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n-Star modules and n-tilting modules

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Abstract

We give some characterizations of (not necessarily selfsmall) n-star modules and prove that (not necessarily finitely generated) n-tilting modules are precisely (not necessarily selfsmall) n-star modules n-presenting all the injectives.

Keywords: Star modules; Tilting modules; Quasi-projective

1. Introduction

The classical tilting modules (simply, finitely 1-tilting modules) were first considered in the early eighties by Brenner–Butler [4], Bongartz [3] and Happel and Ringel [9] etc. Beginning with Miyashita [10], the defining conditions for a classical tilting module were extended to arbitrary rings by many authors, Wakamatsu [13], Colby and Fuller [5], Colpi and Trlifaj [8], and recently, Angeleri Hügel and Coelho [1], Bazzoni [2] and Wei [14]. Among them, Miyashita [10] considered tilting modules of finitely generated projective dimension $\leqslant n$ (simply, finitely n-tilting modules), Colpi and Trlifaj [8] investigated (not necessarily finitely generated) tilting modules of projective dimension $\leqslant 1$ (simply, 1-tilting modules) and then, Angeleri Hügel and Coelho [1] and Bazzoni [2] considered (not

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necessarily finitely generated) tilting modules of projective dimension $\leqslant n$ (simply, n-tilting modules).

One important result in the theory of finitely tilting modules is the famous Brenner–Butler Theorem which shows that a finitely tilting module induces some equivalences between certain subcategories. In this sense, $*$-modules (i.e., selfsmall 1-star modules, see Section 3 for the detailed definition) investigated by Menini and Orsatti [11] and Colpi [6] etc., as well as $*^n$-modules (i.e., selfsmall n-star modules) considered in [16], are also generalizations of the classical tilting modules. In fact, the classical tilting modules are just $*$-modules which generate all the injectives [7] and finitely n-tilting modules are just $*^n$-modules which admit a finitely generated projective resolution and which n-present all the injectives [2,16].

Note that $*$-modules and $*^n$-modules considered above are selfsmall. In particular, $*$-modules are always finitely generated [12] (while it is still an open question whether $*^n$-modules are finitely generated). So (not necessarily finitely generated) n-tilting modules cannot be characterized as a subclass of these selfsmall n-star modules. From this point of view, it is natural to consider n-star modules which are not necessarily selfsmall and to study the relations between them and (not necessarily finitely generated) n-tilting modules.

This rises the problem of characterizing n-star modules in the general setting. The techniques used in the literature to study selfsmall n-star modules do not work here because they heavily depended on the property 'selfsmall' (see [6,16]). Hence we adopt a new method in this paper, and successfully, obtain the desired results.

We now state the main result of this paper.

Theorem. *Let T be an R-module. Then T is an n-tilting module if and only if T is an n-star module which n-presents all the injectives.*

Throughout this paper, R will be an associative ring with nonzero identity and T will be a left R-module. Let R-Mod be the class of left R-modules. If $f: X \to Y$ and $g: Y \to Z$ are homomorphisms, we denote by fg the composition of f and g.

Given an R-module T, we denote by $T^{\perp_{1\leqslant i\leqslant n}} := \{M \in R\text{-Mod} \mid \mathrm{Ext}^i_R(T, M) = 0$ for all $1 \leqslant i \leqslant n\}$. $T^{\perp_{i\geqslant 1}}$ and $T^{\perp_1}$ are defined similarly.

For every R-module T, we denote by $\mathrm{Add}\, T$ the class of modules isomorphic to direct summands of direct sums of copies of T and by $\mathrm{Pres}^n\, T := \{M \in R\text{-Mod} \mid$ there exists an exact sequence $T_n \to \cdots \to T_1 \to M \to 0$ with $T_i \in \mathrm{Add}\, T\}$. Note that there is a clear inclusion between categories: $\mathrm{Pres}^{n+1}\, T \subseteq \mathrm{Pres}^n\, T$. Also note that $\mathrm{Pres}^1\, T$ was usually denoted by $\mathrm{Gen}\, T$ in the literature. Sometimes we also denote by $\mathrm{Pres}^0\, T := R$-Mod.

2. n-Quasi-projective

Definition 2.1 (see also [16]). Let T be an R-module and $n \geqslant 1$. T is said to be n-quasi-projective if for any exact sequence $0 \to L \to T_0 \to N \to 0$ with $T_0 \in \mathrm{Add}\, T$ and $L \in \mathrm{Pres}^{n-1}\, T$, the induced sequence $0 \to \mathrm{Hom}_R(T, L) \to \mathrm{Hom}_R(T, T_0) \to \mathrm{Hom}_R(T, N) \to 0$ is also exact.

We note that the notion of 2-quasi-projective was also known as w-Σ-quasi-projective [6] and the notion of 1-quasi-projective was also known as Σ-quasi-projective.

It is easy to see that if T is n-quasi-projective for some n then T is also m-quasi-projective for all $m \geqslant n$.

The following is the key lemma to obtain our results.

Lemma 2.2. *Let K_1, K_2, T_1, T_2 and N be R-modules such that the following diagram is commutative with exact rows*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \xrightarrow{i_1} & T_1 & \xrightarrow{\pi_1} & N & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \|{\scriptstyle 1_N} & & \\
0 & \longrightarrow & K_2 & \xrightarrow{i_2} & T_2 & \xrightarrow{\pi_2} & N & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \|{\scriptstyle 1_N} & & \\
0 & \longrightarrow & K_1 & \xrightarrow{i_1} & T_1 & \xrightarrow{\pi_1} & N & \longrightarrow & 0
\end{array}$$

Then $K_1 \oplus T_2 \cong K_2 \oplus T_1$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & T_1 & & & & \\
 & & & {\scriptstyle \theta}\swarrow & \downarrow{\scriptstyle 1_{T_1}-gg'} & & & & \\
0 & \longrightarrow & K_1 & \xrightarrow{i_1} & T_1 & \xrightarrow{\pi_1} & N & \longrightarrow & 0
\end{array}$$

Since $(1_{T_1} - gg')\pi_1 = \pi_1 - gg'\pi_1 = 0$ by assumption, $1_{T_1} - gg'$ factors through i_1. Let $\theta : T_1 \to K_1$ be a homomorphism such that $1_{T_1} - gg' = \theta i_1$. Then we check that $i_1\theta i_1 = i_1(1_{T_1} - gg') = i_1 - i_1 gg' = i_1 - ff'i_1 = (1_{K_1} - ff')i_1$ by assumption. Since i_1 is a monomorphism, we deduce that $i_1\theta = 1_{K_1} - ff'$, or equivalently, $i_1\theta + ff' = 1_{K_1}$.

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \xrightarrow{(f,-i_1)} & K_2 \oplus T_1 & \xrightarrow{\binom{i_2}{g}} & T_2 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1_{K_1}} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} f' & i_2\\ -\theta & g\end{smallmatrix}\right)} & & \|{\scriptstyle 1_{T_2}} & & \\
0 & \longrightarrow & K_1 & \xrightarrow{(1,0)} & K_1 \oplus T_2 & \xrightarrow{\binom{0}{1}} & T_2 & \longrightarrow & 0
\end{array}$$

It is straightforward that the above diagram is commutative with exact rows. Hence we obtain that $K_1 \oplus T_2 \cong K_2 \oplus T_1$. □

The preceding lemma yields the following result which turns out to be very useful.

Lemma 2.3. *Let T be an R-module. Assume that $0 \to K_1 \to T_1 \to N \to 0$ and $0 \to K_2 \to T_2 \to N \to 0$ are exact in R-Mod, where $T_1, T_2 \in \operatorname{Add} T$. If both sequences stay exact under the functor $\operatorname{Hom}_R(T, -)$, then*

$$K_1 \oplus T_2 \cong K_2 \oplus T_1.$$

Proof. Under our assumption, one can easily check that there is a diagram as in Lemma 2.2. Hence the conclusion holds. □

We now turn to a characterization of n-quasi-projective modules.

Proposition 2.4. *The following are equivalent for an R-module T.*

(1) *T is n-quasi-projective.*
(2) *If $0 \to L \to T_0 \to N \to 0$ is an exact sequence with $T_0 \in \operatorname{Add} T$ and $N \in \operatorname{Pres}^n T$, then $L \in \operatorname{Pres}^{n-1} T$ if and only if the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, T_0) \to \operatorname{Hom}_R(T, N) \to 0$ is exact.*

Proof. (2) ⇒ (1) is easy.

(1) ⇒ (2). Given an exact sequence $0 \to L \to T_0 \to N \to 0$ with $T_0 \in \operatorname{Add} T$, if $L \in \operatorname{Pres}^{n-1} T$ then the induced sequence $0 \to \operatorname{Hom}_R(T, L) \to \operatorname{Hom}_R(T, T_0) \to \operatorname{Hom}_R(T, N) \to 0$ is clearly exact by Definition 2.1.

On the other hand, assume that $0 \to L \to T_0 \to N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$. Since $N \in \operatorname{Pres}^n T$, we have an exact sequence $\to L' \to T_0' \to N \to 0$ with $T_0' \in \operatorname{Add} T$ and $L' \in \operatorname{Pres}^{n-1} T$. Note that the last sequence stays exact under the functor $\operatorname{Hom}_R(T, -)$ by Definition 2.1, so we can apply Lemma 2.3 to obtain that $L' \oplus T_0 \cong L \oplus T_0'$. It follows that $L \in \operatorname{Pres}^{n-1} T$. □

3. *n*-Star modules

Definition 3.1. An R-module T is said to be an n-star module if T is $(n+1)$-quasi-projective and $\operatorname{Pres}^n T = \operatorname{Pres}^{n+1} T$.

Selfsmall n-star modules are just $*^n$-modules investigated in [16], in particular, selfsmall 1-star modules are just $*$-modules investigated in [6] etc.

It is an easy corollary from the definition of the n-star module T that for any $N \in \operatorname{Pres}^n T$ there is an infinite exact sequence $\cdots \to^{f_k} T_k \to \cdots \to^{f_1} T_1 \to N \to 0$ with $T_k \in \operatorname{Add} T$ and $\operatorname{Ker} f_k \in \operatorname{Pres}^n T$ for all k.

Lemma 3.2. *Let T be an n-star module. Assume that $0 \to L \to^i M \to^\pi N \to 0$ is exact with $L, M \in \operatorname{Pres}^n T$, then $N \in \operatorname{Pres}^n T$, too.*

Proof. By assumption, $L, M \in \mathrm{Pres}^n T$, so we have exact sequences

$$0 \to L' \to T_L \xrightarrow{\alpha} L \to 0 \quad \text{and} \quad 0 \to M_1 \to T_M \xrightarrow{\beta} M \to 0,$$

where $L', M_1 \in \mathrm{Pres}^n T$ and $T_L, T_M \in \mathrm{Add}\, T$. Now we construct the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_L & \xrightarrow{(1,0)} & T_L \oplus T_M & \xrightarrow{\binom{0}{1}} & T_M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \binom{\alpha i}{\beta}} & & \downarrow{\scriptstyle \beta\pi} & & \\
0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{\pi} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that the sequence $0 \to M_1 \to T_M \to M \to 0$ stays exact under the functor $\mathrm{Hom}_R(T, -)$ since T is an n-star module, so the sequence $0 \to M' \to T_L \oplus T_M \to M \to 0$ stays exact under the functor $\mathrm{Hom}_R(T, -)$ by the constructions. By Proposition 2.4, we obtain that $M' \in \mathrm{Pres}^n T$, since $M \in \mathrm{Pres}^{n+1} T$ and T is $(n+1)$-quasi-projective by assumption.

Now by repeating the process to the exact sequence $0 \to L' \to M' \to N' \to 0$, where $L', M' \in \mathrm{Pres}^n T$ by the arguments above, and so on, we obtain that $N \in \mathrm{Pres}^n T$, as desired. □

Proposition 3.3. *Let T be an n-star module. Assume that the exact sequence $0 \to L \to^i M \to^\pi N \to 0$ stays exact under the functor $\mathrm{Hom}_R(T, -)$. If two of the three terms L, M, N are in $\mathrm{Pres}^n T$, so is the third one.*

Proof. In case $L, M \in \mathrm{Pres}^n T$, the assertion follows from Proposition 3.2.

Now assume that $L, N \in \mathrm{Pres}^n T$. Then we have exact sequences

$$0 \to L' \to T_L \xrightarrow{\alpha} L \to 0 \quad \text{and} \quad 0 \to N' \to T_N \xrightarrow{\gamma} N \to 0,$$

where $L', N' \in \operatorname{Pres}^n T$ and $T_L, T_N \in \operatorname{Add} T$. Since the sequence $0 \to L \to^i M \to^\pi N \to 0$ stays exact under the functor $\operatorname{Hom}_R(T, -)$, there is a homomorphism $\theta : T_N \to M$ such that $\theta\pi = \gamma$. Hence we can construct the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_L & \xrightarrow{(1,0)} & T_L \oplus T_N & \xrightarrow{\binom{0}{1}} & T_N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\binom{\alpha i}{\theta}} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{\pi} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By applying the functor $\operatorname{Hom}_R(T, -)$ to the diagram, we obtain that the upper row stays exact under the functor $\operatorname{Hom}_R(T, -)$, since so do the middle row and the left column. Therefore, we can repeat our process to the upper row, and so on, we obtain that $M \in \operatorname{Pres}^n T$.

In the last case, assume that $M, N \in \operatorname{Pres}^n T$. Then we have an exact sequence $0 \to M' \to T_M \to^\beta M \to 0$ with $M' \in \operatorname{Pres}^n T$ and $T_M \in \operatorname{Add} T$. Now we consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M' & = & M' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \longrightarrow & T_M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the bottom row stays exact under the functor $\operatorname{Hom}_R(T, -)$, as well as the middle column, by assumption and the constructions, we have that the induced homomorphism $\operatorname{Hom}_R(T, T_M) \to \operatorname{Hom}_R(T, N)$ is an epimorphism and consequently, the middle row stays exact under the functor $\mathrm{Hom}_R(T,-)$. Thanks to Proposition 2.4, we obtain that $Y \in \mathrm{Pres}^n T$, since $N \in \mathrm{Pres}^{n+1} T$ and T is $(n+1)$-quasi-projective by assumption. Now we can use Lemma 3.2 to conclude that $L \in \mathrm{Pres}^n T$. □

Proposition 3.4. *Let T be an n-star module. Then the functor $\mathrm{Hom}_R(T,-)$ preserves short exact sequences in $\mathrm{Pres}^n T$.*

Proof. Assume that $0 \to L \to^i M \to^\pi N \to 0$ is exact in $\mathrm{Pres}^n T$. So we have exact sequences $0 \to L' \to T_L \to^\alpha L \to 0$ and $0 \to M_1 \to T_M \to^\beta M \to 0$, where $L', M_1 \in \mathrm{Pres}^n T$ and $T_L, T_M \in \mathrm{Add}\, T$. As in the proof of Lemma 3.2, we construct the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L' & \to & M' & \to & N' & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_L & \xrightarrow{(1,0)} & T_L\oplus T_M & \xrightarrow{\binom{0}{1}} & T_M & \to & 0\\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\binom{\alpha i}{\beta}} & & \downarrow{\scriptstyle\beta\pi} & & \\
0 & \to & L & \xrightarrow{i} & M & \xrightarrow{\pi} & N & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

As proved in Lemma 3.2, $N' \in \mathrm{Pres}^n T$ and hence, the right column stays exact under the functor $\mathrm{Hom}_R(T,-)$ since T is $(n+1)$-quasi-projective. Note that the middle row clearly stays exact under the functor $\mathrm{Hom}_R(T,-)$, so we have that the induced homomorphism $\mathrm{Hom}_R(T,M) \to^{\mathrm{Hom}_R(T,\pi)} \mathrm{Hom}_R(T,N)$ is an epimorphism. It follows that the sequence $0 \to L \to M \to N \to 0$ stays exact under the functor $\mathrm{Hom}_R(T,-)$. □

We give now some characterizations of n-star modules.

Theorem 3.5. *The following are equivalent for an R-module T:*

(1) *T is an n-star module.*
(2) *If $0 \to L \to T_0 \to N \to 0$ is an exact sequence with $T_0 \in \mathrm{Add}\, T$ and $N \in \mathrm{Pres}^n T$, then $L \in \mathrm{Pres}^n T$ if and only if the induced sequence $0 \to \mathrm{Hom}_R(T,L) \to \mathrm{Hom}_R(T,T_0) \to \mathrm{Hom}_R(T,N) \to 0$ is exact.*
(3) *If $0 \to L \to M \to N \to 0$ is an exact sequence with $M, N \in \mathrm{Pres}^n T$, then $L \in \mathrm{Pres}^n T$ if and only if the induced sequence $0 \to \mathrm{Hom}_R(T,L) \to \mathrm{Hom}_R(T,M) \to \mathrm{Hom}_R(T,N) \to 0$ is exact.*

Proof. (1) ⇒ (2). If $L \in \mathrm{Pres}^n T$ then the induced sequence $0 \to \mathrm{Hom}_R(T, L) \to \mathrm{Hom}_R(T, T_0) \to \mathrm{Hom}_R(T, N) \to 0$ is clearly exact since T is $(n+1)$-quasi-projective. On the other hand, since $N \in \mathrm{Pres}^n T$ and T is an n-star module, $N \in \mathrm{Pres}^{n+1} T$ too. Hence we obtain the converse implication from Proposition 2.4.

(2) ⇒ (3). The only-if-part follows from Proposition 3.4 and the if-part follows from Proposition 3.3.

(3) ⇒ (1). It is easy to see that T is $(n+1)$-quasi-projective. It remains to show that $\mathrm{Pres}^n T \subseteq \mathrm{Pres}^{n+1} T$. Assume that $N \in \mathrm{Pres}^n T$ and take $T_N := T^{(\mathrm{Hom}_R(T,N))}$. Then we obtain an exact sequence $0 \to N' \to T_N \to N \to 0$ which stays exact under the functor $\mathrm{Hom}_R(T, -)$. Hence we have that $N' \in \mathrm{Pres}^n T$ by assumption. Therefore, $N \in \mathrm{Pres}^{n+1} T$. □

The following result characterizes n-star modules T such that $\mathrm{Pres}^n T$ is closed under extensions.

Proposition 3.6. *The following are equivalent for an R-module T:*

(1) *T is an n-star module and $\mathrm{Pres}^n T$ is closed under extensions.*
(2) $\mathrm{Pres}^n T = \mathrm{Pres}^{n+1} T \subseteq T^{\perp_1}$.

Proof. (1) ⇒ (2). It is sufficient to show that $\mathrm{Pres}^n T \subseteq T^{\perp_1}$. For any $X \in \mathrm{Pres}^n T$ and any extension of T by X: $0 \to X \to Y \to T \to 0$, we have that $Y \in \mathrm{Pres}^n T$ since $\mathrm{Pres}^n T$ is closed under extensions. By Proposition 3.4, the sequence stays exact under the functor $\mathrm{Hom}_R(T, -)$. Hence we obtain that the extension splits.

(2) ⇒ (1). Any exact sequence $0 \to L \to T_N \to N \to 0$ with $L \in \mathrm{Pres}^n T$ and $T_N \in \mathrm{Add}\, T$ stays exact under the functor $\mathrm{Hom}_R(T, -)$ since $\mathrm{Ext}^1_R(T, L) = 0$ by assumption. It follows that T is $(n+1)$-quasi-projective. Since $\mathrm{Pres}^n T = \mathrm{Pres}^{n+1} T$, we obtain that T is an n-star module. Now for any extension $0 \to L \to M \to N \to 0$ of N by L, where $L, N \in \mathrm{Pres}^n T$, we have that it stays exact under the functor $\mathrm{Hom}_R(T, -)$ since $\mathrm{Ext}^1_R(T, L) = 0$. Applying Proposition 3.3, we get that $M \in \mathrm{Pres}^n T$. Hence $\mathrm{Pres}^n T$ is closed under extensions. □

Recall that a class of R-modules $\mathcal{C}$ is said to be closed under n-images if for any exact sequence $C_n \to \cdots \to C_1 \to M \to 0$ in R-Mod with all C_i in $\mathcal{C}$, there holds that $M \in \mathcal{C}$ [15]. The class $\mathrm{Pres}^1 T$ (i.e., $\mathrm{Gen}\, T$) is clearly closed under 1-images (i.e., images). We do not know that whether or not $\mathrm{Pres}^n T$ is closed under n-images in general. But if T is an n-star module such that $\mathrm{Pres}^n T$ is closed under extensions, we have the following result.

Proposition 3.7. *Let T be an n-star module such that $\mathrm{Pres}^n T$ is closed under extensions. Then $\mathrm{Pres}^k(\mathrm{Pres}^n T) = \mathrm{Pres}^k T$ for all $k \geqslant 1$, where $\mathrm{Pres}^k(\mathrm{Pres}^n T)$ denotes the class of R-modules M such that there is an exact sequence $C_k \to \cdots \to C_1 \to M \to 0$ with all C_i in $\mathrm{Pres}^n T$. In particular, $\mathrm{Pres}^n T$ is closed under n-images.*

Proof. It is easy to see that $\mathrm{Pres}^k T \subseteq \mathrm{Pres}^k(\mathrm{Pres}^n T)$. We will show that $\mathrm{Pres}^k(\mathrm{Pres}^n T) \subseteq \mathrm{Pres}^k T$. We proceed it by induction on k.

In case $k = 1$, the conclusion is clear. So we assume now that $\mathrm{Pres}^j(\mathrm{Pres}^n T) = \mathrm{Pres}^j T$ for all $1 \leqslant j \leqslant k$.

Let M be an R-module such that $C_{k+1} \to \cdots \to C_1 \to^{f_1} M \to 0$ is exact with all C_i in $\mathrm{Pres}^n T$. Denote by M_1 the kernel of f_1. Then $M_1 \in \mathrm{Pres}^k T$ by the induction assumption. Note that $C_1 \in \mathrm{Pres}^n T$ and T is an n-star module, so there is an exact sequence $0 \to C' \to T_1 \to C_1 \to 0$ with $T_1 \in \mathrm{Add}\, T$ and $C' \in \mathrm{Pres}^n T$. Consider now the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C' & = & C' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Y & \to & T_1 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M_1 & \to & C_1 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Let $0 \to M_1' \to T_1' \to M_1 \to 0$ be exact with $M_1' \in \mathrm{Pres}^{k-1} T$ and $T_1' \in \mathrm{Add}\, T$. Then we also have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_1' & = & M_1' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C' & \to & X & \to & T_1' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C' & \to & Y & \to & M_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By assumption, $\mathrm{Pres}^n T$ is closed under extensions, so we have that $X \in \mathrm{Pres}^n T$ from the middle row. It follows that $Y \in \mathrm{Pres}^k(\mathrm{Pres}^n T)$ and consequently, $Y \in \mathrm{Pres}^k T$, by the induction assumption. From the exact sequence $0 \to Y \to T_1 \to M \to 0$ we deduce that $M \in \mathrm{Pres}^{k+1} T$. □

4. *n*-Tilting modules

We recall the following definition of (not necessarily finitely generated) n-tilting modules [1,2].

Definition 4.1. An R-module T is said to be n-tilting if it satisfies the following conditions:

(1) p.d. $T \leqslant n$, here p.d. T denotes the projective dimension of T.
(2) $\mathrm{Ext}_R^i(T, T^{(\lambda)}) = 0$ for all $i \geqslant 1$ and all cardinals λ.
(3) There is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$, where T_i's are isomorphic to direct summands of direct sums of copies of T.

Not necessarily finitely generated tilting modules of projective dimension $\leqslant 1$ were already investigated by Colpi and Trlifaj in [8], and it was shown that they are characterized by the condition $\mathrm{Pres}^1 T = T^{\perp_1}$. Not necessarily finitely generated tilting modules of projective dimension $\leqslant n$ were then studied by Angeleri Hügel and Coelho [1] and Bazzoni [2]. Generalizing the result in [8] and the result of [16], Bazzoni [2] showed that T is an n-tilting module if and only if $\mathrm{Pres}^n T = T^{\perp_{i \geqslant 1}}$. In case $n = 1$, the condition is that $\mathrm{Pres}^1 T = T^{\perp_{i \geqslant 1}}$, which is a little different from, but is equivalent to the condition that $\mathrm{Pres}^1 T = T^{\perp_1}$. We will show that an R-module T is n-tilting if and only if $\mathrm{Pres}^n T = T^{\perp_{1 \leqslant i \leqslant n}}$, which completely coincides with the result in [8] in case $n = 1$.

Moreover, as promised before, we will extend characterizations of finitely n-tilting modules in term of $*^n$-modules (see [7] and [16]) to the general case.

Lemma 4.2. *Let T be an n-star module such that $\mathrm{Pres}^n T$ is closed under extensions. Assume that $0 \to L \to M \to N \to 0$ is exact with $M, N \in \mathrm{Pres}^n T$. Then $L \in \mathrm{Pres}^n T$ if and only if $L \in T^{\perp_1}$.*

Proof. The only-if-part follows from Proposition 3.6 and the if-part follows from Proposition 3.3. □

The following is our main result.

Theorem 4.3. *Denote by $\mathcal{I}$ the class of all injective R-modules. The following are equivalent for an R-module T.*

(1) *T is an n-tilting module.*
(2) $\mathrm{Pres}^n T = T^{\perp_{1 \leqslant i \leqslant n}}$.
(3) *T is an n-star module and $\mathcal{I} \subseteq \mathrm{Pres}^n T$.*
(4) $\mathcal{I} \subseteq \mathrm{Pres}^n T = \mathrm{Pres}^{n+1} T \subseteq T^{\perp_1}$.

Proof. (1) ⇒ (2). By [2], T is an n-tilting module if and only if $\mathrm{Pres}^n T = T^{\perp_{i\geqslant 1}}$. Since p.d. $T \leqslant n$, we obtain that $\mathrm{Pres}^n T = T^{\perp_{1\leqslant i\leqslant n}}$.

(2) ⇒ (3). Clearly, $\mathcal{I} \subseteq \mathrm{Pres}^n T$ by assumption. It remains to show that T is an n-star module. Let $0 \to L \to T_N \to N \to 0$ be exact with $T_N \in \mathrm{Add}\, T$ and $N \in \mathrm{Pres}^n T$. By applying the functor $\mathrm{Hom}_R(T, -)$, we obtain that $L \in T^{\perp_{2\leqslant i\leqslant n}}$ and the induced sequence $0 \to \mathrm{Hom}_R(T, L) \to \mathrm{Hom}_R(T, T_N) \to \mathrm{Hom}_R(T, N) \to \mathrm{Ext}^1_R(T, L) \to 0$ is exact, since $\mathrm{Pres}^n T = T^{\perp_{1\leqslant i\leqslant n}}$ by assumption. Then the sequence $0 \to L \to T_N \to N \to 0$ stays exact under the functor $\mathrm{Hom}_R(T, -)$ if and only if $L \in T^{\perp_1}$ if and only if $L \in T^{\perp_{1\leqslant i\leqslant n}} = \mathrm{Pres}^n T$. Now the conclusion follows from Theorem 3.5.

(3) ⇒ (4). By the definition of n-star modules, we need only to show that $\mathrm{Pres}^n T \subseteq T^{\perp_1}$. For any $N \in \mathrm{Pres}^n T$, let then $0 \to N \to I_N \to N' \to 0$ be exact with $I_N \in \mathcal{I}$. By assumption, $\mathcal{I} \subseteq \mathrm{Pres}^n T$, hence $N' \in \mathrm{Pres}^n T$ by Lemma 3.2. It follows that the induced sequence $0 \to \mathrm{Hom}_R(T, N) \to \mathrm{Hom}_R(T, I_N) \to \mathrm{Hom}_R(T, N') \to 0$ is exact by Proposition 3.4. Therefore $N \in T^{\perp_1}$, since I_N is an injective R-module.

(4) ⇒ (1). We need only to show that $\mathrm{Pres}^n T = T^{\perp_{i\geqslant 1}}$ by [2]. Note first that, by assumption and Proposition 3.6, T is an n-star module and $\mathrm{Pres}^n T$ is closed under extensions.

Now for any $M \in \mathrm{Pres}^n T$, letting $0 \to M \to I_1 \to^{g_1} \cdots \to I_k \to^{g_k} \cdots$ be an injective resolution of M, we obtain that $M_i := \mathrm{Im}\, g_i \in \mathrm{Pres}^n T$ by Lemma 3.2, since $I_i \in \mathcal{I} \subseteq \mathrm{Pres}^n T$, for all $i \geqslant 1$. It follows that $M_i \in T^{\perp_1}$, for each i, by assumption. Therefore, we deduce that $\mathrm{Ext}^j_R(T, M) \cong \mathrm{Ext}^1_R(T, M_{j-1}) = 0$, for all $j \geqslant 1$, by the dimension shifting (setting $M_0 := M$). Hence, $\mathrm{Pres}^n T \subseteq T^{\perp_{i\geqslant 1}}$.

On the other hand, for any $M \in T^{\perp_{i\geqslant 1}}$, by taking again an injective resolution $0 \to M \to I_1 \to^{g_1} \cdots \to I_k \to^{g_k} \cdots$ of M, we obtain that $M_i \in T^{\perp_{i\geqslant 1}}$ for all $i \geqslant 1$ by the dimension shifting. By Proposition 3.7, $\mathrm{Pres}^n T$ is closed under n-images. It follows that $M_n \in \mathrm{Pres}^n T$. Since $M_{n-1} \in T^{\perp_{i\geqslant 1}}$ and $I_n \in \mathcal{I} \subseteq \mathrm{Pres}^n T$ too, we have that $M_{n-1} \in \mathrm{Pres}^n T$ by Lemma 4.2. By repeating the process, and so on, we finally obtain that $M \in \mathrm{Pres}^n T$, as desired. □

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