* the class of modules isomorphic to direct summands of direct sums of copies of *T* and by $\operatorname{Pres}^n T := \{M \in R \operatorname{-Mod} \mid \text{there exists} an exact sequence <math>T_n \to \cdots \to T_1 \to M \to 0$ with $T_i \in \operatorname{Add} T\}$. Note that there is a clear inclusion between categories: $\operatorname{Pres}^{n+1} T \subseteq \operatorname{Pres}^n T$. Also note that $\operatorname{Pres}^1 T$ was usually denoted by Gen *T* in the literature. Sometimes we also denote by $\operatorname{Pres}^0 T := R \operatorname{-Mod}$.

2. n-Quasi-projective

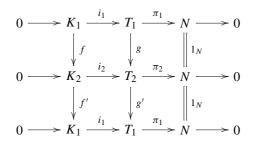
Definition 2.1 (see also [16]). Let *T* be an *R*-module and $n \ge 1$. *T* is said to be *n*-quasiprojective if for any exact sequence $0 \to L \to T_0 \to N \to 0$ with $T_0 \in \text{Add } T$ and $L \in \text{Pres}^{n-1} T$, the induced sequence $0 \to \text{Hom}_R(T, L) \to \text{Hom}_R(T, T_0) \to \text{Hom}_R(T, N) \to 0$ is also exact.

We note that the notion of 2-quasi-projective was also known as w- Σ -quasi-projective [6] and the notion of 1-quasi-projective was also known as Σ -quasi-projective.

It is easy to see that if T is n-quasi-projective for some n then T is also m-quasiprojective for all $m \ge n$.

The following is the key lemma to obtain our results.

Lemma 2.2. Let K_1 , K_2 , T_1 , T_2 and N be R-modules such that the following diagram is commutative with exact rows:



Then $K_1 \oplus T_2 \cong K_2 \oplus T_1$.

Proof. Consider the following diagram:

$$0 \longrightarrow K_1 \xrightarrow{\theta}_{i_1} T_1 \xrightarrow{\pi_1}_{i_1} N \longrightarrow 0$$

Since $(1_{T_1} - gg')\pi_1 = \pi_1 - gg'\pi_1 = 0$ by assumption, $1_{T_1} - gg'$ factors through i_1 . Let $\theta: T_1 \to K_1$ be a homomorphism such that $1_{T_1} - gg' = \theta i_1$. Then we check that $i_1\theta i_1 = i_1(1_{T_1} - gg') = i_1 - i_1gg' = i_1 - ff'i_1 = (1_{K_1} - ff')i_1$ by assumption. Since i_1 is a monomorphism, we deduce that $i_1\theta = 1_{K_1} - ff'$, or equivalently, $i_1\theta + ff' = 1_{K_1}$.

Now we consider the following diagram:

It is straightforward that the above diagram is commutative with exact rows. Hence we obtain that $K_1 \oplus T_2 \cong K_2 \oplus T_1$. \Box

The preceding lemma yields the following result which turns out to be very useful.

Lemma 2.3. Let T be an R-module. Assume that $0 \to K_1 \to T_1 \to N \to 0$ and $0 \to K_2 \to T_2 \to N \to 0$ are exact in R-Mod, where $T_1, T_2 \in \text{Add } T$. If both sequences stay exact under the functor $\text{Hom}_R(T, -)$, then

$$K_1 \oplus T_2 \cong K_2 \oplus T_1.$$

Proof. Under our assumption, one can easily check that there is a diagram as in Lemma 2.2. Hence the conclusion holds. \Box

We now turn to a characterization of *n*-quasi-projective modules.

Proposition 2.4. *The following are equivalent for an R-module T.*

- (1) T is n-quasi-projective.
- (2) If $0 \to L \to T_0 \to N \to 0$ is an exact sequence with $T_0 \in \operatorname{Add} T$ and $N \in \operatorname{Pres}^n T$, then $L \in \operatorname{Pres}^{n-1} T$ if and only if the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, T_0) \to \operatorname{Hom}_R(T, N) \to 0$ is exact.

Proof. (2) \Rightarrow (1) is easy.

 $(1) \Rightarrow (2)$. Given an exact sequence $0 \rightarrow L \rightarrow T_0 \rightarrow N \rightarrow 0$ with $T_0 \in \text{Add} T$, if $L \in \text{Pres}^{n-1} T$ then the induced sequence $0 \rightarrow \text{Hom}_R(T, L) \rightarrow \text{Hom}_R(T, T_0) \rightarrow \text{Hom}_R(T, N) \rightarrow 0$ is clearly exact by Definition 2.1.

On the other hand, assume that $0 \to L \to T_0 \to N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$. Since $N \in \operatorname{Pres}^n T$, we have an exact sequence $\to L' \to T'_0 \to N \to 0$ with $T'_0 \in \operatorname{Add} T$ and $L' \in \operatorname{Pres}^{n-1} T$. Note that the last sequence stays exact under the functor $\operatorname{Hom}_R(T, -)$ by Definition 2.1, so we can apply Lemma 2.3 to obtain that $L' \oplus T_0 \cong L \oplus T'_0$. It follows that $L \in \operatorname{Pres}^{n-1} T$. \Box

3. *n*-Star modules

Definition 3.1. An *R*-module *T* is said to be an *n*-star module if *T* is (n + 1)-quasi-projective and Pres^{*n*} *T* = Pres^{*n*+1} *T*.

Selfsmall *n*-star modules are just $*^n$ -modules investigated in [16], in particular, self-small 1-star modules are just *-modules investigated in [6] etc.

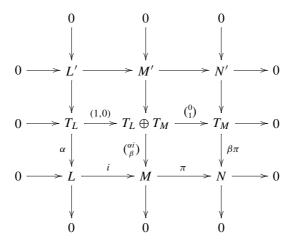
It is an easy corollary from the definition of the *n*-star module *T* that for any $N \in \operatorname{Pres}^n T$ there is an infinite exact sequence $\dots \to f_k T_k \to \dots \to f_1 T_1 \to N \to 0$ with $T_k \in \operatorname{Add} T$ and Ker $f_k \in \operatorname{Pres}^n T$ for all *k*.

Lemma 3.2. Let T be an n-star module. Assume that $0 \to L \to^i M \to^{\pi} N \to 0$ is exact with $L, M \in \operatorname{Pres}^n T$, then $N \in \operatorname{Pres}^n T$, too.

Proof. By assumption, $L, M \in \operatorname{Pres}^n T$, so we have exact sequences

$$0 \to L' \to T_L \xrightarrow{\alpha} L \to 0$$
 and $0 \to M_1 \to T_M \xrightarrow{\beta} M \to 0$,

where L', $M_1 \in \operatorname{Pres}^n T$ and T_L , $T_M \in \operatorname{Add} T$. Now we construct the following exact commutative diagram:



Note that the sequence $0 \to M_1 \to T_M \to M \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$ since *T* is an *n*-star module, so the sequence $0 \to M' \to T_L \oplus T_M \to M \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$ by the constructions. By Proposition 2.4, we obtain that $M' \in \operatorname{Pres}^n T$, since $M \in \operatorname{Pres}^{n+1} T$ and *T* is (n + 1)-quasi-projective by assumption.

Now by repeating the process to the exact sequence $0 \to L' \to M' \to N' \to 0$, where $L', M' \in \operatorname{Pres}^n T$ by the arguments above, and so on, we obtain that $N \in \operatorname{Pres}^n T$, as desired. \Box

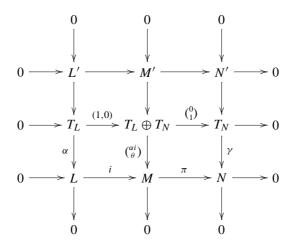
Proposition 3.3. Let T be an n-star module. Assume that the exact sequence $0 \to L \to^i M \to^{\pi} N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$. If two of the three terms L, M, N are in $\operatorname{Pres}^n T$, so is the third one.

Proof. In case $L, M \in \operatorname{Pres}^n T$, the assertion follows from Proposition 3.2. Now assume that $L, N \in \operatorname{Pres}^n T$. Then we have exact sequences

$$0 \to L' \to T_L \xrightarrow{\alpha} L \to 0$$
 and $0 \to N' \to T_N \xrightarrow{\gamma} N \to 0$,

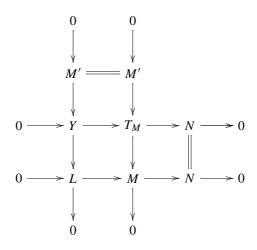
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where $L', N' \in \operatorname{Pres}^n T$ and $T_L, T_N \in \operatorname{Add} T$. Since the sequence $0 \to L \to^i M \to^{\pi} N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$, there is a homomorphism $\theta : T_N \to M$ such that $\theta \pi = \gamma$. Hence we can construct the following exact commutative diagram:



By applying the functor $\operatorname{Hom}_R(T, -)$ to the diagram, we obtain that the upper row stays exact under the functor $\operatorname{Hom}_R(T, -)$, since so do the middle row and the left column. Therefore, we can repeat our process to the upper row, and so on, we obtain that $M \in \operatorname{Pres}^n T$.

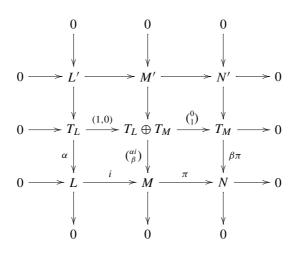
In the last case, assume that $M, N \in \operatorname{Pres}^n T$. Then we have an exact sequence $0 \to M' \to T_M \to {}^{\beta} M \to 0$ with $M' \in \operatorname{Pres}^n T$ and $T_M \in \operatorname{Add} T$. Now we consider the following pullback diagram:



Since the bottom row stays exact under the functor $\operatorname{Hom}_R(T, -)$, as well as the middle column, by assumption and the constructions, we have that the induced homomorphism $\operatorname{Hom}_R(T, T_M) \to \operatorname{Hom}_R(T, N)$ is an epimorphism and consequently, the middle row stays exact under the functor $\text{Hom}_R(T, -)$. Thanks to Proposition 2.4, we obtain that $Y \in \text{Pres}^n T$, since $N \in \text{Pres}^{n+1} T$ and T is (n + 1)-quasi-projective by assumption. Now we can use Lemma 3.2 to conclude that $L \in \text{Pres}^n T$. \Box

Proposition 3.4. Let T be an n-star module. Then the functor $\operatorname{Hom}_R(T, -)$ preserves short exact sequences in $\operatorname{Pres}^n T$.

Proof. Assume that $0 \to L \to^i M \to^{\pi} N \to 0$ is exact in Pres^{*n*} *T*. So we have exact sequences $0 \to L' \to T_L \to^{\alpha} L \to 0$ and $0 \to M_1 \to T_M \to^{\beta} M \to 0$, where $L', M_1 \in$ Pres^{*n*} *T* and $T_L, T_M \in$ Add *T*. As in the proof of Lemma 3.2, we construct the following exact commutative diagram:



As proved in Lemma 3.2, $N' \in \operatorname{Pres}^n T$ and hence, the right column stays exact under the functor $\operatorname{Hom}_R(T, -)$ since T is (n + 1)-quasi-projective. Note that the middle row clearly stays exact under the functor $\operatorname{Hom}_R(T, -)$, so we have that the induced homomorphism $\operatorname{Hom}_R(T, M) \to^{\operatorname{Hom}_R(T, \pi)} \operatorname{Hom}_R(T, N)$ is an epimorphism. It follows that the sequence $0 \to L \to M \to N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$. \Box

We give now some characterizations of *n*-star modules.

Theorem 3.5. *The following are equivalent for an R-module T*:

- (1) T is an n-star module.
- (2) If $0 \to L \to T_0 \to N \to 0$ is an exact sequence with $T_0 \in \operatorname{Add} T$ and $N \in \operatorname{Pres}^n T$, then $L \in \operatorname{Pres}^n T$ if and only if the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, T_0) \to \operatorname{Hom}_R(T, N) \to 0$ is exact.
- (3) If $0 \to L \to M \to N \to 0$ is an exact sequence with $M, N \in \operatorname{Pres}^n T$, then $L \in \operatorname{Pres}^n T$ if and only if the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, M) \to \operatorname{Hom}_R(T, N) \to 0$ is exact.

Proof. (1) \Rightarrow (2). If $L \in \operatorname{Pres}^n T$ then the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, T_0) \to \operatorname{Hom}_R(T, N) \to 0$ is clearly exact since T is (n + 1)-quasi-projective. On the other hand, since $N \in \operatorname{Pres}^n T$ and T is an *n*-star module, $N \in \operatorname{Pres}^{n+1} T$ too. Hence we obtain the converse implication from Proposition 2.4.

 $(2) \Rightarrow (3)$. The only-if-part follows from Proposition 3.4 and the if-part follows from Proposition 3.3.

 $(3) \Rightarrow (1)$. It is easy to see that T is (n + 1)-quasi-projective. It remains to show that $\operatorname{Pres}^n T \subseteq \operatorname{Pres}^{n+1} T$. Assume that $N \in \operatorname{Pres}^n T$ and take $T_N := T^{(\operatorname{Hom}_R(T,N))}$. Then we obtain an exact sequence $0 \to N' \to T_N \to N \to 0$ which stays exact under the functor $\operatorname{Hom}_R(T, -)$. Hence we have that $N' \in \operatorname{Pres}^n T$ by assumption. Therefore, $N \in \operatorname{Pres}^{n+1} T$. \Box

The following result characterizes *n*-star modules T such that $\operatorname{Pres}^n T$ is closed under extensions.

Proposition 3.6. *The following are equivalent for an R-module T*:

(1) T is an n-star module and $\operatorname{Pres}^n T$ is closed under extensions.

(2) $\operatorname{Pres}^n T = \operatorname{Pres}^{n+1} T \subseteq T^{\perp_1}$.

Proof. (1) \Rightarrow (2). It is sufficient to show that $\operatorname{Pres}^n T \subseteq T^{\perp_1}$. For any $X \in \operatorname{Pres}^n T$ and any extension of T by $X: 0 \to X \to Y \to T \to 0$, we have that $Y \in \operatorname{Pres}^n T$ since $\operatorname{Pres}^n T$ is closed under extensions. By Proposition 3.4, the sequence stays exact under the functor $\operatorname{Hom}_R(T, -)$. Hence we obtain that the extension splits.

 $(2) \Rightarrow (1)$. Any exact sequence $0 \to L \to T_N \to N \to 0$ with $L \in \operatorname{Pres}^n T$ and $T_N \in \operatorname{Add} T$ stays exact under the functor $\operatorname{Hom}_R(T, -)$ since $\operatorname{Ext}^1_R(T, L) = 0$ by assumption. It follows that T is (n + 1)-quasi-projective. Since $\operatorname{Pres}^n T = \operatorname{Pres}^{n+1} T$, we obtain that T is an *n*-star module. Now for any extension $0 \to L \to M \to N \to 0$ of N by L, where $L, N \in \operatorname{Pres}^n T$, we have that it stays exact under the functor $\operatorname{Hom}_R(T, -)$ since $\operatorname{Ext}^1_R(T, L) = 0$. Applying Proposition 3.3, we get that $M \in \operatorname{Pres}^n T$. Hence $\operatorname{Pres}^n T$ is closed under extensions. \Box

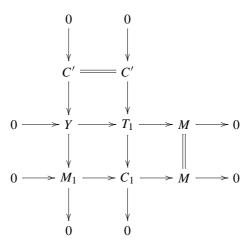
Recall that a class of *R*-modules *C* is said to be closed under *n*-images if for any exact sequence $C_n \to \cdots \to C_1 \to M \to 0$ in *R*-Mod with all C_i in *C*, there holds that $M \in C$ [15]. The class $\text{Pres}^1 T$ (i.e., Gen T) is clearly closed under 1-images (i.e., images). We do not know that whether or not $\text{Pres}^n T$ is closed under *n*-images in general. But if *T* is an *n*-star module such that $\text{Pres}^n T$ is closed under extensions, we have the following result.

Proposition 3.7. Let T be an n-star module such that $\operatorname{Pres}^n T$ is closed under extensions. Then $\operatorname{Pres}^k(\operatorname{Pres}^n T) = \operatorname{Pres}^k T$ for all $k \ge 1$, where $\operatorname{Pres}^k(\operatorname{Pres}^n T)$ denotes the class of R-modules M such that there is an exact sequence $C_k \to \cdots \to C_1 \to M \to 0$ with all C_i in $\operatorname{Pres}^n T$. In particular, $\operatorname{Pres}^n T$ is closed under n-images.

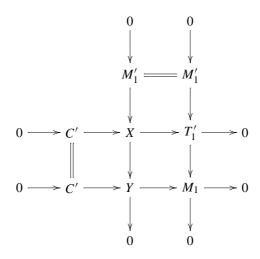
Proof. It is easy to see that $\operatorname{Pres}^{k} T \subseteq \operatorname{Pres}^{k}(\operatorname{Pres}^{n} T)$. We will show that $\operatorname{Pres}^{k}(\operatorname{Pres}^{n} T) \subseteq \operatorname{Pres}^{k} T$. We proceed it by induction on k.

In case k = 1, the conclusion is clear. So we assume now that $\operatorname{Pres}^{j}(\operatorname{Pres}^{n} T) = \operatorname{Pres}^{j} T$ for all $1 \leq j \leq k$.

Let *M* be an *R*-module such that $C_{k+1} \to \cdots \to C_1 \to {}^{f_1} M \to 0$ is exact with all C_i in Pres^{*n*} *T*. Denote by M_1 the kernel of f_1 . Then $M_1 \in \operatorname{Pres}^k T$ by the induction assumption. Note that $C_1 \in \operatorname{Pres}^n T$ and *T* is an *n*-star module, so there is an exact sequence $0 \to C' \to T_1 \to C_1 \to 0$ with $T_1 \in \operatorname{Add} T$ and $C' \in \operatorname{Pres}^n T$. Consider now the following pullback diagram:



Let $0 \to M'_1 \to T'_1 \to M_1 \to 0$ be exact with $M'_1 \in \operatorname{Pres}^{k-1} T$ and $T'_1 \in \operatorname{Add} T$. Then we also have the following pullback diagram:



By assumption, $\operatorname{Pres}^n T$ is closed under extensions, so we have that $X \in \operatorname{Pres}^n T$ from the middle row. It follows that $Y \in \operatorname{Pres}^k(\operatorname{Pres}^n T)$ and consequently, $Y \in \operatorname{Pres}^k T$, by the

induction assumption. From the exact sequence $0 \to Y \to T_1 \to M \to 0$ we deduce that $M \in \operatorname{Pres}^{k+1} T$. \Box

4. *n*-Tilting modules

We recall the following definition of (not necessarily finitely generated) *n*-tilting modules [1,2].

Definition 4.1. An *R*-module *T* is said to be *n*-tilting if it satisfies the following conditions:

- (1) p.d. $T \leq n$, here p.d. T denotes the projective dimension of T.
- (2) $\operatorname{Ext}_{R}^{i}(T, T^{(\lambda)}) = 0$ for all $i \ge 1$ and all cardinals λ .
- (3) There is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$, where T_i 's are isomorphic to direct summands of direct sums of copies of T.

Not necessarily finitely generated tilting modules of projective dimension ≤ 1 were already investigated by Colpi and Trlifaj in [8], and it was shown that they are characterized by the condition $\operatorname{Pres}^1 T = T^{\perp_1}$. Not necessarily finitely generated tilting modules of projective dimension $\leq n$ were then studied by Angeleri Hügel and Coelho [1] and Bazzoni [2]. Generalizing the result in [8] and the result of [16], Bazzoni [2] showed that T is an n-tilting module if and only if $\operatorname{Pres}^n T = T^{\perp_i \geq 1}$. In case n = 1, the condition is that $\operatorname{Pres}^1 T = T^{\perp_i \geq 1}$, which is a little different from, but is equivalent to the condition that $\operatorname{Pres}^1 T = T^{\perp_i \geq 1}$. We will show that an R-module T is n-tilting if and only if $\operatorname{Pres}^n T = T^{\perp_i \geq i}$, which completely coincides with the result in [8] in case n = 1.

Moreover, as promised before, we will extend characterizations of finitely *n*-tilting modules in term of $*^n$ -modules (see [7] and [16]) to the general case.

Lemma 4.2. Let T be an n-star module such that $\operatorname{Pres}^n T$ is closed under extensions. Assume that $0 \to L \to M \to N \to 0$ is exact with $M, N \in \operatorname{Pres}^n T$. Then $L \in \operatorname{Pres}^n T$ if and only if $L \in T^{\perp_1}$.

Proof. The only-if-part follows from Proposition 3.6 and the if-part follows from Proposition 3.3. \Box

The following is our main result.

Theorem 4.3. Denote by \mathcal{I} the class of all injective *R*-modules. The following are equivalent for an *R*-module *T*.

- (1) T is an n-tilting module.
- (2) $\operatorname{Pres}^n T = T^{\perp_1 \leq i \leq n}$.
- (3) *T* is an *n*-star module and $\mathcal{I} \subseteq \operatorname{Pres}^n T$.
- (4) $\mathcal{I} \subseteq \operatorname{Pres}^n T = \operatorname{Pres}^{n+1} T \subseteq T^{\perp_1}$.

Proof. (1) \Rightarrow (2). By [2], *T* is an *n*-tilting module if and only if $\operatorname{Pres}^n T = T^{\perp_i \ge 1}$. Since p.d. $T \le n$, we obtain that $\operatorname{Pres}^n T = T^{\perp_1 \le i \le n}$.

 $(2) \Rightarrow (3)$. Clearly, $\mathcal{I} \subseteq \operatorname{Pres}^n T$ by assumption. It remains to show that T is an n-star module. Let $0 \to L \to T_N \to N \to 0$ be exact with $T_N \in \operatorname{Add} T$ and $N \in \operatorname{Pres}^n T$. By applying the functor $\operatorname{Hom}_R(T, -)$, we obtain that $L \in T^{\perp_{2 \leq i \leq n}}$ and the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, T_N) \to \operatorname{Hom}_R(T, N) \to \operatorname{Ext}^1_R(T, L) \to 0$ is exact, since $\operatorname{Pres}^n T = T^{\perp_{1 \leq i \leq n}}$ by assumption. Then the sequence $0 \to L \to T_N \to N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$ if and only if $L \in T^{\perp_1}$ if and only if $L \in T^{\perp_{1 \leq i \leq n}} = \operatorname{Pres}^n T$. Now the conclusion follows from Theorem 3.5.

 $(3) \Rightarrow (4)$. By the definition of *n*-star modules, we need only to show that $\operatorname{Pres}^n T \subseteq T^{\perp_1}$. For any $N \in \operatorname{Pres}^n T$, let then $0 \to N \to I_N \to N' \to 0$ be exact with $I_N \in \mathcal{I}$. By assumption, $\mathcal{I} \subseteq \operatorname{Pres}^n T$, hence $N' \in \operatorname{Pres}^n T$ by Lemma 3.2. It follows that the induced sequence $0 \to \operatorname{Hom}_R(T, N) \to \operatorname{Hom}_R(T, I_N) \to \operatorname{Hom}_R(T, N') \to 0$ is exact by Proposition 3.4. Therefore $N \in T^{\perp_1}$, since I_N is an injective *R*-module.

 $(4) \Rightarrow (1)$. We need only to show that $\operatorname{Pres}^n T = T^{\perp_i \ge 1}$ by [2]. Note first that, by assumption and Proposition 3.6, *T* is an *n*-star module and $\operatorname{Pres}^n T$ is closed under extensions.

Now for any $M \in \operatorname{Pres}^n T$, letting $0 \to M \to I_1 \to g_1 \cdots \to I_k \to g_k \cdots$ be an injective resolution of M, we obtain that $M_i := \operatorname{Im} g_i \in \operatorname{Pres}^n T$ by Lemma 3.2, since $I_i \in \mathcal{I} \subseteq \operatorname{Pres}^n T$, for all $i \ge 1$. It follows that $M_i \in T^{\perp_1}$, for each i, by assumption. Therefore, we deduce that $\operatorname{Ext}^j_R(T, M) \cong \operatorname{Ext}^1_R(T, M_{j-1}) = 0$, for all $j \ge 1$, by the dimension shifting (setting $M_0 := M$). Hence, $\operatorname{Pres}^n T \subseteq T^{\perp_i \ge 1}$.

On the other hand, for any $M \in T^{\perp_i \ge 1}$, by taking again an injective resolution $0 \to M \to I_1 \to g^{g_1} \cdots \to I_k \to g^{g_k} \cdots$ of M, we obtain that $M_i \in T^{\perp_i \ge 1}$ for all $i \ge 1$ by the dimension shifting. By Proposition 3.7, $\operatorname{Pres}^n T$ is closed under *n*-images. It follows that $M_n \in \operatorname{Pres}^n T$. Since $M_{n-1} \in T^{\perp_i \ge 1}$ and $I_n \in \mathcal{I} \subseteq \operatorname{Pres}^n T$ too, we have that $M_{n-1} \in \operatorname{Pres}^n T$ by Lemma 4.2. By repeating the process, and so on, we finally obtain that $M \in \operatorname{Pres}^n T$, as desired. \Box

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