Integral operators with variable kernels on weak Hardy spaces ✤

Yong Ding a,*, Shanzhen Lu a, Shuanglin Shao b

a School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
b Department of Mathematics, UCLA, CA 90095-1555, USA

Received 19 November 2003
Available online 19 December 2005
Submitted by J. Conway

Abstract

In this note the authors study the mapping properties of a class of integral operators with variable kernels on the weak Hardy spaces.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Singular integral; Fractional integral; Variable kernel; Weak Hardy space

1. Introduction

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) with normalized Lebesgue measure $d\sigma$. A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \geq 1$, if $\Omega(x, z)$ satisfies the following conditions:

(i) for any $x, z \in \mathbb{R}^n$ and $\lambda > 0$, $\Omega(x, \lambda z) = \Omega(x, z)$;

(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} := \sup_{x \in \mathbb{R}^n} (\int_{S^{n-1}} |\Omega(x, z')|^q \, d\sigma(z'))^{1/q} < \infty$, where $z' = z/|z|$ for any $z \in \mathbb{R}^n \setminus \{0\}$.

✩ The research was supported by NSF of China (Grant Nos. 10571014 and 10571015) and DPFIHE of China (Grant No. 20050027025).

* Corresponding author.

E-mail addresses: dingy@bnu.edu.cn (Y. Ding), lusz@bnu.edu.cn (S. Lu), slshao@ucla.edu (S. Shao).
If $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ and satisfies
\[
\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0 \quad \text{for any } x \in \mathbb{R}^n,
\] (1.1)
then the singular integral operator with variable kernel is defined by
\[
T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x - y)}{|x - y|^n} f(y) \, dy.
\]

In 1955, Calderón and Zygmund found that the operator $T_\Omega$ connects closely with the problem about the second order linear elliptic equations with variable coefficients. In [1], Calderón and Zygmund obtained the $L^2$ boundedness of $T_\Omega$ (see also [2]).

**Theorem A.** (See [1].) If $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > 2(n-1)/n$, satisfies (1.1), then there is a constant $C > 0$ such that $\|T_\Omega f\|_{L^2} \leq C \|f\|_{L^2}$.

In 1971, Muckenhoupt and Wheeden [5] gave the weighted boundedness of the operator $T_\Omega$ for power weight. In the same paper, Muckenhoupt and Wheeden considered also the similar question for the factional integral operator $T_{\Omega, \alpha}$ with variable kernel. Here $T_{\Omega, \alpha}$ is defined by
\[
T_{\Omega, \alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} f(y) \, dy,
\]
where $0 < \alpha < n$ and $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q \geq 1$.

In this note we shall study the mapping properties of $T_\Omega$ and $T_{\Omega, \alpha}$ on the weak Hardy spaces $H^{1, \infty}(\mathbb{R}^n)$. The space $H^{1, \infty}(\mathbb{R}^n)$ was defined first by Fefferman and Soria in [3].

**Definition 1.** Suppose that $\phi \in C^\infty_0(\mathbb{R}^n)$ with $\int \phi \neq 0$. Denote $f^*_+(x) = \sup_{t > 0} |(\phi_t \ast f)(x)|$, where $\phi_t(x) = t^{-n} \phi(x/t)$. Then
\[
H^{1, \infty}(\mathbb{R}^n) = \left\{ f: \sup_{\beta > 0} \beta \left| \left\{ x \in \mathbb{R}^n: f^*_+(x) > \beta \right\} \right| \leq C < \infty \right\}. \tag{1.2}
\]
The smallest constant $C$ in (1.2) is called the $H^{1, \infty}(\mathbb{R}^n)$ norm of $f$, which is denoted by $\|f\|_{H^{1, \infty}}$.

Before stating our results, let us recall the definition of the integral modulus of continuity.

**Definition 2.** Let $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \geq 1$. Then the integral modulus $\omega_q(\delta)$ of continuity of order $q$ of $\Omega$ is defined by
\[
\omega_q(\delta) = \sup_{\|\rho\| \leq \delta} \left( \int_{S^{n-1}} \sup_{x \in \mathbb{R}^n} \left| \Omega(x, \rho z') - \Omega(x, z') \right|^q d\sigma(z') \right)^{1/q}
\]
and $\rho$ is a rotation in $\mathbb{R}^n$ with $\|\rho\| = \sup_{z' \in S^{n-1}} |\rho z' - z'|$. We simply denote $\omega_q(\delta)$ by $\omega(\delta)$ when $q = 1$. Our first result shows that $T_\Omega$ is a bounded operator from the weak Hardy space $H^{1, \infty}(\mathbb{R}^n)$ to the weak $L^1$ space $L^{1, \infty}(\mathbb{R}^n)$ if $\Omega$ satisfies a weaker smoothness condition on the unit sphere.
Theorem 1. Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > 2(n - 1)/n$, satisfies (1.1). If

$$\int_0^1 \frac{\omega(\delta)}{\delta} \left(1 + |\log \delta|\right)^\sigma d\delta < \infty \quad \text{for some } \sigma > 1,$$

then there exists a constant $C > 0$, independent of $\Omega$, such that for any $f \in H^{1,\infty}(\mathbb{R}^n)$ and $\beta > 0$,

$$\left|\left\{ x : |T_{\Omega f}(x)| > \beta \right\}\right| \leq \frac{C}{\beta} \| f \|_{H^{1,\infty}}. \quad (1.4)$$

Remark 1. Denote by $M$ the Hardy–Littlewood maximal operator. It is known that if $\phi \in C_0^\infty(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$, then $f_+(x) \leq CM(f)(x)$ (see [6, p. 62, Theorem 2]). Hence, by the weak (1,1) boundedness of $M$, it is easy to see that the space $L^1(\mathbb{R}^n)$ is continuously embedded as a subspace of the space $H^{1,\infty}(\mathbb{R}^n)$, and $\| f \|_{H^{1,\infty}} \leq C \| f \|_{L^1}$ for any $f \in L^1(\mathbb{R}^n)$. Thus we get immediately the following corollary of Theorem 1.

Corollary 1. If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > 2(n - 1)/n$, satisfies (1.1) and (1.3), then the operator $T_{\Omega}$ is of weak type $(1,1)$. That is, there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}^n)$ and $\beta > 0$,

$$\left|\left\{ x : |T_{\Omega f}(x)| > \beta \right\}\right| \leq \frac{C}{\beta} \| f \|_{L^1}. \quad (1.4)$$

Remark 2. By the conclusions of Theorem A and Corollary 1 and applying the interpolation theorem of linear operator (see [7]), we obtain the $L^p$ boundedness of the operator $T_{\Omega}$ for $1 < p < 2$.

Corollary 2. If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > 2(n - 1)/n$, satisfies (1.1) and (1.3), then for $1 < p < 2$, $\| T_{\Omega f} \|_{L^p} \leq C \| f \|_{L^p}$, where $C > 0$ is independent of $f$.

Now let us turn to stating the mapping property of the factional integral $T_{\Omega,\alpha}$ on $H^{1,\infty}(\mathbb{R}^n)$.

Theorem 2. Suppose $0 < \alpha < n$, $r > n/(n - \alpha)$. If $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfying

$$\int_0^1 \frac{\omega_{n/(n-\alpha)}(\delta)}{\delta} \left(1 + |\log \delta|\right)^\sigma d\delta < \infty, \quad \text{for some } \sigma > 1,$$

then there exists a constant $C > 0$ such that for any $f \in H^{1,\infty}(\mathbb{R}^n)$ and $\beta > 0$,

$$\left|\left\{ x : |T_{\Omega,\alpha} f(x)| > \beta \right\}\right| \leq C \left(\| f \|_{H^{1,\infty}} / \beta\right)^{n/(n-\alpha)}. \quad (1.6)$$

Remark 3. Through Remark 1, we easily obtain the following corollary of Theorem 2.

Corollary 3. Suppose $0 < \alpha < n$, $r > n/(n - \alpha)$. If $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfying (1.5), then there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}^n)$ and $\beta > 0$,

$$\left|\left\{ x : |T_{\Omega,\alpha} f(x)| > \beta \right\}\right| \leq C \left(\| f \|_{L^1} / \beta\right)^{n/(n-\alpha)}. \quad (1.6)$$
2. Proof of Theorem 1

We need the following Fefferman–Soria’s decomposition theorem of function in $H^{1,\infty}(\mathbb{R}^n)$.

**Theorem B.** (See [3].) Given a function $f \in H^{1,\infty}(\mathbb{R}^n)$, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^{\infty}$ with the following properties:

(a) $f - \sum_{|k| \leq N} f_k$ tends to zero in the sense of distributions;
(b) each $f_k$ may be further decomposed as $f_k = \sum_i h_i^k$ in $L^1$ and $\{h_i^k\}$ satisfies
   (i) $\text{supp}(h_i^k) \subset B_i^k := B(x_i^k, r_i^k)$, where $B(x, r)$ denotes the ball in $\mathbb{R}^n$ with the center at $x$ and radius $r$. Moreover, $\sum_i |B_i^k| \leq C_1 2^{-k}$ and $\sum_i \chi_{B_i^k}(x) \leq C$, where $C_1 \sim \|f\|_{H^{1,\infty}}$;
   (ii) $\|h_i^k\|_{\infty} \leq C 2^k$, where $C$ is independent of $i$ and $k$;
   (iii) $\int h_i^k(x) \, dx = 0$ for every $i$ and $k$.

Now let us return to the proof of Theorem 1. We need to show that there exists a constant $C > 0$ such that (1.4) holds for any $f \in H^{1,\infty}(\mathbb{R}^n)$ and $\beta > 0$. To do this, we choose $k_0$ satisfying $2^{k_0} \leq \beta < 2^{k_0+1}$. By Theorem B, we may write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2 \quad \text{and} \quad f_k = \sum_i h_i^k,$$

where $h_i^k$ satisfies (i)–(iii). Denote $A_k = \text{supp}(f_k)$. Then $A_k = \bigcup_i B_i^k$ and $|A_k| \leq \sum_i |B_i^k| \leq C 2^{-k} \|f\|_{H^{1,\infty}}$. Note that $\|f_k\|_{\infty} \leq C 2^k$. We have

$$\|F_1\|_2 \leq \sum_{k=-\infty}^{k_0} \|f_k\|_2 \leq C \sum_{k=-\infty}^{k_0} 2^k |A_k|^{1/2} \leq C \sum_{k=-\infty}^{k_0} 2^{k/2} \|f\|_{H^{1,\infty}}^{1/2} \leq C \|f\|_{H^{1,\infty}} \beta^{1/2}.$$

By Theorem A, it is easy to see that

$$|\{x: |T_\Omega(F_1)(x)| > \beta\}| \leq \beta^{-2} \|T_\Omega(F_1)\|_2^2 \leq C \beta^{-2} \|F_1\|_2^2 \leq C \beta^{-1} \|f\|_{H^{1,\infty}}. \quad (2.1)$$

On the other hand, denote

$$B_i^k = B(x_i^k, 2(3/2)^{(k-k_0)/n} r_i^k) \quad \text{and} \quad B_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i B_i^k.$$

We have

$$|B_{k_0}| \leq C \sum_{k=k_0+1}^{\infty} \sum_i |B_i^k| \leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^n (3/2)^{k-k_0} |B_i^k| \leq C \sum_{k=k_0+1}^{\infty} (3/2)^{k-k_0} 2^{-k} \|f\|_{H^{1,\infty}} \leq C \beta^{-1} \|f\|_{H^{1,\infty}}. \quad (2.2)$$

Thus, in order to complete the proof of (1.4), it suffices to show

$$|\{x \in (B_{k_0})^c: |T_\Omega(F_2)(x)| > \beta\}| \leq C \beta^{-1} \|f\|_{H^{1,\infty}}. \quad (2.3)$$
Notice

\[
\int_{(B_k^0)^c} \left| T_{\Omega}(F_2)(x) \right| \, dx = \int_{(B_k^0)^c} \left| \frac{\Omega(x, x-y) - \Omega(x, x-x_i^k)}{|x-y|^n} \sum_{k=k_0+1} \sum_i h_i^k(y) \right| \, dx \\
\leq \sum_{k=k_0+1} \sum_i (I_1 + I_2), \tag{2.4}
\]

where

\[
I_1 = \int_{(B_k^0)^c} \left| \int_{B_i^k} h_i^k(y) \frac{\Omega(x, x-y) - \Omega(x, x-x_i^k)}{|x-y|^n} \, dy \right| \, dx
\]

and

\[
I_2 = \int_{(B_k^0)^c} \left| \int_{B_i^k} h_i^k(y) \Omega(x, x-x_i^k) \left( \frac{1}{|x-y|^n} - \frac{1}{|x-x_i^k|^n} \right) \right| \, dx.
\]

It is easy to see that \( \frac{1}{3} |x-y| \leq |x-x_i^k| \leq 3|x-y| \) for all \( i \) and \( k \), since \( y \in B_i^k \) and \( x \in (B_k^0)^c \).

Thus

\[
I_2 \leq C \int_{(B_k^0)^c} \frac{|\Omega(x, x-x_i^k)|}{|x-x_i^k|^{n+1}} \int_{B_i^k} |h_i^k(y)| \, dy \, dx \\
\leq C 2^k r_i^k |B_i^k| \int_{2(3/2)^k}^\infty \frac{1}{t^2} \int_{S^{n-1}} |\Omega(t x' + x_i^k, x')| \, d\sigma(x') \, dt \\
\leq C 2^k |B_i^k| (2/3)^{(k-k_0)/n}.
\]

Therefore

\[
\sum_{k=k_0+1} \sum_i I_2 \leq C \sum_{k=k_0+1} \sum_i (2/3)^{(k-k_0)/n} \|f\|_{H^1, \infty} \leq C \|f\|_{H^1, \infty}. \tag{2.5}
\]

For \( I_1 \), we have still \( |x-y| \sim |x-x_i^k| \). Denote simply \( 2^j (3/2)^{(k-k_0)/n} r_i^k \) by \( R_i^k \),

\[
I_1 \leq C 2^k \int_{B_i^k} \sum_{j=1}^\infty \int_{R_i^k, j \leq |x-x_i^k| \leq R_i^k, j+1} \frac{|\Omega(x, x-y) - \Omega(x, x-x_i^k)|}{|x-x_i^k|^n} \, dx \, dy \\
\leq C 2^k \int_{B_i^k} \sum_{j=1}^\infty \int_{R_i^k, j \leq |x| \leq R_i^k, j+1} \frac{|\Omega(x + x_i^k, x + x_i^k - y) - \Omega(x + x_i^k, x)|}{|x|^n} \, dx \, dy \\
\leq C 2^k \int_{B_i^k} \sum_{j=1}^\infty \int_{R_i^k, j \leq |x'| \leq R_i^k, j+1} \frac{1}{t} \int_{S^{n-1}} |\Omega(t x' + x_i^k, t x' + x_i^k - y) - \Omega(t x' + x_i^k, t x')| \, d\sigma(x') \, dt \, dy.
\]
Setting \( y - x^k_i = t\xi \), then

\[
\int_{S^{n-1}} \left| \Omega(tx' + x^k_i, tx' + x^k_i - y) - \Omega(tx' + x^k_i, tx') \right| d\sigma(x') \\
= \int_{S^{n-1}} \left| \Omega(tx' + x^k_i, \frac{x'}{|x' - \xi|}) - \Omega(tx' + x^k_i, x') \right| d\sigma(x') \\
\leq C \sup_{\|\rho\| \leq |\xi|} \int_{S^{n-1}} \sup_{w \in \mathbb{R}^n} \left| \Omega(w, \rho x') - \Omega(w, x') \right| d\sigma(x') \\
\leq C\omega \left( \frac{|y - x^k_i|}{t} \right).
\]

Thus

\[
I_1 \leq C 2^k \sum_{R^k_{i,j+1}} \int_{B^k_i \cap R^k_{i,j}} \frac{1}{t} \omega \left( \frac{|y - x^k_i|}{t} \right) dt \ dy = C 2^k \sum_{R^k_{i,j}} \int_{B^k_i \cap \frac{|y - x^k_i|}{R^k_{i,j}}} \frac{\omega(\delta)}{\delta} d\delta \ dy
\]

\[
\leq C 2^k \int_{B^k_i} \int_0^{(2/3)^{(k-k_0)/n}} \frac{\omega(\delta)}{\delta} d\delta \ dy. \tag{2.6}
\]

By (1.3) we have

\[
\int_0^{(2/3)^{(k-k_0)/n}} \frac{\omega(\delta)}{\delta} d\delta \leq C \frac{1}{[1 + (k - k_0) \log(3/2)]^\sigma} \frac{1}{\delta} \left( 1 + |\log \delta| \right)^\sigma \leq C \frac{1}{(k - k_0)^\sigma},
\]

which and (2.6) imply

\[
\sum_{k=k_0+1}^{\infty} \sum_{i} I_1 \leq C \sum_{k=k_0+1}^{\infty} \sum_{i} 2^k |B^k_i| \left( \frac{1}{(k-k_0)^\sigma} \right) \leq C \| f \|_{H^{1,\infty}}. \tag{2.7}
\]

Then (2.3) follows from (2.4), (2.5) and (2.7). Combining (2.3) with (2.1) and (2.2), we complete the proof of Theorem 1.

3. Proof of Theorem 2

Before proving Theorem 2, let us recall some known results.

**Theorem C.** (See [5].) Suppose that \( 0 < \alpha < n \), \( 1 < p < n/\alpha \), \( 1/q = 1/p - \alpha/n \) and \( r > p' \).

If \( \Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1}) \), then if and only if \(-n/q < \gamma < n/q' - (n - 1)/r - \alpha \), there exists a constant \( C > 0 \) independent of \( f \) and \( \Omega \), such that

\[
\left\| \left| x \right|^\gamma \int_{\mathbb{R}^n} \frac{\Omega(x, y) f(x - y)}{|y|^{n-\alpha}} dy \right\|_{L^q(\mathbb{R}^n)} \leq C \| \Omega \|_{L^\infty \times L^r(S^{n-1})} \left\| \left| x \right|^\gamma f(x) \right\|_{L^p(\mathbb{R}^n)}.
\]

If \( r < p' \), for any \( \gamma \) there does not exist any constant \( C \) satisfying the above.
Remark 4. If $r > p'$, then $n/q - (n - 1)/r - \alpha > 0$. Therefore under the conditions of Theorem C, we have $\|T_{\Omega, \alpha}(f)\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$, where $C$ is a constant independent of $f$.

Lemma 3.1. (See [4].) Let $0 \leq \alpha < n$. Suppose $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ satisfies the $L^q$-Dini condition for $q \geq 1$. If there exists a constant $0 < \beta < 1/2$ such that $|y| < \beta R$, then

$$
\left( \int_{R < |x| < 2R} \left| \frac{\Omega(x, x - y)}{|y|^{n-\alpha}} - \frac{\Omega(x, x)}{|x|^{n-\alpha}} \right|^q dx \right)^{1/q} \leq C R^{n-q(n-\alpha)} \left( \frac{|y|}{R} \right)^{\alpha/n} \left( \int \frac{|y/R|^q \omega_q(\delta)}{\delta} d\delta \right),
$$

where the constant $C > 0$ is independent of $R$ and $y$.

Now let us turn to the proof of Theorem 2. We need to show that there exists a constant $C > 0$ such that (1.6) holds for any $f \in H^{1, \infty}(\mathbb{R}^n)$ and $\beta > 0$. To do this, we choose $k_0$ satisfying $2^{k_0} \leq \beta n/(n-\alpha)/\|f\|_{H^{1, \infty}} < 2^{k_0+1}$. By Theorem B, we may write

$$
f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1} f_k := F_1 + F_2 \quad \text{and} \quad f_k = \sum_i h_i^k,
$$

where $h_i^k$ satisfies (i)–(iii). Denote $A_k = \text{supp}(f_k)$. Then $A_k = \bigcup_i B_i^k$ and $|A_k| \leq \sum_i |B_i^k| \leq C 2^{-k} \|f\|_{H^{1, \infty}}$. Since $r > n/(n-\alpha)$, equivalently $1 \leq r' < n/\alpha$ we can choose $p$ such that $1 \leq r' < p < n/\alpha$. We then choose $q$ such that $1/q = 1/p - \alpha/n$. By Remark 4, we can obtain $T_{\Omega, \alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Note that $\|f_k\|_{L^p} \leq C 2^k$, we have

$$
\|F_1\|_{L^p} \leq \sum_{k=-\infty}^{k_0} \|f_k\|_{L^p} \leq C \sum_{k=-\infty}^{k_0} 2^{k} |A_k|^{1/p} \leq C \sum_{k=-\infty}^{k_0} 2^{k} (2^{-k} \|f\|_{H^{1, \infty}})^{1/p} \leq C \sum_{k=-\infty}^{k_0} 2^{(1-1/p)k} \|f\|_{H^{1, \infty}}^{1/p} \leq C 2^{k_0 (1-1/p)} \|f\|_{H^{1, \infty}}^{1/p} \leq C \beta^{1-n/q(n-\alpha)} \|f\|_{H^{1, \infty}}^{n/q(n-\alpha)}.
$$

Thus

$$
\left| \left\{ x \left| T_{\Omega, \alpha}(F_1)(x) \right| > \beta \right\} \right| \leq \beta^{-q} \|T_{\Omega, \alpha}(F_1)\|_{L^q} \leq C \beta^{-q} \|F_1\|_{L^p}^q \leq C \beta^{-q} (\beta^{1-n/q(n-\alpha)} \|f\|_{H^{1, \infty}}^{n/q(n-\alpha)} q) \leq C (\|f\|_{H^{1, \infty}}/\beta)^{n/(n-\alpha)}.
$$

On the other hand, denote

$$
\overline{B}_i^k = B(x_i^k, 2(3/2)^{(k-k_0)/n} r_i^k) \quad \text{and} \quad B_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \overline{B}_i^k.
$$
We have

\[
|B_{k_0}| \leq C \sum_{k=k_0+1}^{\infty} \sum_i |B_i^k| \leq C \sum_{k=k_0+1}^{\infty} 2^n (3/2)^{k-k_0} |B_i^k| \\
\leq C \sum_{k=k_0+1}^{\infty} (3/2)^{k-k_0} 2^{-k} \|f\|_{H^{1,\infty}} \\
\leq C2^{-(k_0+1)} \|f\|_{H^{1,\infty}} \\
\leq C \left(\|f\|_{H^{1,\infty}}/\beta\right)^{n/(n-\alpha)}.
\] (3.2)

Thus, in order to complete the proof of (1.6), it suffices to show

\[
\left|\left\{x \in (B_{k_0})^c : \left|T_\Omega(F_2)(x)\right| > \beta\right\}\right| \leq C \left(\|f\|_{H^{1,\infty}}/\beta\right)^{n/(n-\alpha)}.
\] (3.3)

Setting \(s = n/(n-\alpha)\), by the Minkowski inequality and Lemma 3.1, since \(n/s - (n-\alpha) = 0\), we get

\[
J := \left( \int_{(B_{k_0})^c} \left|T_\Omega(F_2)(x)\right|^s dx \right)^{1/s} \\
= \left( \int_{(B_{k_0})^c} \left( \int_{\mathbb{R}^n} \sum_{k=k_0+1}^{\infty} \sum_i h_i^k(y) \Omega(x, x-y) \frac{dx}{|x-y|^{n-\alpha}} \right)^s dy \right)^{1/s} \\
\leq \sum_{k=k_0+1}^{\infty} \sum_i \left( \int_{(B_{k_0})^c} \left( \int_{B_i^k} h_i^k(y) \left( \frac{\Omega(x, x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^{n-\alpha}} \right) \right)^s dy \right)^{1/s} \\
\leq \sum_{k=k_0+1}^{\infty} \sum_i \int_{B_i^k} \left| h_i^k(y) \right| \left( \int_{(B_{k_0})^c} \left( \frac{\Omega(x, x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^{n-\alpha}} \right)^s dx \right)^{1/s} dy \\
\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{B_i^k} \left| h_i^k(y) \right| \\
\times \sum_{j=1}^{\infty} \left( \int_{R_{i,j}^k \leq |x-x_i^k| \leq R_{i,j+1}^k} \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^{n-\alpha}} \right| \right)^{1/s} dy \\
\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{B_i^k} \left| h_i^k(y) \right| \sum_{j=1}^{\infty} \left( \frac{|y-x_i^k|}{R_{i,j}^k} \right) + \int_{|y-x_i^k|/R_{i,j+1}^k \leq \delta} \frac{\omega_3(\delta)}{\delta} d\delta \right) dy \\
\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{B_i^k} \left| h_i^k(y) \right| \left( \frac{2/3}{(k-k_0)/n} + \int_0^{(2/3)(k-k_0)/n} \frac{\omega_3(\delta)}{\delta} d\delta \right) dy.
\] (3.4)

By (1.5) we have
\[
\left(\frac{2}{3}\right)^{(k-k_0)/n} \int_0^1 \frac{\omega_s(\delta)}{\delta} \, d\delta \leq C \frac{1}{\left[1 + (k-k_0)\log(3/2)\right]^\sigma} \int_0^1 \frac{\omega_s(\delta)}{\delta} \left(1 + |\log \delta|\right)^\sigma \, d\delta
\]

which and (3.4) imply

\[
J \leq C \sum_{k=k_0+1}^{\infty} \sum_{i} 2^k |B_i^k| \left(\left(\frac{2}{3}\right)^{(k-k_0)/n} + \frac{1}{(k-k_0)^\sigma}\right) \leq C \|f\|_{H^{1,\infty}}.
\]

Then (3.3) follows from (3.5). Combining (3.3) with (3.1) and (3.2), we therefore complete the proof of Theorem 2.

References