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Very weak solutions and large uniqueness classes of stationary Navier–Stokes equations in bounded domains of $\mathbb{R}^2$

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Abstract

Extending the notion of very weak solutions, developed recently in the three-dimensional case, to bounded domains $\Omega \subset \mathbb{R}^2$ we obtain a new class of unique solutions u in $L^q(\Omega)$, $q > 2$, to the stationary Navier–Stokes system $-\Delta u + u \cdot \nabla u + \nabla p = f$, $\operatorname{div} u = k$, $u|_{\partial\Omega} = g$ with data f, k, g of low regularity. As a main consequence we obtain a new uniqueness class also for classical weak or strong solutions. Indeed, such a solution is unique if its L^q-norm is sufficiently small or the data satisfy the uniqueness condition of a very weak solution.

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1. Introduction and main results

Throughout this paper, $\Omega \subset \mathbb{R}^2$ denotes a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and unit outer normal vector $N(x) = (N_1(x), N_2(x))$ at $x = (x_1, x_2) \in \partial\Omega$. Then we consider the stationary Navier–Stokes system

$$-\Delta u + u \cdot \nabla u + \nabla p = f, \qquad \operatorname{div} u = k \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g \tag{1.1}$$

with nonhomogeneous data $f = \operatorname{div} F$, k and g satisfying

$$\begin{gathered} F = (F_{ij})_{i,j=1,2} \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in W^{-1/q,q}(\partial\Omega), \\ \int_\Omega k\,dx = \int_{\partial\Omega} g \cdot N\,do, \end{gathered} \tag{1.2}$$

where $2 < q < \infty$, $q' = \frac{q}{q-1} < r \leqslant q$, $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$; the surface integral in (1.2) is well defined in the generalized sense $\int_{\partial\Omega} g \cdot N\,do = \langle g, N\rangle_{\partial\Omega} = \langle N \cdot g, 1\rangle_{\partial\Omega}$.

Definition 1.1. Given data F, k and g as in (1.2) a vector field $u = (u_1, u_2) \in L^q(\Omega)$ is called a *very weak solution* of (1.1) if and only if for every test function

$$w \in C^2_{0,\sigma}(\bar{\Omega}) = \left\{v \in C^2(\bar{\Omega}){:}\ \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0\right\}$$

the well defined relation

$$-\langle u, \Delta w\rangle + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle - \langle ku, w\rangle = -\langle F, \nabla w\rangle \tag{1.3}$$

and the equations

$$\operatorname{div} u = k \quad \text{in } \Omega, \qquad N \cdot u|_{\partial\Omega} = N \cdot g \tag{1.4}$$

are satisfied.

Here $C^2(\bar{\Omega}) = \{v|_{\bar{\Omega}}{:}\ v \in C^2(\mathbb{R}^2)\}$, $\langle\cdot,\cdot\rangle$ denotes the usual L^q–$L^{q'}$-pairing on Ω and $\langle g, N \cdot \nabla w\rangle_{\partial\Omega}$ means the value of the boundary distribution $g \in W^{-1/q,q}(\partial\Omega)$ applied to the test function $N \cdot \nabla w$; for more details see Section 2.1. The relation (1.3) is formally obtained from (1.1) by applying the test function $w \in C^2_{0,\sigma}(\bar{\Omega})$, using integration by parts and the equation $u \cdot \nabla u = \operatorname{div}(uu) - ku$ where $uu = (u_i u_j)_{i,j=1,2}$. The boundary condition $N \cdot u|_{\partial\Omega} = N \cdot g$ is well defined since $u \in L^q(\Omega)$ and $k = \operatorname{div} u = L^r(\Omega)$. On the other hand, an elementary calculation proves that

$$N \cdot \nabla w = (\operatorname{rot} w)\tau \text{ on } \partial\Omega \quad \text{for all } w \in C^2_{0,\sigma}(\bar{\Omega}),$$

where $\tau = (-N_2, N_1) \perp N$ is the unit tangential vector at $x \in \partial\Omega$ and $\operatorname{rot} w = \partial_1 w_2 - \partial_2 w_1$. Hence, the term

$$\langle g, N \cdot \nabla w\rangle_{\partial\Omega} = \bigl\langle g, (\operatorname{rot} w)\tau\bigr\rangle_{\partial\Omega}$$

in (1.3) contains only the tangential component $g \cdot \tau = u|_{\partial\Omega} \cdot \tau$ of g. Therefore, the condition on the normal component of u on $\partial\Omega$ in (1.4) must be prescribed in addition to (1.3). In principle, we follow the notion of very weak solutions introduced by Amann [2,3] for the three-dimensional nonstationary case with $k = 0$ and extended in [6,10] to the stationary and nonstationary 3D-case with $k \neq 0$.

To prove the main existence result for the Navier–Stokes equations we first consider the stationary Stokes system

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = k \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g \tag{1.5}$$

with data $f = \operatorname{div} F, k$ and g as in (1.2) where now $1 < r \leqslant q < \infty$, $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$.

Theorem 1.2. *Suppose the data* $f = \operatorname{div} F, k, g$ *satisfy* (1.2) *with* $1 < r \leqslant q < \infty$, $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$. *Then there exists a unique very weak solution* $u \in L^q(\Omega)$ *of the Stokes system* (1.5), *i.e.,*

$$\begin{gathered} -\langle u, \Delta w\rangle + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} = -\langle F, \nabla w\rangle \quad \text{for all } w \in C^2_{0,\sigma}(\bar{\Omega}) \\ \operatorname{div} u = k \quad \text{in } \Omega, \qquad N \cdot u|_{\partial\Omega} = N \cdot g. \end{gathered} \tag{1.6}$$

Moreover, there exists a pressure $p \in W^{-1,q}(\Omega)$ *such that* $-\Delta u + \nabla p = f$ *in the sense of distributions, and* (u, p) *satisfy the estimate*

$$\|u\|_q + \|p\|_{-1,q} \leqslant C\bigl(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\bigr) \tag{1.7}$$

with a constant $C = C(\Omega, q, r) > 0$.

For the Navier–Stokes system the nonlinear term $u \cdot \nabla u$ causes the additional restrictions $q > 2$ and $q' < r$. Now our main result reads as follows:

Theorem 1.3. *Suppose the data* $f = \operatorname{div} F, k, g$ *satisfy* (1.2) *with* $2 < q < \infty$, $q' < r \leqslant q$ *and* $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$. *There exists a constant* $K = K(\Omega, q, r) > 0$ *such that if*

$$\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega} \leqslant K, \tag{1.8}$$

then the Navier–Stokes system (1.1) *has a unique very weak solution* $u \in L^q(\Omega)$. *Moreover, there exists a pressure* $p \in W^{-1,q}(\Omega)$ *such that* (1.1) *is satisfied in the sense of distributions.*

Furthermore, under the smallness condition (1.8) *the solution pair* (u, p) *of* (1.1) *satisfies the a priori estimates*:

$$\|u\|_q \leqslant C\big(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big), \tag{1.9}$$

$$\|p\|_{-1,q} \leqslant C\big(\|F\|_r + \|u\|_q + \|u\|_q^2 + \|u\|_q \, \|k\|_r\big) \tag{1.10}$$

with $C = C(\Omega, q, r) > 0$.

As an application we consider the classical Navier–Stokes equations with data $F \in L^2(\Omega)$, $k = 0$ and $g \in W^{1/2,2}(\partial\Omega)$ such that $\int_{\partial\Omega} g \cdot N\, do = 0$ and a weak solution $u \in W^{1,2}(\Omega)$, i.e.,

$$-\Delta u + u \cdot \nabla u + \nabla p = \operatorname{div} F, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g \tag{1.11}$$

in the usual weak L^2-sense. As is well known, see [9, VIII, Theorem 4.1], there exists at least one weak solution $u \in W^{1,2}(\Omega)$ if Ω is simply connected or if $\int_{\Gamma_i} g \cdot N\, do = 0$ for every boundary component Γ_i of $\partial\Omega$ in the case of a multiply-connected domain. Moreover, there exists a constant $K_1 = K_1(\Omega) > 0$ such that the smallness assumption

$$\|F\|_2 + \|g\|_{1/2,2,\partial\Omega} \leqslant K_1 \tag{1.12}$$

guarantees the uniqueness of the weak solution u, cf. [9, VIII, Theorem 4.2].

The following corollaries are an obvious consequence of Theorem 1.3. First we obtain a *weaker uniqueness condition* and therefore a *larger uniqueness class for weak solutions* $u \in W^{1,2}(\Omega)$ of (1.11).

Corollary 1.4. *Let* $F \in L^2(\Omega)$, $g \in W^{1/2,2}(\partial\Omega)$, *and let* $u \in W^{1,2}(\Omega)$ *be a weak solution of* (1.11) *in the weak* L^2*-sense. Moreover, let* $2 < q < \infty$, $q' < r \leqslant 2$ *and* $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$. *There exists a constant* $K = K(\Omega, q, r) > 0$ *such that if*

$$\|F\|_r + \|g\|_{-1/q,q,\partial\Omega} \leqslant K, \tag{1.13}$$

then u *is unique in the class of such weak solutions with the same data* $f = \operatorname{div} F$ *and* g.

Note that the weakest integrability condition on F in (1.13) is obtained when $q = 4$ and $r > \frac{4}{3}$ is chosen arbitrarily close to $\frac{4}{3}$; concerning g the embedding $L^2(\partial\Omega) \subset W^{-1/4,4}(\partial\Omega)$ shows that a weak solution of (1.11) is unique provided that $\|u\|_4$ or $\|F\|_r + \|g\|_{2,\partial\Omega}$ with $r > \frac{4}{3}$ are sufficiently small.

Corollary 1.4 on weak L^2-solutions may easily be extended to weak L^q-solutions. As in (1.11) a vector field $u \in W^{1,q}(\Omega)$ is called a weak L^q-solution of (1.1) if

$$-\Delta u + u \cdot \nabla u + \nabla p = \operatorname{div} F, \qquad \operatorname{div} u = k \text{ in } \Omega$$

holds with some $p \in L^q(\Omega)$ in the sense of distributions and if $u|_{\partial\Omega} = g$ is satisfied in the sense of classical trace theorems.

The next corollary follows from Theorem 1.3 and the regularity property in Proposition 2.4(1).

Corollary 1.5. *Assume that the data $f = \operatorname{div} F$, k and g from* (1.2) *additionally satisfy the conditions $F \in L^q(\Omega)$, $k \in L^q(\Omega)$ and $g \in W^{1-1/q,q}(\partial\Omega)$. Then there exists a constant $K = K(\Omega, q, r) > 0$ such that the smallness condition*

$$\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega} \leqslant K$$

implies the existence of a unique weak solution $u \in W^{1,q}(\Omega)$ in the usual weak L^q-sense.

The proofs in Section 2.3 below will show that the previous results can be improved concerning the assumptions on $f = \operatorname{div} F$:

Remark 1.6. The condition $f = \operatorname{div} F$, $F \in L^r(\Omega)$, in (1.2) may be replaced by the slightly weaker condition $A_q^{-1} P_q f \in L^q_\sigma(\Omega)$ in the sense of (2.10) below. In this case, the term $-\langle F, \nabla w\rangle = \langle \operatorname{div} F, w\rangle$ in (1.3) and (1.6) is replaced by

$$\left\langle A_q^{-1} P_q f, A_{q'} w\right\rangle, \quad w \in C^2_{0,\sigma}(\bar{\Omega}).$$

Then both Theorems 1.2 and 1.3 remain valid if we replace $\|F\|_r$ by $\|A_q^{-1} P_q f\|_q$ in the smallness assumption (1.8) and in the a priori estimates (1.7), (1.9) and (1.10). This extension follows from the proofs in Sections 2.2 and 2.3 and the explicit representation formulae (2.12) using (2.13), (2.18), (2.22) which are written in a form easily leading to this more general result.

2. Proofs

2.1. Preliminaries

Let $1 < q < \infty$ and $q' = \frac{q}{q-1}$. For the bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ of class $C^{2,1}$ we need the usual Lebesgue and Sobolev spaces $L^q(\Omega)$, $W^{m,q}(\Omega)$, $W_0^{m,q}(\Omega)$, $m = 1, 2$, with norms $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{m,q}(\Omega)} = \|\cdot\|_{m,q}$, respectively. The space $W^{-m,q}(\Omega) = W_0^{m,q'}(\Omega)'$ denotes the dual space of $W_0^{m,q'}(\Omega)$ with pairing $\langle f, v\rangle$ for any functional $f \in W^{-m,q}(\Omega)$ and test function $v \in W_0^{m,q'}(\Omega)$; the norm in $W^{-m,q}(\Omega)$ is denoted by $\|\cdot\|_{W^{-m,q}(\Omega)} = \|\cdot\|_{-m,q}$. Analogously, on the boundary $\partial\Omega$ we introduce the spaces $L^q(\partial\Omega)$, $W^{\alpha,q}(\partial\Omega)$ and $W^{-\alpha,q}(\partial\Omega) = W^{\alpha,q'}(\partial\Omega)'$ with pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$, $0 \leqslant \alpha \leqslant 2$. The corresponding norms are $\|\cdot\|_{q,\partial\Omega}$, $\|\cdot\|_{\alpha,q,\partial\Omega}$ and $\|\cdot\|_{-\alpha,q,\partial\Omega}$. Note that we will use the same notation for function spaces of scalar-, vector- or matrix-valued fields.

The spaces of smooth functions on Ω are denoted by $C_0^m(\Omega)$, $C^m(\Omega)$, $C^m(\bar{\Omega})$ for $m = 0, 1, 2, \ldots$ and $m = \infty$. Moreover,

$$C_0^m(\bar{\Omega}) = \left\{v \in C^m(\bar{\Omega}):\ v|_{\partial\Omega} = 0\right\}, \qquad C^m_{0,\sigma}(\Omega) = \left\{u \in C_0^m(\Omega):\ \operatorname{div} u = 0\right\},$$

and—as the main space of test functions—

$$C^m_{0,\sigma}(\bar{\Omega}) = \left\{u \in C_0^m(\bar{\Omega}):\ \operatorname{div} u = 0\right\}.$$

Concerning distributions $d \in C_0^\infty(\Omega)'$ on Ω we again use the symbol $\langle \cdot, \cdot \rangle$ for the duality pairing; on the boundary the test function space $C^m(\partial\Omega)$, $m = 1, 2$, allows for distributions in $C^m(\partial\Omega)'$ with pairing $\langle \cdot, \cdot \rangle_{\partial\Omega}$.

For $1 < q < \infty$ let $L^q_\sigma(\Omega)$ be the closure of $C^\infty_{0,\sigma}(\Omega)$ with the norm $\|\cdot\|_q$. As is well known, $L^q_\sigma(\Omega)$ is the space of solenoidal vector fields in $L^q(\Omega)$ with vanishing normal trace on $\partial\Omega$. Then the dual space $L^q_\sigma(\Omega)'$ can be identified with $L^{q'}_\sigma(\Omega)$ using the canonical pairing $\langle f, v \rangle = \int_\Omega f \cdot v\, dx$; thus we will write $L^q_\sigma(\Omega)' = L^{q'}_\sigma(\Omega)$. Similarly we use the space $L^q(\partial\Omega)'$ with canonical pairing $\langle f, v \rangle_{\partial\Omega} = \int_{\partial\Omega} f \cdot v\, do$ where $\int_{\partial\Omega} \ldots do$ denotes the boundary integral on $\partial\Omega$ with surface measure do.

Let us recall some classical trace and extension properties for Sobolev spaces. For $m = 1, 2$ there exists a well defined boundary trace operator from $W^{m,q}(\Omega)$ onto $W^{m-1/q,q}(\partial\Omega)$. Conversely, there exist linear bounded extension operators

$$E_1 : W^{1-1/q,q}(\partial\Omega) \to W^{1,q}(\Omega), \tag{2.1}$$

$$E_2 : W^{2-1/q,q}(\partial\Omega) \times W^{1-1/q,q}(\partial\Omega) \to W^{2,q}(\Omega) \tag{2.2}$$

such that

$$E_1(h)\big|_{\partial\Omega} = h \quad \text{and} \quad E_2(h_1, h_2)\big|_{\partial\Omega} = h_1, \qquad N \cdot \nabla E_2(h_1, h_2) = h_2. \tag{2.3}$$

We note that the operator norms of E_1 and E_2 depend only on Ω and q.

Let $1 < r \leqslant q$, $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$, and let $f \in L^q(\Omega)$, $\operatorname{div} f \in L^r(\Omega)$. Then by Green's identity $\langle \operatorname{div} f, E_1(h) \rangle = \langle N \cdot f, h \rangle_{\partial\Omega} - \langle f, \nabla E_1(h) \rangle$ and the embedding estimate $\|E_1(h)\|_{r'} \leqslant c(\|E_1(h)\|_{q'} + \|\nabla E_1(h)\|_{q'})$, we obtain that

$$\big|\langle N \cdot f, h \rangle_{\partial\Omega}\big| \leqslant c\big(\|f\|_q + \|\operatorname{div} f\|_r\big)\|h\|_{1/q,q',\partial\Omega}, \quad h \in W^{1/q,q'}(\partial\Omega),$$

with $c = c(\Omega, q, r) > 0$. Hence the normal component $N \cdot f|_{\partial\Omega}$ of f at $\partial\Omega$ is well defined in $W^{-1/q,q}(\partial\Omega)$ and satisfies the estimate

$$\|N \cdot f\|_{-1/q,q,\partial\Omega} \leqslant c\big(\|f\|_q + \|\operatorname{div} f\|_r\big). \tag{2.4}$$

Conversely, there exists a bounded linear extension operator

$$\hat{E} : W^{-1/q,q}(\partial\Omega) \to \big\{f \in L^q(\Omega):\ \operatorname{div} f \in L^r(\Omega)\big\}$$

such that $N \cdot \hat{E}(h)|_{\partial\Omega} = h$; in particular,

$$\big\|\hat{E}(h)\big\|_q + \big\|\operatorname{div} \hat{E}(h)\big\|_r \leqslant c\|h\|_{-1/q,q,\partial\Omega} \tag{2.5}$$

with $c = c(\Omega, q, r) > 0$; cf. [15, Corollary 4.6, (4.10)].

By analogy, for $f \in L^q(\Omega)$ such that $\operatorname{rot} f = \partial_1 f_2 - \partial_2 f_1 \in L^r(\Omega)$, i.e., $\operatorname{div} \tilde{f} \in L^r(\Omega)$ for $\tilde{f} = (f_2, -f_1)$, we conclude that the tangential component

$$\tau \cdot f \in W^{-1/q,q}(\partial\Omega), \quad \tau = (-N_2, N_1),$$

of f at $\partial\Omega$ is well defined; moreover, by (2.4)

$$\|\tau \cdot f\|_{-1/q,q,\partial\Omega} \leqslant c\big(\|f\|_q + \|\operatorname{rot} f\|_r\big). \tag{2.6}$$

We recall that there exists a linear bounded operator

$$B : L^q_0(\Omega) := \left\{ f \in L^q(\Omega) \colon \int_\Omega f\,dx = 0 \right\} \to W^{1,q}_0(\Omega),$$

$$B : L^q_0(\Omega) \cap W^{1,q}_0(\Omega) \to W^{2,q}_0(\Omega),$$

satisfying $\operatorname{div} B(f) = f$; in particular, there exists $c = c(\Omega, q) > 0$ such that

$$\big\|B(f)\big\|_{1,q} \leqslant c\|f\|_q, \quad \big\|B(f)\big\|_{2,q} \leqslant c\|f\|_{1,q} \tag{2.7}$$

for $f \in L^q_0(\Omega)$ and $f \in L^q_0(\Omega) \cap W^{1,q}_0(\Omega)$, respectively; see [5], [8, Theorem III 3.2], [16, p. 68].

Let $f \in L^q(\Omega)$, $1 < q < \infty$. Then the weak Neumann problem $\Delta H = \operatorname{div} f$ in Ω, $N \cdot (\nabla H - f)|_{\partial\Omega} = 0$, has a unique solution $\nabla H \in L^q(\Omega)$ such that

$$\|\nabla H\|_q \leqslant c\|f\|_q, \quad c = c(\Omega, q) > 0; \tag{2.8}$$

cf. [7,15]. Setting $P_q f = f - \nabla H$ we get the bounded Helmholtz projection $P_q : L^q(\Omega) \to L^q(\Omega)$ with range $\mathcal{R}(P_q) = L^q_\sigma(\Omega)$, satisfying $P_q^2 = P_q$ and $P_q' = P_{q'}$ for the dual operator.

The Stokes operator

$$A_q = -P_q\Delta : \mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \to L^q_\sigma(\Omega)$$

is a closed bijective operator on the dense domain $\mathcal{D}(A_q) \subset L^q_\sigma(\Omega)$ with the following properties: The fractional powers $A_q^\beta : \mathcal{D}(A_q^\beta) \to L^q_\sigma(\Omega)$, $0 \leqslant \beta \leqslant 1$, with dense domain $\mathcal{D}(A_q^\beta) \subset L^q_\sigma(\Omega)$ are well defined and injective, and $A_q^{-\beta} = (A_q^\beta)^{-1} : L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$ are bounded operators with range $\mathcal{R}(A_q^{-\beta}) = \mathcal{D}(A_q^\beta)$. The norms $\|u\|_{1,q}$ and $\|A_q^{1/2}u\|_q$ are equivalent for $u \in \mathcal{D}(A_q^{1/2})$, and the norms $\|u\|_{2,q}$ and $\|A_q u\|_q$ are equivalent for $u \in \mathcal{D}(A_q)$; in particular, $C^\infty_{0,\sigma}(\Omega)$ is dense in $\mathcal{D}(A_q^{1/2})$ with norm $\|A_q^{1/2}\|_q$, and $C^2_{0,\sigma}(\bar{\Omega})$ is dense in $\mathcal{D}(A_q)$ with norm $\|A_q\|_q$. Moreover, the embedding estimate

$$\|u\|_q \leqslant c\big\|A_r^\beta u\big\|_r, \quad u \in \mathcal{D}\big(A_r^\beta\big),\ 1 < r \leqslant q,\ \beta + \frac{1}{q} \geqslant \frac{1}{r}, \tag{2.9}$$

holds with a constant $c = c(\Omega, \beta, q, r) > 0$. Finally, $A_q u = A_r u$ for $u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_r)$, $1 < q, r < \infty$, and $(A_q)' = A_{q'}$ for the dual operator of A_q; cf. [1,4,8,11–15, 16–18].

To solve the Stokes and Navier–Stokes equations in their very weak formulation we introduce a generalized meaning of the operator $A_q^{-\beta} P_q$, $0 \leqslant \beta \leqslant 1$, $1 < q < \infty$. Given a distribution $u = (u_1, u_2) \in C_0^\infty(\Omega)'$ we say that its restriction $P_q u := u|_{C_{0,\sigma}^\infty}$ satisfies

$$A_q^{-\beta} P_q u \in L^q_\sigma(\Omega): \iff \begin{array}{l} \langle P_q u, w\rangle \text{ is well defined} \quad \text{for all } w \in \mathcal{D}\big(A_{q'}^{\beta}\big) \\ \text{and} \quad \big|\langle P_q u, w\rangle\big| \leqslant C \big\| A_{q'}^{\beta} w \big\|_{q'} \end{array} \tag{2.10}$$

with a constant $C = C(u)$. In other words,

$$\big|\big\langle P_q u, A_{q'}^{-\beta} v\big\rangle\big| \leqslant C \|v\|_{q'} \quad \text{for all } v \in L^{q'}_\sigma(\Omega).$$

Hence there exists an element $A_q^{-\beta} P_q u := u^* \in L^q_\sigma(\Omega)$ with norm $\|A_q^{-\beta} P_q u\|_q \leqslant C$ such that formally

$$\big\langle A_q^{-\beta} P_q u, v\big\rangle = \langle u^*, v\rangle = \big\langle P_q u, A_{q'}^{-\beta} v\big\rangle = \big\langle u, P_{q'} A_{q'}^{-\beta} v\big\rangle, \quad v \in L^{q'}_\sigma(\Omega). \tag{2.11}$$

2.2. *Proof of Theorem 1.2*

The idea of the proof is based on an explicit representation of the very weak solution u in the form

$$u = R + S + \nabla H, \tag{2.12}$$

where $\nabla H = (I - P_q)u$ carries the information of $k = \operatorname{div} u$ and $g \cdot N = u|_{\partial\Omega} \cdot N$, see (2.13) below, where $S = A_q^{-1} P_q \operatorname{div} F$ solves a homogeneous Stokes equation with external force $f = \operatorname{div} F$, and R mainly carries the information of the tangential component of g (plus a correction due to ∇H), see (2.21) below.

In the following we construct R, S and ∇H step by step using only the data f, k, g; then we show that $u = R + S + \nabla H$ is the desired very weak solution. First we define ∇H as a solution of the weak Neumann problem

$$\Delta H = k \quad \text{in } \Omega, \qquad N \cdot \nabla H|_{\partial\Omega} = N \cdot g. \tag{2.13}$$

For this purpose we define $v = \hat{E}(N \cdot g)$ as in Section 2.1 satisfying $v \in L^q(\Omega)$, $\operatorname{div} v \in L^r(\Omega)$ and $N \cdot v|_{\partial\Omega} = N \cdot g$. Moreover, since $\int_\Omega (\operatorname{div} v - k)\, dx = \int_{\partial\Omega} N \cdot g\, do - \int_\Omega k\, dx = 0$ by (1.2), we find $b = B(\operatorname{div} v - k) \in W_0^{1,r}(\Omega)$ satisfying $\operatorname{div} b = \operatorname{div} v - k$ and

$$\|b\|_q \leqslant c_1 \|\nabla b\|_r \leqslant c_2 \big(\|\operatorname{div} v\|_r + \|k\|_r\big) \tag{2.14}$$

with $c_j = c_j(\Omega, q, r) > 0$, $j = 1, 2$, see (2.7). Then we solve the weak Neumann problem

$$\Delta H = \operatorname{div}(v - b), \qquad N \cdot (\nabla H - v + b)|_{\partial\Omega} = 0 \tag{2.15}$$

and obtain by (2.5), (2.8), (2.14) the estimate

$$\|\nabla H\|_q \leqslant c_1 \|v - b\|_q \leqslant c_2\big(\|g\|_{-1/q,q,\partial\Omega} + \|k\|_r\big) \tag{2.16}$$

with $c_j = c_j(\Omega, q, r) > 0$, $j = 1, 2$. For later use, we remark that $\nabla H|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ is well defined. Actually, $\operatorname{div}(\nabla H) = k \in L^r(\Omega)$ and $\operatorname{rot}(\nabla H) = 0$; hence by (2.4), (2.6), (2.16)

$$\begin{aligned}\|\nabla H\|_{-1/q,q,\partial\Omega} &\leqslant c_1\big(\|N \cdot \nabla H\|_{-1/q,q,\partial\Omega} + \|\tau \cdot \nabla H\|_{-1/q,q,\partial\Omega}\big)\\ &\leqslant c_2\big(\|g\|_{-1/q,q,\partial\Omega} + \|k\|_r\big)\end{aligned} \tag{2.17}$$

with $c_j = c_j(\Omega, q, r) > 0$, $j = 1, 2$.

Next we define

$$S = A_q^{-1} P_q \operatorname{div} F. \tag{2.18}$$

Note that for all $w \in D(A_{q'})$

$$\begin{aligned}\big|\langle \operatorname{div} F, w\rangle\big| &= \big|-\langle F, \nabla w\rangle\big| \leqslant \|F\|_r \|\nabla w\|_{r'}\\ &\leqslant c_1 \|F\|_r \big\|A_{r'}^{1/2} w\big\|_{r'} \leqslant c_2 \|F\|_r \|A_{q'} w\|_{q'}\end{aligned}$$

with $c_j = c_j(\Omega, q, r) > 0$, $j = 1, 2$. Hence $A_q^{-1} P_q \operatorname{div} F \in L^q_\sigma(\Omega)$ is well defined and satisfies

$$\big\|A_q^{-1} P_q \operatorname{div} F\big\|_q \leqslant c_2 \|F\|_r, \tag{2.19}$$

cf. (2.10). Moreover, by (2.11), for all $w \in C^2_{0,\sigma}(\bar\Omega)$

$$-\langle S, \Delta w\rangle = \big\langle A_q^{-1} P_q \operatorname{div} F, A_{q'} w\big\rangle = \langle P_q \operatorname{div} F, w\rangle = -\langle F, \nabla w\rangle. \tag{2.20}$$

Comparing this identity with (1.6) we conclude that $S = A_q^{-1} P_q \operatorname{div} F$ is a very weak solution of the Stokes system with $S|_{\partial\Omega} = 0$, $\operatorname{div} S = 0$ in Ω and external force $\operatorname{div} F$.

Now it remains to find the remainder term R $(= u - S - \nabla H)$ as the very weak solution of the Stokes system

$$-\Delta R + \nabla p = 0, \qquad \operatorname{div} R = 0 \quad \text{in } \Omega, \quad R|_{\partial\Omega} = g - \nabla H|_{\partial\Omega}. \tag{2.21}$$

Thus for all $w \in C^2_{0,\sigma}(\bar\Omega)$

$$-\langle R, \Delta w\rangle + \langle g - \nabla H, N \cdot \nabla w\rangle_{\partial\Omega} = 0. \tag{2.22}$$

By (2.17) and using properties of the trace map we get for $\tilde g = g - \nabla H$ and all $w \in \mathcal{D}(A_{q'})$

$$\begin{aligned}\bigl|\langle \tilde g, N \cdot \nabla w\rangle_{\partial\Omega}\bigr| &\leqslant c\|\tilde g\|_{-1/q,q,\partial\Omega}\|\nabla w\|_{1/q,q',\partial\Omega} \leqslant c\|\tilde g\|_{-1/q,q,\partial\Omega}\|w\|_{2,q'} \\ &\leqslant c\bigl(\|g\|_{-1/q,q,\partial\Omega} + \|k\|_r\bigr)\|A_{q'}w\|_{q'}.\end{aligned}$$

Since $\mathcal{R}(A_{q'}) = L^{q'}_\sigma(\Omega)$, this inequality may be written in the form

$$\bigl|\bigl\langle \tilde g, N \cdot \nabla\bigl(A_{q'}^{-1}v\bigr)\bigr\rangle_{\partial\Omega}\bigr| \leqslant c\bigl(\|g\|_{-1/q,q,\partial\Omega} + \|k\|_r\bigr)\|v\|_{q'}, \quad v \in L^{q'}_\sigma(\Omega).$$

Hence there exists a unique $R \in L^q_\sigma(\Omega)$ satisfying $\langle R, v\rangle = \langle \tilde g, N \cdot \nabla A_{q'}^{-1} v\rangle$ for all $v \in L^{q'}_\sigma(\Omega)$ and consequently also (2.22); moreover,

$$\|R\|_q \leqslant c\bigl(\|g\|_{-1/q,q,\partial\Omega} + \|k\|_r\bigr), \quad c = c(\Omega, q, r) > 0. \tag{2.23}$$

Finally we have to show that $u := R + S + \nabla H$ is a very weak solution of (1.5). By (2.13), (2.20), (2.22) it suffices to show the identity

$$\langle \nabla H, \Delta w\rangle = \langle \nabla H, N \cdot \nabla w\rangle_{\partial\Omega} \quad \text{for all } w \in C^2_{0,\sigma}(\bar\Omega). \tag{2.24}$$

For its proof we approximate k, g in (2.13) by smooth functions k_n, g_n, $n \in \mathbb{N}$, such that $\|k - k_n\|_r \to 0$, $\|g - g_n\|_{-1/q,q,\partial\Omega} \to 0$ as $n \to \infty$, and let $\nabla H_n \in L^q(\Omega)$ be the solution of (2.13) with k, g replaced by k_n, g_n. Then, by (2.16), (2.17) we obtain $\|\nabla H - \nabla H_n\|_q \to 0$, $\|\nabla H - \nabla H_n\|_{-1/q,q,\partial\Omega} \to 0$ as $n \to \infty$; hence the identity

$$\begin{aligned}\langle \nabla H_n, \Delta w\rangle &= \langle \nabla H_n, N \cdot \nabla w\rangle_{\partial\Omega} - \bigl\langle \nabla(\nabla H_n), \nabla w\bigr\rangle \\ &= \langle \nabla H_n, N \cdot \nabla w\rangle_{\partial\Omega} + \bigl\langle \Delta(\nabla H_n), w\bigr\rangle \\ &= \langle \nabla H_n, N \cdot \nabla w\rangle_{\partial\Omega} - \langle \Delta H_n, \operatorname{div} w\rangle \\ &= \langle \nabla H_n, N \cdot \nabla w\rangle_{\partial\Omega}\end{aligned}$$

converges to (2.24) as $n \to \infty$.

Note that a very weak solution $u \in L^q(\Omega)$ of (1.5) is unique. Indeed, in the case $F = 0$, $k = 0$, $g = 0$ the defining identity (1.6) implies that $u \in L^q_\sigma(\Omega)$ satisfies $-\langle u, \Delta w\rangle = \langle u, A_{q'}w\rangle = 0$ for all $w \in C^2_{0,\sigma}(\bar\Omega)$; since $\mathcal{R}(A_{q'}) = L^{q'}_\sigma(\Omega)$ we conclude that $u = 0$. Moreover, in the general case, (2.12) and (2.16), (2.19), (2.23) yield the a priori estimate (1.7) for u.

Concerning the pressure, we consider test functions $w \in C^\infty_{0,\sigma}(\Omega)$ in (1.6) and are led to the identity

$$\langle \operatorname{div} F + \Delta u, w\rangle = 0$$

in the sense of distributions. Then de Rham's argument proves the existence of a distribution $p \in C^\infty_0(\Omega)'$ satisfying $\operatorname{div} F + \Delta u = \nabla p$. Furthermore, we get $\nabla p \in W^{-2,q}(\Omega)$ and

$\|\nabla p\|_{-2,q} \leqslant c_1(\|F\|_r + \|u\|_q)$. From [16, II, (2.3.3)], we get that there exists $M \in \mathbb{R}$ such that

$$\|p - M\|_{-1,q} \leqslant c_2\|\nabla p\|_{-2,q} \leqslant c_3\big(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big)$$

with $c_j = c_j(\Omega, q, r) > 0$, $j = 1, 2, 3$. Replacing p by $p - M$ we complete the proof of Theorem 1.2. □

2.3. Proof of Theorem 1.3

We write the Navier–Stokes system (1.1) in the form

$$-\Delta u + \nabla p = \hat{f}(u), \qquad \operatorname{div} u = k \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g, \tag{2.25}$$

where

$$\hat{f}(u) = f - u \cdot \nabla u = f - \operatorname{div}(uu) + ku, \tag{2.26}$$

and use the representation formula (2.12) in the form

$$u = \mathcal{F}(u) := \nabla H + R + A_q^{-1} P_q \hat{f}(u); \tag{2.27}$$

here ∇H, R are defined by (2.13), (2.22), respectively. At this point, it is necessary to show that $A_q^{-1} P_q \hat{f}(u) \in L^q(\Omega)$ for $u \in L^q(\Omega)$, see (2.10).

Lemma 2.1. *Let $2 < q < \infty$, $q' < r \leqslant q$ and $\frac{1}{2} + \frac{1}{q} \geqslant \frac{1}{r}$, and let $u, v \in L^q(\Omega)$, $k \in L^r(\Omega)$.*

(i) *There exists a constant $c = c(\Omega, q, r) > 0$ such that*

$$\big\|A_q^{-1} P_q \operatorname{div}(uv)\big\|_q \leqslant c\|u\|_q\|v\|_q, \qquad \big\|A_q^{-1} P_q(ku)\big\|_q \leqslant c\|u\|_q\|k\|_r.$$

(ii) *Let $w \in L^{q_0}(\Omega)$, $q_0 > 2$ and $\tilde{q} = \frac{q_0 q}{q_0 + q}$. Then there exists a constant $c = c(\Omega, q, q_0) > 0$ such that*

$$\big\|A_{\tilde{q}}^{-1/2} P_{\tilde{q}} \operatorname{div}(vw)\big\|_{\tilde{q}} + \big\|A_{\tilde{q}}^{-1/2} P_{\tilde{q}} \operatorname{div}(wv)\big\|_{\tilde{q}} \leqslant c\|v\|_q\|w\|_{q_0},$$
$$\big\|A_2^{-1/2} P_2(ku)\big\|_2 \leqslant c\|u\|_q\|k\|_r.$$

Proof. (i) For $\varphi \in C^2_{0,\sigma}(\bar{\Omega}) \subset \mathcal{D}(A_{q'})$

$$\begin{aligned}\big|\langle \operatorname{div}(uv), \varphi\rangle\big| &= \big|\langle uv, \nabla\varphi\rangle\big| \leqslant \|u\|_q\|v\|_q\|\nabla\varphi\|_{(q/2)'} \\ &\leqslant c_1\|u\|_q\|v\|_q\big\|A^{1/2}_{(q/2)'}\varphi\big\|_{(q/2)'} \leqslant c_2\|u\|_q\|v\|_q\|A_{q'}\varphi\|_{q'}\end{aligned}$$

by (2.9) with $\beta = \frac{1}{2}$ and q, r replaced by $(\frac{q}{2})'$, q'. Hence (2.10) yields the first estimate of (i). To prove the second estimate, define $s = (1 - \frac{1}{r} - \frac{1}{q})^{-1} \in (1, \infty)$ and use (2.9) with $\beta = 1$ and q, r replaced by s, q' to get that

$$\bigl|\langle ku, \varphi\rangle\bigr| \leqslant \|k\|_r \|u\|_q \|\varphi\|_s \leqslant c_3 \|k\|_r \|u\|_q \|A_{q'}\varphi\|_{q'}.$$

(ii) For $\varphi \in C_{0,\sigma}^\infty(\Omega)$ Hölder's inequality yields the estimate

$$\bigl|\langle \operatorname{div}(vw), \varphi\rangle\bigr| = \bigl|\langle vw, \nabla\varphi\rangle\bigr| \leqslant c\|v\|_q \|w\|_{q_0} \bigl\|A_{\tilde{q}'}^{1/2}\varphi\bigr\|_{\tilde{q}'}.$$

The term $|\langle \operatorname{div}(wv), \varphi\rangle|$ will be estimated similarly. Since $C_{0,\sigma}^\infty(\Omega)$ is dense in $\mathcal{D}(A_{\tilde{q}'}^{1/2})$, the first inequality is proved, see (2.10). Moreover, using the continuous embedding $\mathcal{D}(A_2^{1/2}) \subset L_\sigma^s(\Omega)$ for every $s \in (1, \infty)$, the estimate

$$\bigl|\langle ku, \varphi\rangle\bigr| \leqslant \|k\|_r \|u\|_q \|\varphi\|_s \leqslant c\|k\|_r \|u\|_q \bigl\|A_2^{1/2}\varphi\bigr\|_2,$$

where $s = (1 - \frac{1}{r} - \frac{1}{q})^{-1}$, proves the second inequality. $\square$

By Lemma 2.1 a vector field $u \in L^q(\Omega)$ is a very weak solution of (1.1) if and only if u is a very weak solution of (2.25). Moreover, u may be found as a fixed point of the nonlinear equation (2.27). To solve (2.27), we use (2.16), (2.19), (2.23) and Lemma 2.1 to get the inequality

$$\bigl\|\mathcal{F}(u)\bigr\|_q \leqslant C_0\bigl(\|u\|_q^2 + \|u\|_q \|k\|_r + \|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\bigr) \tag{2.28}$$

where $C_0 = C_0(\Omega, q, r) > 0$. Setting $\alpha = C_0$, $\beta = C_0\|k\|_r$ and $\gamma = C_0(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega})$, the previous inequality may be written in the form

$$\bigl\|\mathcal{F}(u)\bigr\|_q \leqslant \alpha\|u\|_q^2 + \beta\|u\|_q + \gamma, \quad u \in L^q(\Omega).$$

Analogously, we obtain that

$$\bigl\|\mathcal{F}(u) - \mathcal{F}(v)\bigr\|_q \leqslant \bigl(\alpha\|u\|_q + \alpha\|v\|_q + \beta\bigr)\|u - v\|_q, \quad u, v \in L^q(\Omega).$$

Now Banach's fixed point theorem applied to $\mathcal{F}$ on a closed ball $\mathcal{B}_\rho(0) \subset L^p(\Omega)$, $\rho > 0$, proves the existence of a unique fixed point $u \in \mathcal{B}_\rho(0)$ of (2.27) provided that the data F, k, g satisfy the smallness condition (1.8) with a suitable constant $K = K(C_0)$, C_0 as in (2.28). Moreover, the unique solution $u \in \mathcal{B}_\rho(0)$ satisfies the a priori estimate (1.9) with $C = 2C_0$; for more details of this standard procedure see, e.g., [10, Proof of Theorem 4]. As in the proof of Theorem 1.2 we get a pressure $p \in W^{-1,q}(\Omega)$ such that (1.1) holds in the sense of distributions and satisfying (1.10).

It remains to prove the uniqueness of u in the class of *all* very weak solutions of (1.1). Assume that $u, v \in L^q(\Omega)$ are very weak solutions of (1.1) with the same data f, k, g. Then the representation formula (2.27) (with $\nabla H = 0$, $R = 0$) yields for $u - v$ the identity

$$u - v = A_q^{-1} P_q \operatorname{div}\bigl(v(v-u) + (v-u)u\bigr) + A_q^{-1} P_q\bigl(k(u-v)\bigr), \tag{2.29}$$

which can be considered as a linear equation in $u - v$ keeping u, v fixed. Applying $A_q^{1/2}$ formally we get for $w = u - v$ that

$$A_2^{1/2} w = -A_2^{-1/2} P_2 \operatorname{div}(vw + wu) + A_2^{-1/2} P_2(kw). \tag{2.30}$$

Actually, if $q \geqslant 4$, then Lemma 2.1(ii) shows that both terms $A_2^{-1/2} P_2 \operatorname{div}(vw + wu)$ and $A_2^{-1/2} P_2(kw)$ are well defined elements in $L^2_\sigma(\Omega)$ yielding $w = u - v \in \mathcal{D}(A_2^{1/2})$ in (2.29). However, if $2 < q < 4$, then $A_2^{-1/2} P_2(kw) \in L^2_\sigma(\Omega)$ as before, but by Lemma 2.1(ii) using $q_0 = q > 2$ we only get that $A_{\tilde q}^{-1/2} P_{\tilde q} \operatorname{div}(vw + wu) \in L^{\tilde q}_\sigma$ where $\tilde q = \frac{q_0 q}{q_0 + q} < 2$. Hence by (2.29)

$$w \in \mathcal{D}\bigl(A_{\tilde q}^{1/2}\bigr) \subset W^{1,\tilde q}(\Omega) \subset L^{q_1}(\Omega), \qquad \frac{1}{q_1} = \frac{1}{\tilde q} - \frac{1}{2} = \frac{1}{q_0} - \left(\frac{1}{2} - \frac{1}{q}\right).$$

This step will be repeated finitely many times implying that

$$w \in L^{q_j}(\Omega), \qquad \frac{1}{q_j} = \frac{1}{q_0} - j\left(\frac{1}{2} - \frac{1}{q}\right), \quad j = 1, 2, \ldots,$$

until $\frac{q_0 q_j}{q_0 + q_j} \geqslant 2$ will be guaranteed. Then the case $q \geqslant 4$ considered just before applies and proves $w \in \mathcal{D}(A_2^{1/2})$ and (2.30).

Next take the L^2-scalar product of (2.30) with $A_2^{1/2} w$ and note the identity $\int_\Omega A_2^{1/2} w \cdot A_2^{-1/2} P_2 \operatorname{div}(vw)\, dx = -\int_\Omega (v \cdot \nabla w) \cdot w \, dx = \frac{1}{2} \int_\Omega k|w|^2 \, dx$. Then Lemma 2.1(ii) and (1.9) imply that

$$\begin{aligned}\bigl\|A_2^{1/2} w\bigr\|_2^2 &\leqslant C_1\bigl(\|u\|_q + \|k\|_r\bigr)\bigl\|A_2^{1/2} w\bigr\|_2^2 \\ &\leqslant C_2\bigl(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\bigr)\bigl\|A_2^{1/2} w\bigr\|_2^2\end{aligned}$$

with $C_i = C_i(\Omega, q, r) > 0$, $i = 1, 2$. Assuming that the smallness condition (1.8) even implies $KC_2 < 1$, we conclude that $A_2^{1/2} w = 0$ and that $u = v$.

Now Theorem 1.3 is completely proved. $\square$

2.4. *Further results*

Remark 2.2 *(Representation formula).* (1) The representation $u = R + S + \nabla H$, see (2.12), of the very weak solution u of the Stokes system (1.5) describes u as the sum of three terms each of which is a very weak solution of a related Stokes system. Concerning ∇H, note that by (2.13) $v = \nabla H$ solves the equation

$$-\Delta v + \nabla p = 0, \qquad \operatorname{div} v = k \quad \text{in } \Omega, \quad v|_{\partial\Omega} = \nabla H|_{\partial\Omega},$$

where $\nabla H|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ is well defined, cf. (2.17).

(2) Consider a very weak solution $u \in L^q(\Omega)$ of the Navier–Stokes system (1.1), (1.2). By (2.26), (2.27) u has a representation

$$u = R + S + \nabla H - A_q^{-1} P_q u \cdot \nabla u$$

where R, S and ∇H are defined by (2.21), (2.18) and (2.13), respectively. Let

$$E = E(f, k, g) = R + S + \nabla H \in L^q(\Omega),$$

i.e., E is a very weak solution of the inhomogeneous Stokes system (1.5). Then $U = u - E$ is a very weak solution of the nonlinear system

$$-\Delta U + \nabla p + (U + E) \cdot \nabla(U + E) = 0, \qquad \operatorname{div} U = 0, \quad U|_{\partial\Omega} = 0 \tag{2.31}$$

of Navier–Stokes type with homogeneous data; here Definition 1.1 must be modified correspondingly. We may solve (2.31) directly with Banach's fixed point theorem when E is considered to be known. In this case the weaker smallness condition

$$\|E\|_q + \|k\|_r < K_1 = K_1(\Omega, q, r) \tag{2.32}$$

instead of (1.8) yields existence and (global) uniqueness of the very weak solution of (2.31) and therefore also of (1.1).

(3) The term $\|F\|_r + \|g\|_{-1/q,q,\partial\Omega} + \|k\|_r$ in the smallness condition (1.8) may be arbitrarily large even when the smallness condition (2.32) is satisfied. Actually, we consider data of the type $F_n = \nabla\rho_n$, $\rho_n \in C_{0,\sigma}^\infty(\Omega)$, and $k_n = 0$, $g_n = 0$, $n \in \mathbb{N}$, only. Obviously the unique very weak solution of the Stokes system (1.5) is $E_n = -\rho_n$ so that we have to compare the norms

$$\|F_n\|_r = \|\nabla\rho_n\|_r \quad \text{and} \quad \|E_n\|_q = \|\rho_n\|_q.$$

To be more precise, let $0 \neq \rho \in C_{0,\sigma}^\infty(B_1(0))$ be fixed and assume that for every $n \in \mathbb{N}$ the domain Ω admits the choice of n^2 points $\{x_1^{(n)}, \ldots, x_{n^2}^{(n)}\} \subset \Omega$ such that the balls $B_{1/n}(x_k^{(n)})$, $1 \leqslant k \leqslant n^2$, are pairwise disjoint. Now define

$$\rho_n(x) = \sum_{k=1}^{n^2} \rho\big(n\big(x - x_k^{(n)}\big)\big), \quad x \in \Omega.$$

Then $\|\rho_n\|_q^q = \sum_k \|\rho(n(\cdot - x_k^{(n)}))\|_q^q = M_0^q$ where $M_0 = \|\rho\|_q > 0$ and

$$\|\nabla\rho_n\|_r^r = n^r \sum_k \big\|(\nabla\rho)\big(n\big(\cdot - x_k^{(n)}\big)\big)\big\|_r^r = n^r M_1^r,$$

where $M_1 = \|\nabla\rho\|_r^r > 0$. Hence $\|F_n\|_r / \|E_n\|_q \sim n$ as $n \to \infty$.

Remark 2.3 *(Traces).* Consider a very weak solution $u \in L^q(\Omega)$ of the Stokes system (1.5). Then the normal component $N \cdot u|_{\partial\Omega} = N \cdot g \in W^{-1/q,q}(\partial\Omega)$ is well defined, cf. (2.4). Since there is no trace theorem of the type $u \in L^q(\Omega) \Rightarrow u|_{\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$, we have to consider the tangential component $\tau \cdot u|_{\partial\Omega} = -N_2 g_1 + N_1 g_2$ more carefully.

Let $h \in W^{1/q,q'}(\partial\Omega)$ with $N \cdot h = 0$ be given and define using (2.2), (2.7)

$$w = w(h) = (I - B\,\mathrm{div}) \cdot E_2(0, h).$$

Obviously $\mathrm{div}\, w = 0$ in Ω, $w|_{\partial\Omega} = 0$ and $N \cdot \nabla w|_{\partial\Omega} = h$, since $\mathrm{div}\, E_2(0,h)|_{\partial\Omega} = N \cdot h = 0$; note that here h and $E_2(0,h)$ are vector-valued. Moreover, $w(\cdot)$ defines a bounded linear operator from the space $\{h \in W^{1/q,q'}(\partial\Omega)\colon\ N \cdot h = 0\}$ into $W^{2,q'}(\Omega)$ satisfying the estimate

$$\big\|w(h)\big\|_{2,q'} \leqslant c\|h\|_{1/q,q',\partial\Omega}, \quad c = c(\Omega, q) > 0.$$

Inserting w as a test function into (1.6) we get the well defined relation

$$\langle g, h\rangle_{\partial\Omega} = \big\langle u, \Delta w(h)\big\rangle + \big\langle A_q^{-1} P_q f, A_{q'} w(h)\big\rangle. \tag{2.33}$$

Since $N \cdot h = 0$, we may replace $\langle g, h\rangle_{\partial\Omega}$ by $\langle(\tau \cdot g)\tau, h\rangle_{\partial\Omega}$ and interpret the right-hand side of (2.33) as the precise meaning of the tangential component $\tau \cdot u|_{\partial\Omega}$ of the very weak solution u. Moreover, $\tau \cdot u|_{\partial\Omega}$ satisfies the estimate

$$\begin{aligned}\|\tau \cdot u\|_{-1/q,q,\partial\Omega} &\leqslant c_1\big(\|u\|_q + \big\|A_q^{-1} P_q f\big\|_q\big) \\ &\leqslant c_2\big(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big)\end{aligned} \tag{2.34}$$

with $c_j = c_j(\Omega, q, r) > 0$, $j = 1, 2$.

Analogously, for a very weak solution $u \in L^q(\Omega)$ of the Navier–Stokes system (1.1), (1.2), the tangential component $\tau \cdot u|_{\partial\Omega} = \tau \cdot g \in W^{-1/q,q}(\Omega)$ is well defined and satisfies (2.33), (2.34) with f replaced by $f - u \cdot \nabla u$.

Proposition 2.4 *(Regularity).* (1) *Assume that the data* $f = \mathrm{div}\, F, k, g$ *in Theorems* 1.2 *and* 1.3 *satisfy*

$$F \in L^q(\Omega), \quad k \in L^q(\Omega), \quad g \in W^{1-1/q,q}(\partial\Omega).$$

Then the very weak solution u *in Theorems* 1.2 *and* 1.3 *has the regularity property* $u \in W^{1,q}(\Omega)$ *and there exists a corresponding pressure* $p \in L^q(\Omega)$*. Moreover, estimate* (1.7) *in Theorem* 1.2 *can be replaced by*

$$\|u\|_{1,q} + \|p\|_q \leqslant c\big(\|F\|_q + \|k\|_q + \|g\|_{1-1/q,q,\partial\Omega}\big) \tag{2.35}$$

with $c = c(\Omega, q) > 0$.

(2) *Assume that*

$$F \in W^{1,q}(\Omega), \quad k \in W^{1,q}(\Omega), \quad g \in W^{2-1/q,q}(\partial\Omega).$$

Then $u \in W^{2,q}(\Omega)$ *and* $p \in W^{1,q}(\Omega)$ *in Theorems* 1.2 *and* 1.3. *Moreover, in Theorem* 1.2

$$\|u\|_{2,q} + \|p\|_{1,q} \leqslant c\big(\|F\|_{1,q} + \|k\|_{1,q} + \|g\|_{2-1/q,q,\partial\Omega}\big). \tag{2.36}$$

Proof. (1) Concerning the linear case of Theorem 1.2 let $w = E_1(g) \in W^{1,q}(\Omega)$, and $b = B(k - \operatorname{div} w) \in W_0^{1,q}(\Omega)$, see (2.1) and (2.7); note that $\int_\Omega (k - \operatorname{div} w)\,dx = \int_\Omega k\,dx - \int_{\partial\Omega} N \cdot g\,do = 0$. Then $\tilde{u} = u - w - b$ solves the Stokes system

$$-\Delta\tilde{u} + \nabla p = \tilde{f}, \qquad \operatorname{div}\tilde{u} = 0 \quad \text{in } \Omega, \quad \tilde{u}|_{\partial\Omega} = 0 \tag{2.37}$$

with $\tilde{f} = f + \Delta w + \Delta b \in W^{-1,q}(\Omega)$ in the usual weak L^q-sense. Using the estimates of Section 2.1 we obtain the well-defined equation

$$A_q^{1/2}\tilde{u} = A_q^{-1/2} P_q \tilde{f} \in L^q_\sigma(\Omega) \tag{2.38}$$

leading to a unique solution $\tilde{u} \in \mathcal{D}(A_q^{1/2}) \subset W_0^{1,q}(\Omega)$. Then $u = \tilde{u} + w + b \in W^{1,q}(\Omega)$ is the (unique) very weak solution of (1.5) satisfying the estimate (2.35). Moreover, de Rham's argument yields the existence of a unique pressure $p \in L_0^q(\Omega)$ satisfying (2.35) as well.

In the nonlinear case we formally get that $\tilde{u} = u - w - b$ satisfies the identity

$$A_q^{1/2}\tilde{u} = A_q^{-1/2} P_q(\tilde{f} - u \cdot \nabla u), \tag{2.39}$$

cf. (2.38). However, we need an argument as at the end of the proof of Theorem 1.3 to show that all terms in (2.39) are well defined, i.e., that $u \in L^q(\Omega)$ yields $\nabla u \in L^q(\Omega)$ under the assumptions given on f, k and g.

(2) The proof follows the same lines as before. In this case $u \in W^{2,q}(\Omega)$ is a (classical) strong L^q-solution. □

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