Eigenvalues of Matrices with Tree Graphs

Clark Jeffries*
Centre de recherches mathématiques
Université de Montréal
Montréal, P.Q. H3C 3J7, Canada

and

P. van den Driessche†
Department of Mathematics
University of Victoria
Victoria, B.C. V8W 2Y2, Canada

Submitted by Richard A. Brualdi

ABSTRACT

When the undirected graph of a real square matrix is a tree or forest, we establish finitely computable tests yielding information about the magnitudes and multiplicities of the eigenvalues of the matrix. Applying these tests to system designs expressed as signed directed graphs can be sufficient to guarantee controllability of the associated linear dynamical systems.

1. INTRODUCTION

We work throughout with real matrices and begin by introducing some notation and definitions needed to develop our results. We generally follow the conventions in [4]. When $A = [a_{ij}]$ is a real matrix of order n, its signed digraph SD(A) has node set $\{1,2,\dots,n\}$ and a directed edge from i to j iff $a_{ji} \neq 0$. This edge is signed as the sign of a_{ji}. The set of all matrices with the

*Some of the research of this author was carried out while visiting the University of Victoria.
†The work of this author was partially supported by NSERC grant A-8965. Please address all correspondence to this author.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/88/$3.50

same sign pattern (and thus the same signed digraph) as A is denoted by $Q(A)$. We also use the undirected graph $G(A)$ which has the same node set as SD(A) with edge set $\{\{i, j\} : i \neq j$ and $a_{ij} \neq 0 \neq a_{ji}\}$. An edge of $G(A)$ thus corresponds to a 2-cycle in SD(A). A 2-cycle in SD(A) is positive if $a_{ij}a_{ji} > 0$ and negative if $a_{ij}a_{ji} < 0$. A node i with a 1-cycle is called *distinguished* and corresponds to a nonzero diagonal entry in A, $a_{ii} \neq 0$. Since we seek to characterize eigenvalues, we work throughout with matrices A which are irreducible. Moreover, for the validity of many constructions and reduction arguments, we require SD(A) to have a 2-cycle but no k-cycle for $k > 2$; equivalently, $G(A)$ is a tree and A is combinatorially symmetric.

We consider the differential equation $\dot{x}(t) = \tilde{A}x(t)$ with $\tilde{A} \in Q(A)$, and we seek to detect the possibility of constant or sinusoidal trajectories. Here a *constant* $\langle$*strictly constant*$\rangle$ *trajectory* $x(t) \in \mathbb{R}^n$ for $\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j$ satisfies $\dot{x}_i = 0$ and $x_i \neq 0$ for some $\langle$all$\rangle$ i. A *sinusoidal* $\langle$*strictly sinusoidal*$\rangle$ *trajectory* for our equation satisfies $\ddot{x}_i = -x_i$ and $x_i \not\equiv 0$ for some $\langle$all$\rangle$ i. (We use $\not\equiv 0$ to denote "not the constant function with the value zero.") It is well known that $\dot{x} = \tilde{A}x$ admits a constant $\langle$sinusoidal$\rangle$ trajectory iff $\tilde{A}$ has a zero $\langle$purely imaginary$\rangle$ eigenvalue (see, for example, [3]).

Suppose $G(A)$ is a tree. We define SD(A) to be λ-*consistent* if there exist nonzero constants $\{\lambda_1, \dots, \lambda_n\}$ such that $\lambda_i a_{ij} = -\lambda_j a_{ji}$ for $i \neq j$; all $\lambda_i a_{ii} \geqslant 0$; and some $\lambda_i a_{ii} > 0$. For example, matrices with $a_{ij}a_{ji} \leqslant 0$ for all $i \neq j$, $a_{ii} \leqslant 0$ for all i, and $a_{ii} < 0$ for at least one i have SD(A) λ-consistent with all λ_i negative. These matrices are candidates for sign stability [4].

We now show that λ-consistency can be expressed in finitely computable terms. Suppose node 1 is a distinguished node in λ-consistent SD(A), so $a_{11} \neq 0$. Choose $\lambda_1 = \pm 1$, so $a_{11}\lambda_1 > 0$. Then use the signs of 2-cycles along the chain to specify all other $\{\lambda_j\}$ signs; this can be done because of the tree structure of $G(A)$. Then SD(A) is λ-consistent iff each $\lambda_i a_{ii} \geqslant 0$. Thus λ-consistency is a property of SD(A), rather than A itself.

We define a *subchain* of SD(A) as a subgraph which is a straight chain of 2-cycles; thus the undirected graph of a subchain is a simple path (that is, an unbranched tree). Clearly when SD(A) has at least two 1-cycles, then SD(A) is not λ-consistent iff some subchain of SD(A) with distinguished end nodes and no other distinguished nodes is not λ-consistent. We call a subchain with distinguished end nodes and no other distinguished nodes a *proper subchain*.

2. STRICTLY CONSTANT TRAJECTORIES

We first consider zero eigenvalues and have the following characterization.

THEOREM 1. *Suppose* A *is an irreducible matrix of order* $\geqslant 2$ *and* SD(A) *has no k-cycle,* $k > 2$. *Then there exist* $\tilde{A} \in Q(A)$ *and a strictly constant trajectory satisfying* $\dot{x} = \tilde{A}x = 0$ *iff each end node of* SD(A) *is distinguished and* SD(A) *is not* λ*-consistent.*

Proof. Suppose x is a strictly constant trajectory. Obviously each end node must be distinguished. Furthermore, if SD(A) were λ-consistent, then the derivative of $\Lambda = \sum_{i=1}^{n}\lambda_i x_i^2/2$ along the constant trajectory x would yield $0 = \dot{\Lambda} = \sum_{i=1}^{n}\lambda_i a_{ii}x_i^2 > 0$, a contradiction.

For the converse, let us assume SD(A) is itself a proper subchain. Labeling the nodes in the obvious way, $\tilde{A}$ is a tridiagonal matrix; we fix all entries except $\tilde{a}_{nn}$ ($\neq 0$). Considering the disjoint cycles of $\tilde{A}$ and setting $\alpha_i = \tilde{a}_{i\,i+1}\tilde{a}_{i+1\,i}$ for $i = 1,2,\ldots,n-1$ gives

$$\det \tilde{A} = \begin{cases} (-1)^{n/2}[-\tilde{a}_{11}\alpha_2\alpha_4\cdots\alpha_{n-2}\tilde{a}_{nn} + \alpha_1\alpha_3\cdots\alpha_{n-1}] & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2}[\tilde{a}_{11}\alpha_2\alpha_4\cdots\alpha_{n-1} + \alpha_1\alpha_3\cdots\alpha_{n-2}\tilde{a}_{nn}] & \text{if } n \text{ is odd.} \end{cases}$$

The sign of $\tilde{a}_{nn}$ is either $+$ or $-$, and it is either possible to adjust the magnitude of $\tilde{a}_{nn}$ so $\det \tilde{A} = 0$ or not, depending only on that sign. If SD(A) were λ-consistent, it would be impossible to have a constant trajectory with $\det \tilde{A} = 0$. Since we are assuming that SD(A) is not λ-consistent, $\tilde{a}_{nn}$ must be of the other sign, so for some choice of $|\tilde{a}_{nn}|$, $\det \tilde{A} = 0$.

Now let x be a nontrivial solution of $\tilde{A}x = 0$ with $\tilde{A}$ as above. The equations $\sum_{j=1}^{n}\tilde{a}_{ij}x_j = 0$ with $\tilde{a}_{11} \neq 0$ show that $x_1 = 0$ implies $x_2 = 0$, $x_3 = 0$, and so on through the chain; therefore $x_1 \neq 0$. But $x_1 \neq 0$ implies, by the same argument, that each component of x is nonzero; and so x is a strictly constant trajectory. (Note that if an end node is not distinguished, then the argument fails, as some component of x is zero).

Next suppose that SD(A) may be partitioned into a proper subchain which is not λ-consistent and a second subchain with exactly one distinguished node, the end node (not the node of attachment); see Figure 1 for an example. Starting at the node of attachment q, let the nodes of the second subchain be labeled $q, q+1, \ldots, q+m$, with $q+m$ the end node and $\tilde{a}_{q+m\,q+m} \neq 0$. Let $\tilde{A}$, x be specified as above for the proper subchain, and let other $\tilde{A}$ entries be arbitrary in magnitude except $\tilde{a}_{q+1\,q}$. Tentatively set $x_{q+m} = 1$. Then x_{q+m} can be used to specify x_{q+m-1}, which in turn can be used to specify x_{q+m-2}, and so on down the chain. Finally x_{q+1} is specified. If there is a sign conflict in the equation at node $q+1$, start over with $x_{q+m} = -1$. Then specify $\tilde{a}_{q+1\,q}$. At node q we must modify $\tilde{A}$-values so that

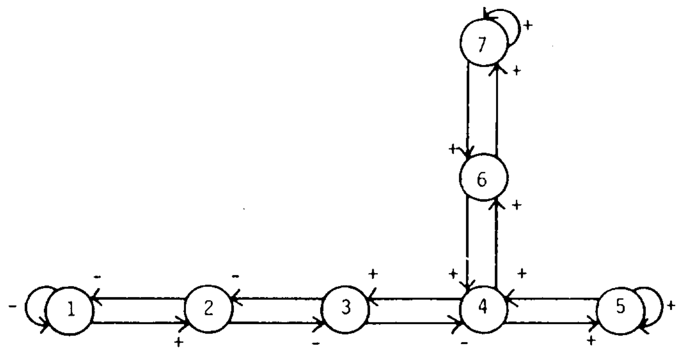

FIG. 1. An example to illustrate method of proof of Theorem 1. Nodes 1,2,3,4,5 are in a proper subchain; nodes 4,6,7 are in a second subchain with the end node distinguished.

$\tilde{a}_{q\alpha}x_{\alpha} + \tilde{a}_{q\beta}x_{\beta} + \tilde{a}_{q\,q+1}x_{q+1} = 0$, where nodes α, β are neighbors of node q in the proper subchain. The first two summands are already of opposite signs, so adjustment of the magnitudes of $\tilde{a}_{q\alpha}$ and $\tilde{a}_{q\beta}$ can clearly be carried out so $\tilde{A}x = 0$, all $x_i \neq 0$, $i = q,\dots, q+m$. The case in which the node of attachment q is also distinguished follows in a similar way, with the additional term $\tilde{a}_{qq}x_q$ in the equation at node q.

A simple extension of the above sequence shows that any number of subchains with distinguished end nodes can be accommodated. Lastly, additional nodes can acquire (small magnitude) 1-cycles by local adjustment of $\tilde{A}$ values, since each node clearly has inputs of opposite signs. ■

The case when A is of order 1 is trivial: there exists $\tilde{A} \in Q(A)$ and a strictly constant trajectory x satisfying $\dot{x} = \tilde{A}x = 0$ iff SD(A) consists of a single undistinguished node.

3. STRICTLY SINUSOIDAL TRAJECTORIES

We now consider detection of purely imaginary eigenvalues of $\tilde{A}$, that is, the possibility of a sinusoidal trajectory x for $\dot{x} = \tilde{A}x$, not all components of x being zero, with $\tilde{A} \in Q(A)$. We obviously restrict considerations to matrices of order $n \geqslant 2$.

LEMMA 1. *Suppose A is irreducible, all 2-cycles in* SD(A) *are positive, and* SD(A) *contains no k-cycle for $k > 2$. Then there exist no $\tilde{A} \in Q(A)$ and sinusoidal x solving $\dot{x} = \tilde{A}x$.*

Proof. When all 2-cycles in SD(A) are positive, there exist $\{\lambda_i\}$, $\lambda_i > 0$, such that $\lambda_i a_{ij} = \lambda_j a_{ji}$ for all $i \neq j$. Thus the matrix $[\lambda_i^{1/2} a_{ij} \lambda_j^{-1/2}]$ is symmetric and so has only real eigenvalues. Thus A, being diagonally similar to this matrix, also has no nonzero purely imaginary eigenvalues and so no sinusoidal trajectory. ■

LEMMA 2. *Suppose A is irreducible and* SD(A) *has no 1-cycle or k-cycle, $k > 2$, but at least one negative 2-cycle. Then there exist $\tilde{A} \in Q(A)$ and strictly sinusoidal x solving $\dot{x} = \tilde{A}x$.*

Proof. Suppose SD(A) contains a negative 2-cycle. The question of existence is resolved by showing that it is always possible to attach straight chains to any subsystem with strictly sinusoidal nodes. In fact only $\pm \sin t$ and $\pm \cos t$ are required as node values; the entries of $\tilde{A}$ are specified and modified as needed. Consider attachment of a straight chain with node set $\{2,3,\dots,p\}$ to a subsystem with strictly sinusoidal node values at node 1. Assume node 1 has the value $\sin t$. The idea is illustrated in Figure 2. We tentatively assign node p the value $\cos t$ if p is even, or $\sin t$ if p is odd. Then let $|\tilde{a}_{p\,p-1}| = 1$, so that the sign of $\tilde{a}_{p\,p-1}$ determines whether node $p-1$ in $\dot{x} = \tilde{A}x$ is $\pm \dot{x}_p$, that is, $\pm \sin t$ if p is even, $\pm \cos t$ if p is odd. Consider the equation at row $p-1$. Specifying either $|\tilde{a}_{p-1\,p}| = |\tilde{a}_{p-1\,p-2}| = \frac{1}{2}$ or $|\tilde{a}_{p-1\,p}| = 1$, $|\tilde{a}_{p-1\,p-2}| = 2$, as needed according to edge signs, allows us to keep $x_{p-1} = \pm \dot{x}_p$. This procedure extends down to row 1 of $\dot{x} = Ax$. At row 1 there might be a sign inconsistency. If so, then start over with the opposite sign for node p to correct this. If node 1 has the value $\cos t$, then assign node p the value $\sin t$ if p is even or $\cos t$ if p is odd, and proceed as previously. Finally, adjust the magnitudes of $\tilde{a}_{1j}$ as needed. ■

LEMMA 3. *Suppose A is irreducible,* SD(A) *contains no k-cycle for $k > 2$,* SD(A) *contains at least one negative 2-cycle and at least one 1-cycle, and* SD(A) *is not λ-consistent. Then there exists $\tilde{A} \in Q(A)$ and a strictly sinusoidal trajectory x for $\dot{x} = \tilde{A}x$.*

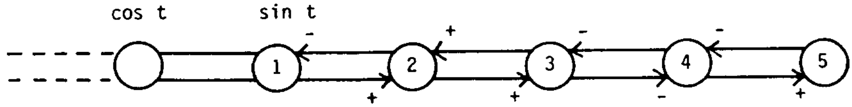

FIG. 2. An illustration of the idea in the proof of Lemma 2. Attachment of a straight chain with node set $\{2,3,4,5\}$ to subsystem at node 1.

Proof. Let us define an associated matrix sign pattern $Q(\bar{A})$ in terms of $Q(A)$ by replacing each diagonal entry in $Q(A)$ with 0. Apply Lemma 2 to $Q(\bar{A})$ to obtain matrix values $\bar{A}$ and strictly sinusoidal trajectory $y(t)$ for $\dot{y} = \bar{A}y$.

We shall employ some of the machinery of [4, Section 5]. If $x_i(t)$ is any function satisfying $\ddot{x} = -x$, define a complex number γ_i by $x_i(t) = \mathrm{Re}[\gamma_i e^{\iota t}]$. It will suffice to prove the existence of n complex numbers $\{\gamma_i\}$, each nonzero, satisfying for some $\tilde{A} \in Q(A)$

$$(-\tilde{a}_{ii} + \iota)\gamma_i = \sum_{i \neq j} \tilde{a}_{ij}\gamma_j. \tag{1}$$

Lemma 2 implies the existence of nonzero complex numbers $\{\delta_i\}$ satisfying $\iota\delta_i = \sum_{i \neq j}\bar{a}_{ij}\delta_j$, where $y_i(t) = \mathrm{Re}[\delta_i e^{\iota t}]$. If nodes i and j are connected by a 2-cycle in $\mathrm{SD}(A)$, then the ratio δ_i/δ_j is a nonzero purely imaginary number.

Regard $G(A)$ as a tree rooted at a distinguished node r with neighbor s. Define $\tilde{a}_{ij} = \bar{a}_{ij}$ for all index pairs $i \neq j$, except define $\tilde{a}_{rs}$ as below. Choose a node q such that $a_{qq} \neq 0$ and such that $\mathrm{SD}(B)$ is λ-consistent, where the matrix $B = [b_{ij}]$ has $b_{ij} = \tilde{a}_{ij}$ for $i \neq j$, $b_{rr} = -a_{rr}$, $b_{qq} = a_{qq}$, and all other $b_{ii} = 0$. Considering the Λ argument in the proof of Theorem 1, no trajectory $z(t)$ for $\dot{z} = Bz$ could be strictly sinusoidal.

Using the tree structure of $G(A)$, we can solve all but the rth equation in (1) starting at the ends of branches and computing "down the tree" until each γ_i, $i \neq r$, is given as some $\gamma_i = \alpha_i\gamma_r$. If all $|\tilde{a}_{ii}|$, $i \neq r$, are sufficiently small, the ratios of γ-values for connected nodes are all nonzero complex numbers, close to the corresponding δ-ratios. In particular $\mathrm{Im}(\alpha_s) \neq 0$. Define $\gamma_r = 1$ (so $x_r(t) = \cos t$). For sufficiently small $|\tilde{a}_{ii}|$ it will suffice to solve $-\tilde{a}_{rr} + \iota = \tilde{a}_{rs}\alpha_s$ with real $\tilde{a}_{rr}, \tilde{a}_{rs}$. Since $\mathrm{Im}(\alpha_s) \neq 0$, this is possible. The sign of $\tilde{a}_{rs}$ is correct, since each $\tilde{a}_{ij}$ is close to $\bar{a}_{ij}$, $i \neq j$, which follows from the continuity of complex inversion and complex multiplication. Suppose the sign of $\tilde{a}_{rr}$ is 0 or the opposite of a_{rr}. Rechoose all $|\tilde{a}_{ii}|$ much smaller for $i \neq q$, retaining $|a_{qq}|$. Recalculate all $\tilde{a}_{ij}$, $i \neq j$, and $\tilde{a}_{rr}$. To avoid a positive integral of $\dot{\Lambda}$ over the interval $[0, 2\pi]$, such a rechoice must lead to $\tilde{a}_{rr}$ of the correct sign. (The integral of $\dot{\Lambda}$ over $[0, 2\pi]$ must be zero for $x = \mathrm{Re}[\gamma e^{\iota t}]$.) ■

THEOREM 2. *Suppose A is irreducible and $\mathrm{SD}(A)$ has no k-cycle, $k > 2$. Then there exists a strictly sinusoidal trajectory x solving $\dot{x} = \tilde{A}x$ for some $\tilde{A} \in Q(A)$ iff $\mathrm{SD}(A)$ has at least one negative 2-cycle and $\mathrm{SD}(A)$ is not λ-consistent.*

Proof. Consider $\tilde{A} \in Q(A)$ and a strictly sinusoidal trajectory x with associated constants $\{\lambda_j\}$, and assume there is a 1-cycle at node i. Define

$\Lambda = \Sigma_{i=1}^{n}\lambda_i x_i^2/2$. If SD($A$) were λ-consistent, then along x the derivative of Λ would be $\dot{\Lambda} = \Sigma_{i=1}^{n}\lambda_i \tilde{a}_{ii} x_i^2 > 0$. This would contradict the fact that $\Lambda(x(t)) = \Lambda(x(t+2\pi))$. Lemma 1 establishes that SD(A) must have a negative 2-cycle.

On the other hand, Lemmas 2 and 3 establish the existence of x when SD(A) has a negative 2-cycle and is not λ-consistent. ■

COROLLARY 1. *If A is irreducible and* SD(A) *has no k-cycle, $k > 2$, and exactly one 1-cycle, then no $\tilde{A} \in Q(A)$ admits a strictly sinusoidal trajectory.*

Proof. Since in this case SD(A) is λ-consistent, Theorem 2 precludes a strictly sinusoidal trajectory. ■

4. CONSTANT TRAJECTORIES

We now study solutions of $Ax = 0$ with $x \not\equiv 0$ but some $x_i = 0$. Note that the case $n = 1$ (A is the 0 matrix) is trivial, so we take $n \geqslant 2$. Suppose our previous assumptions hold, namely that A is irreducible and SD(A) has no k-cycles, $k > 2$, and also that $Ax = 0$, $x \not\equiv 0$. Then there is natural way to partition SD(A) and $G(A)$ into subgraphs. Let a white block be a maximal connected subgraph on the nodes of SD(A) which correspond to nonzero components of x. All nodes not in white blocks are black and in black blocks. This arrangement is expressed in the following color test. A *0-coloring* is a scheme for coloring all nodes of SD(A) which has no k-cycle, $k > 2$, black or white, so that:

(i) no black node is a neighbor of exactly one white node;

(ii) each maximal white block as a subgraph is either: a single undistinguished node; or a digraph which has at least two nodes, which has each end node distinguished, and which is not λ-consistent.

THEOREM 3. *Suppose A is an irreducible matrix of order $\geqslant 2$ and* SD(A) *contains no k-cycle, $k > 2$. Then there exists $\tilde{A} \in Q(A)$ and a vector $x \not\equiv 0$ satisfying $\tilde{A}x = 0$ iff* SD(A) *admits a 0-coloring with at least one white node.*

Proof. Suppose $n \geqslant 2$, $Ax = 0$, $x \not\equiv 0$. Color white all nodes corresponding to nonzero entries in x, and color all other nodes black. Theorem 1 implies condition (ii) when all $x_i \neq 0$ and so all nodes are white. When both black and white nodes are present, considering the jth row equation in $Ax = 0$ for some $x_j = 0$, we see condition (i) must be fulfilled. Any white block satisfies a subsystem of equations $\bar{A}\bar{x} = 0$ which conforms to the conditions of Theorem 1, and so must fulfill condition (ii).

For the converse, suppose SD(A) admits a 0-coloring with some white node. We proceed to construct $x \not\equiv 0$ satisfying $\tilde{A}x = 0$. Consider the subsystem of equations $\tilde{A}x = 0$ associated with a white block. By Theorem 1 such a subsystem admits a solution with each component nonzero. Let the components of the full vector x corresponding to the subsystem be so defined. Suppose j is a black node connected to a node in this white block; then by (i) node j is connected to other white nodes $k_1, k_2, \dots, k_q$, $q \geqslant 1$. Let $|\tilde{a}_{jk_i}|$ be arbitrary positive numbers. Choose nonzero $\{x_{k_i}\}$ values satisfying the jth row equation in $\tilde{A}x = 0$. Using Theorem 1, extend these x-values through their respective white blocks; since $G(A)$ is a tree, this procedure can be carried out for all white nodes and black nodes connected to white nodes. Let all other entries in $\tilde{A}$ corresponding to edges in SD(A) from black nodes have arbitrary magnitudes, and components of x corresponding to black nodes be zero. This completes the construction of $\tilde{A}$ and x with $\tilde{A}x = 0$ and $x \not\equiv 0$. ■

To consider multiple eigenvalues of A, we use the idea of an undirected *block graph* $B(A)$. Suppose we have a nontrivial 0-coloring of SD(A). Delete from SD(A) all black nodes not connected to any white nodes and all edges to or from such black nodes. The nodes of $B(A)$ consist of the remaining black nodes $\{b_1, b_2, \dots\}$ and the (maximal) white blocks $\{w_1, w_2, \dots\}$. An edge $\{(b_i, w_j)\}$ belongs to $B(A)$ precisely when some node of b_i is connected by a 2-cycle to some node of w_j. We say that $B(A)$ is *branched* at a black node if some black node in $B(A)$ is connected to more than two white nodes. We make use of ideas developed in [2, 6, 7] for sign symmetric matrices.

THEOREM 4. *Suppose A is irreducible and* SD(A) *contains no k-cycle, $k > 2$. Then* 0 *is an eigenvalue in at least two Jordan blocks of some $\tilde{A} \in Q(A)$ iff* SD(A) *admits a 0-coloring for which $B(A)$ is branched at a black node.*

Proof. Suppose there exists $\tilde{A} \in Q(A)$ and that 0 is an eigenvalue in two or more Jordan blocks of $\tilde{A}$. Then there must exist two linearly independent solutions x and y for $\tilde{A}x = \tilde{A}y = 0$; choose x so that the number of components with $x_i = 0$ is maximal. If the 0-colorings associated with x and y are the same, then $x_i = 0$ iff $y_i = 0$. By rescaling and renumbering we can achieve $x_1 = y_1 \neq 0$. Thus the 0-coloring associated with $x - y$ has more zero components than x, a contradiction. So we may assume without loss of generality that the 0-coloring associated with x has a minimal number of white nodes and that the 0-colorings associated with x and y are distinct.

Consider a white block in the 0-coloring for x, with submatrix $\bar{A}$ and subvector $\bar{x}$, so $\bar{A}\bar{x}=0$. Suppose the white block is attached at node i to exactly one black node, node j (there must always exist such a block and node). Suppose node j is white in the 0-coloring for y; then we also have $\bar{A}\bar{y}+\xi=0$ where ξ is a vector with exactly one nonzero entry corresponding to the attachment of node i to white node j. Since any vector in the kernel of $\bar{A}$ must be proportional to $\bar{x}$, we have $\bar{y}_k=\alpha\bar{x}_k$ for all nodes in the white block, even node i. The ith row equation is then $0=\Sigma\bar{a}_{ik}\bar{x}_k=\Sigma\bar{a}_{ik}\bar{y}_k+\xi_i=\alpha\Sigma\bar{a}_{ik}\bar{x}_k+\xi_i=\xi_i$. This contradiction shows that node j must be black in the 0-coloring for y. Using the tree structure of $G(A)$, this argument extends to all nodes which are black in the 0-coloring for x and attached to white nodes; and shows that no white block of the 0-coloring for y can properly contain a white block of the 0-coloring for x. Thus if the two colorings differ, some white block and its adjacent black nodes of the coloring for x lie entirely within a black block of the coloring for y. Now color a node of SD(A) white if it is white in either the x or the y 0-coloring, and black otherwise. Clearly this is a 0-coloring for SD(A), and the associated $B(A)$ is branched.

Conversely, suppose that there is a 0-coloring for SD(A) with $B(A)$ branched. Then we use the proof of the first part of Theorem 3 to construct y and $\tilde{A}$ so $\tilde{A}y=0$ and so $y_i\neq 0$ if and only if node i is white. Thus the vector components of y corresponding to black nodes are zero, the edge values from black nodes are arbitrary. Some branched component of $B(A)$ contains a straight path with white block end nodes. Recolor black all nodes of SD(A) not in that straight path to achieve a distinct 0-coloring. Use Theorem 3 again to construct a new x (same $\tilde{A}$) which is not proportional to y but which satisfies $\tilde{A}x=0$. ■

5. SINUSOIDAL TRAJECTORIES

We now give a color test associated with sinusoidal trajectories and imaginary eigenvalues. An Im-*coloring* is a scheme for coloring all nodes of SD(A) which has no k-cycle, $k>2$, black or white so that:

(i) no black node is a neighbor of exactly one white node;

(ii) each maximal white block as a subgraph contains at least one negative 2-cycle and is not λ-consistent.

Clearly, starting with an Im-coloring, we can derive a block graph $B(A)$ just as in the previous section.

THEOREM 5. *Suppose A is an irreducible matrix of order $\geqslant 2$, and* SD(A) *has no k-cycles, $k>2$. Then there exists a sinusoidal trajectory for*

$\dot{x} = \tilde{A}x$, $x \not\equiv 0$, for some $\tilde{A} \in Q(A)$ iff SD(A) *admits an* Im-*coloring with at least one white node.*

Proof. If $\dot{x} = \tilde{A}x$ is a sinusoidal trajectory ($x \not\equiv 0$), color node i white if $x_i \not\equiv 0$; otherwise color node i black. Theorem 2 together with a line of reasoning parallel to that in the first part of the proof of Theorem 3 show that such a coloring is a nontrivial Im-coloring.

Suppose SD(A) admits an Im-coloring with at least one white node. Using Lemma 3 and balancing edge values as in the latter part of the proof of Theorem 3 establishes the existence of a sinusoidal trajectory. ■

THEOREM 6. *Suppose A is irreducible and* SD(A) *contains no k-cycle, $k > 2$. Then ι is an eigenvalue in at least two Jordan blocks of some $\tilde{A} \in Q(A)$ iff* SD(A) *admits an* Im-*coloring for which $B(A)$ is branched at a black node.*

Proof. The proof is completely analogous to the proof of Theorem 4 (using Theorem 5) and is omitted. ■

6. SIGN CONTROLLABILITY

For an irreducible matrix A with SD(A) having no k-cycles, $k > 2$, we can detect the possibility that a real number λ is an eigenvalue of A or is the real part of a complex eigenvalue of A as follows. For any A there are a finite number of digraphs SD($A - \lambda I$) for $-\infty < \lambda < \infty$. First we apply Theorem 3 to determine whether or not some matrix in $Q(A - \lambda I)$ has 0 as an eigenvalue. Applying Theorem 4 then gives us in addition a criterion for λ to occur in two or more Jordan blocks for some $\tilde{A} \in Q(A)$. Theorems 5 and 6 give us analogous conditions for the occurrence of $\lambda + \iota$ or, by rescaling, $\lambda + \kappa\iota$ ($\kappa > 0$), as an eigenvalue of some $\tilde{A} \in Q(A)$ or as an eigenvalue in two or more Jordan blocks. This is a characterization with implications in control theory; for background information and related results see [1, 2, 5]. Generalizing control theory concepts, we define A to be *controllable* if distinct Jordan blocks of A have distinct eigenvalues. We call A *sign controllable* if every $\tilde{A} \in Q(A)$ is controllable.

We express the above observations formally as follows.

COROLLARY 2. *Suppose A is irreducible and* SD(A) *has no k-cycle, $k > 2$. Then A is sign controllable if no block graph $B(\tilde{A} - \lambda I)$ obtained from*

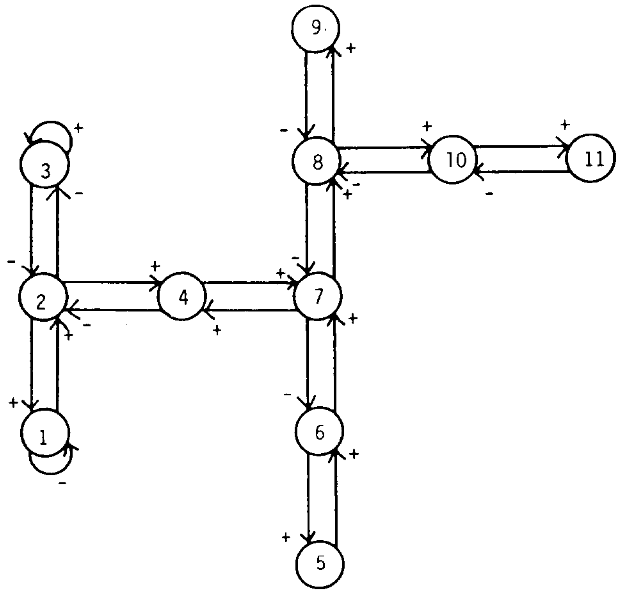

FIG. 3. An example of a sign controllable signed digraph.

any 0*-coloring or any* Im*-coloring of any* $\mathrm{SD}(\tilde{A}-\lambda I)$, $\tilde{A}\in Q(A)$, *is branched at a black node.* ■

To conclude, we consider as an example A with $\mathrm{SD}(A)$ in Figure 3. By inspection the 0-coloring possibilities can be enumerated as follows. If node 7 is black, then nodes 5, 6, 8, 10, 9, 11, 4, 2, 1, and 3 are forced to be black (in that order, using the rules for 0-colorings). However, if node 7 is white, then nodes $4,6,8$ are black, nodes $5,9$ are white, nodes $10,11$ are black, and nodes $1,2,3$ are white. This is the only nontrivial 0-coloring of $\mathrm{SD}(A)$, and the associated block graph is not branched at a black node, so 0 is an eigenvalue in at most one Jordan block of any $\tilde{A}\in Q(A)$.

By inspection the Im-coloring possibilities can be enumerated as follows. If node 7 is black, then the node sets $\{1,2,3,4\}$ and $\{8,9,10,11\}$ can be white with nodes 5 and 6 black. The block graph $B(A)$ associated with this coloring is not branched. If node 7 is white, then all nodes white except 8,9 is an Im-coloring but also has no branching in $B(A)$. The only other Im-coloring with node 7 white is all nodes white; again $B(A)$ is without branching. We conclude from Theorems 4 and 6 that no $\tilde{A}\in Q(A)$ can have either 0 or $\kappa\iota$ as eigenvalues in more than one Jordan block.

There are six signed digraphs of the form $\mathrm{SD}(\tilde{A}-\lambda I)$ aside from $\mathrm{SD}(A)$ itself. The only possible $B(\tilde{A}-\lambda I)$ branchings must occur at nodes 2 or 7.

(Branching cannot occur at node 8 because node 9 cannot be white for $\lambda \neq 0$.) If all of nodes 8, 9, 10, 11 have 1-cycles of one sign, then there is no proper subgraph among those nodes with nontrivial 0-coloring. Hence no 0-coloring could have 7 black and 8 white. Likewise no 0-coloring could have 2 black and 1 or 3 white. Hence no $B(\tilde{A} - \lambda I)$ graph from a 0-coloring of $\mathrm{SD}(\tilde{A} - \lambda I)$ branches at a black node. The subgraph containing nodes 5, 6 has no negative 2-cycle, so in no Im-coloring can node 7 be black and node 6 white. Likewise in no Im-coloring can node 2 be black and node 1 or 3 be white. Hence no $B(\tilde{A} - \lambda I)$ graph from an Im-coloring of $\mathrm{SD}(\tilde{A} - \lambda I)$ branches at a black node. In summary, Theorems 4 and 6 imply A with $\mathrm{SD}(A)$ in Figure 3 is sign controllable.

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