Description of the Optimal Solution Set of the Linear Programming Problem and the Dimension Formula

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ABSTRACT

We give a definition of the *normal form* of an optimal solution of a linear programming problem and propose an algorithm to reduce the optimal solution to its normal form. The number of steps required to execute the proposed algorithm is slightly higher than in the standard simplex method. However, the algorithm enables us to describe the entire optimal solution set and find its dimension. One important particular case of this problem is the description and determination of the dimension of the set of points belonging to a convex polyhedron specified by linear constraints.

1. INTRODUCTION

In this paper we answer the following questions. Assume a linear programming problem is solved by the simplex algorithm. How to describe, using this solution, the entire optimal solution set, and how to find its dimension? These problems are solved as follows. We define the normal form of an optimal basic solution and propose to consider the LPP solved when its solution is in the normal form. In order to reduce an arbitrary optimal basic solution to its normal form we need to carry out some additional computations in the simplex method. We call these additional computations the *N-algorithm*.

One important particular case of this problem is the description of the set of points belonging to a convex polyhedron specified by linear constraints. First we solve the problem in this special case (Section 2), then we show that the general case can be reduced to this specific case (Section 3).

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The normal form of an optimal solution allows one to describe the entire set of optimal solutions and derive the formula for the dimension of this set in terms of the parameters of the normal form.

In Section 4, the optimal solution sets of the primal and dual LPPs are treated simultaneously.

The problems that are studied in the present paper have been considered by other authors [2, 3, 5]. However, as it seems to me, the approach adopted in this paper is better directed toward applications: we propose a numerical algorithm to describe the optimal solution set of the LPP and find the dimension of this set. On the other hand, from the purely theoretical point of view, our approach allows one to simplify the proofs of some known theorems and derive new ones (see Sections 3 and 4).

2. NONNEGATIVE SOLUTIONS OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

We consider a problem of finding nonnegative solutions of a homogeneous system of linear equations that of the form

$$x_d = Ax_c, \qquad x_d, x_c \geqslant 0, \tag{1}$$

where

$$x_c = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{pmatrix}, \qquad x_d = \begin{pmatrix} x_{l+1} \\ x_{l+2} \\ \vdots \\ x_{l+k} \end{pmatrix}$$

are columns of nonbasic and basic variables respectively and A is a $k \times l$ matrix.

It is easy to see that an arbitrary homogeneous system can be reduced to the form (1) where k is the number of independent linear equations.

Definition 1. We will say that the system (1) is *reduced to the normal* *form* if the matrix A is block upper triangular, i.e.,

$$A = \begin{pmatrix} b_1 & * & \cdots & * & * & * & \cdots & * \\ 0 & b_2 & \cdots & * & * & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_p & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & O & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & c_q & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & c_1 \end{pmatrix}, \qquad (2)$$

where the blocks b_i, $i = 1, 2, \ldots, p$, on the "main diagonal" are strictly positive columns, the blocks c_j, $j = 1, 2, \ldots, q$, are strictly negative rows, and O is the zero $r \times s$ matrix (including the cases $r = 0$, $s \neq 0$ and $r \neq 0$, $s = 0$). All blocks below the "main diagonal" are zero.

REMARK 1. In the cases $r = 0$, $s \neq 0$ and $r \neq 0$, $s = 0$ the matrix O becomes the empty set and the "main diagonal" has a gap. For example,

$$A = \begin{pmatrix} b_1 & A_1 & A_2 \\ 0 & 0 & c_1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} b_1 & A_1 \\ 0 & A_2 \\ 0 & c_1 \end{pmatrix}.$$

Denote the set of solutions of the system (1) by W.

LEMMA 1. *If the matrix A is of the form* (2), *then*

$$\dim W = p + s. \qquad (3)$$

Proof. We will consider the equations of the system (1) consecutively, going from the bottom to the top. The last equation is

$$x_{k+1} = -\nu_{l'+1}x_{l'+1} - \nu_{l'+2}x_{l'+2} - \cdots - \nu_l x_l, \qquad \nu_i > 0, \qquad (4)$$

where the ν_i's are coefficients of the row c_1. It follows from (4) that

$$x_{k+1} = 0,$$

$$x_{l'+1} = x_{l'+2} = \cdots = x_{l-1} = x_l = 0, \qquad l' < l. \qquad (5)$$

Then we consider the next to last equation of the system (1). Taking into account (5) and reasoning similarly, we conclude that some more variables are zero. Continuing the process, we will reach the block c_q and obtain

$$\begin{aligned} x_{k+l-q+1} &= \cdots = x_{k+l} = 0, \\ x_{p+s+1} = x_{p+s+2} &= \cdots = x_l = 0. \end{aligned} \tag{6}$$

Thus, it is sufficient to consider the solutions of the system (1) with the condition $q = 0$.

Assign to the variables $x_{p+1}, x_{p+2}, \ldots, x_{p+s}$ the arbitrary nonnegative values $x^0_{p+1}, x^0_{p+2}, \ldots, x^0_{p+s}$ respectively. Then the basic variables

$$x_{k+l-r-q+1}, \ldots, x_{k+l-q}$$

that correspond to the rows of the matrix O have to be equal to zero.

Next we consider the basic variables that correspond to the rows of the column b_p. The conditions of nonnegativity of these variables are

$$x_p \geqslant D^1_p, \qquad x_p \geqslant D^2_p, \ldots, \qquad x_p \geqslant D^{s_p}_p, \tag{7}$$

where s_p is the length of the column b_p. These inequalities hold because all the entries of the column b_p are positive. (A negative ith entry of b_p would give rise to the opposite inequality: $x_p \leqslant D^i_p$.) Denoting $\max_i D^i_p$ by D_p, we obtain

$$x_p \geqslant D_p. \tag{8}$$

Next, we fix a value x^0_p that satifies (8) and consider the basic variables that correspond to the rows of the column b_{p-1}. Similarly to the previous argument, we obtain

$$x_{p-1} \geqslant D_{p-1}, \tag{9}$$

and so on. As a result we get a set of nonnegative values

$$x_i = x^0_i, \qquad i = 1, 2, \ldots, p + s, \tag{10}$$

that defines a solution of the system (1), and every solution can be obtained in this way.

If these values are chosen such that the inequalities (8), (9), et. are strict, then, under small perturbations of the values x_i^0, $i = 1, 2, \dots p + s$, the inequalities still hold, which means that the dimension of the solution space of the system (1) is equal to $p + s$. ■

In fact, in the course of the proof of the lemma we presented a method of computing the consecutive values of variables. We state it as a separate lemma.

LEMMA 2. *If the matrix A has the form* (2), *then all the solutions of the system* (1) *can be found as follows*:

(I) $x_i = 0$, $i = p + s + 1, p + s + 2, \dots l$;

(II) *for* $i = p + 1, p + 2, \dots, p + s$ *the values of* x_i *can be chosen arbitrarily nonnegative*;

(III) *the values of the remaining variables can be found consecutively by the following procedure*: *if the values* $x = x_i^0$, $i = t + 1, t + 2, \dots, p, p + 1, \dots, p + s$, *are already fixed, we substitute these values into the equations that correspond to the column* b_t *and obtain the inequality*

$$x_t \geqslant D_t; \tag{11}$$

then let x_t *be equal to any positive value* x_t^0 *that satisfies* (11).

Now we present a method for reducing an arbitrary system (1) to the normal form. Our approach is based on phase 1 of the two-phase method. Omitting the details, we will recollect only the following fact. Given is a system of equations similar to (1) but whose constant terms are not necessarily zero; then the method of artificial basis reduces the system to a new form where either (I) all constants are nonnegative or (II) one of the equations has the form

$$x_j = -f_0 - f_{i_1} x_{i_1} - f_{i_2} x_{i_2} - \cdots - f_{i_t} x_{i_t}, \tag{12}$$

where $f_0, f_{i_\alpha} > 0$.

ALGORITHM *for reducing the system* (1) *to the normal form* (*the N-algorithm*). If the matrix $A = 0$, then $A = O$ and the system is already in the normal form. If $A \neq 0$, we consider an unknown on the right-hand side that has at least one nonzero coefficient. Let it be x_1. Set $x_1 = 1$. We obtain a nonhomogeneous system, to which we apply the method of artificial basis.

After this, if the system in the new form has nonnegative constants [case (I)], we renumber the unknowns on the left-hand side so that strictly positive constants precede the zeros. We denote the subcolumn formed by strictly positive constants by b_1. Then we exclude from further consideration all the rows passing through b_1 and the column of "constants."

If case (II) takes place, we renumber the unknowns so that strictly negative coefficients in the equation (12) follow the zeros, and this equation becomes the last. We denote the subrow formed by strictly negative coefficients by c_1. Then we exclude from further consideration tha last row and the columns that pass through the subrow c_1.

In both cases we arrive at a new system with a reduced matrix A. Continuing this process (which will stop after a finite number of steps, since each step reduces the matrix A), we reduce the system (1) to the normal form.

REMARK 2. The normal form of the sytem (1) is not unique. However, as follows from Lemma 1 and Lemma 5 (see below), the numbers $p + s$ and $q + r$ are invariants of the normal form.

Our presentation of the algorithm contains, in fact, a proof of the following lemma.

LEMMA 3. *Any system* (1) *can be reduced to the normal form by the N-algorithm. To carry out this reduction it suffices to solve* $p + q \leqslant \min(m, n)$ *LPPs, and each of these LPPs contains fewer basic variables and fewer nonbasic variables than the previous LPP.*

Summarizing the results of Lemmas 1, 2, and 3 we arrive at the following

THEOREM 1. *To solve the system* (1) *it suffices to reduce the system to the form where the matrix A is in the normal form* (2). *Then the solution set is described by Lemma* 2, *and its dimension is given by* (3).

3. THE NORMAL FORM OF AN OPTIMAL BASIC SOLUTION OF AN LPP AND THE SET OF OPTIMAL SOLUTIONS

From now on we will consider the LPP. One can understand the results of Section 2 as solving the dimension problem and describing the set of optimal solutions in the most degenerate case of the LPP. This case is the case when all constants terms of the equations and coefficients of the objective form vanish.

The main objective of Section 3 is to show that solving the problems mentioned above in the general case is reduced to solving these problems in this special case.

Assume that we have an optimal basic solution of an LPP. This means that we have a partition of the variables into basic and nonbasic variables such that the original system of equations and the objective form C take the following form:

$$
\boxed{\begin{array}{l}
x_{n+1} = 0 + \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1l}x_l \\
x_{n+2} = 0 + \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2l}x_l \\
\quad \vdots \\
x_{n+k} = 0 + \alpha_{k1}x_1 + \alpha_{k2}x_2 + \cdots + \alpha_{kl}x_l
\end{array}}
\begin{array}{l}
+\beta_{1,l+1}x_{l+1} + \cdots + \beta_{1n}x_n, \\
+\beta_{2,l+1}x_{l+1} + \cdots + \beta_{2n}x_n, \\
\quad \vdots \\
+\beta_{k,l+1}x_{l+1}
\end{array}
$$

$$
\begin{aligned}
&\qquad + \cdots + \beta_{kn}x_n, \\
x_{n+k+1} &= a_{k+1} + \alpha_{k+1,1}x_1 + \cdots + \alpha_{k+1,l}x_l + \beta_{k+1,l+1}x_{l+1} \\
&\qquad + \cdots + \beta_{k+1,n}x_n, \\
&\qquad \vdots \\
x_{n+m} &= a_m + \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{ml}x_l + \beta_{m,l+1}x_{l+1} \\
&\qquad + \cdots + \beta_{mn}x_n, \\
&\qquad C = c_0 + 0x_1 + \cdots + 0x_l + c_{l+1}x_{l+1} + \cdots + c_nx_n,
\end{aligned} \tag{13}
$$

where

$$a_i > 0, \; c_j < 0, \quad k+1 \leqslant i \leqslant m, \; l+1 \leqslant j \leqslant n, \quad x_t \geqslant 0, \quad 1 \leqslant t \leqslant n+m.$$

DEFINITION 2. The set of restrictions with respect to the nonnegative variables $x_1, \ldots, x_l, x_{n+1}, \ldots, x_{n+k}$ is called the *zero system of the given basic optimal solution* if it is obtained from the system (13) by the following procedure:

(I) all equations containing nonzero constants are left out;

(II) all nonbasic variables that have nonzero coefficients in the objective function are set to zero.

In other words, the new system is the system enclosed in the rectangle in

(13) with nonnegativity conditions for the new variables:

$$\begin{aligned}
x_{n+1} &= \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1l}x_l,\\
x_{n+2} &= \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2l}x_l,\\
&\ \vdots\\
x_{n+k} &= \alpha_{k1}x_1 + \alpha_{k2} + \cdots + \alpha_{kl}x_l,\\
x_i &\geqslant 0, \qquad i = 1, 2, \ldots, l, n+1, n+2, \ldots, n+k. \qquad (14)
\end{aligned}$$

REMARK 3. If $k = 0$, then the zero system consists of the inequalities

$$x_1, x_2, \ldots, x_l \geqslant 0.$$

If $l = 0$, then the zero system reduces to the equations

$$x_{n+1} = x_{n+2} = \cdots = x_{n+k} = 0.$$

DEFINITION 3. We will say that a basic solution *has normal form* if its zero system is in normal form.

Applying the N-algorithm to the zero system of an optimal solution and then expressing the new basic variables in terms of the nonbasic ones for the whole basic solution, we obtain an optimal basic solution in the normal form. All the numerical values of the variables and the coefficients of the linear form, obviously, remain unchanged because all the new basic variables that appear in the course of computation have constants equal to zero. We will also call these computations the *N-algorithm.*

Let V be the set of optimal solutions of an LPP, and W be the solution set of the zero system of one of its basic optimal solutions. One can describe the set V in terms of the set W.

Consider the following homogeneous system of linear equations

$$\begin{aligned}
x_{n+1} &= \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \beta_{1n}x_n,\\
&\ \vdots\\
x_{n+m} &= \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \beta_{mn}x_n, \qquad (15)
\end{aligned}$$

which is obtained from (13) by setting the values of the constant terms to zero.

DEFINITION 4. Let an optimal basic solution of an LPP be given, and let $x \in W$ be a solution of its zero system. Denote by $\tilde{x}$ the solution of the system (15) that is obtained by substitution of the values of $x_1, x_2, \ldots, x_l$ of the given solution x of the zero system and the values $x_{l+1} = x_{l+2} = \cdots = x_n = 0$ into the system (15). We shall call this solution $\tilde{x}$ the *transfer of the optimal basic solution*.

LEMMA 3. *Let a be an optimal basic solution of an LPP, and $\tilde{x}$ be its transfer. Then there exists a constant $\lambda(x) > 0$ that depends only on x such that the expression*

$$a + \lambda \tilde{x} \tag{16}$$

is an optimal solution if and only if $0 \leqslant \lambda \leqslant \lambda(x)$.

Conversely, any optimal solution of the given LPP can be obtained this way.

Proof. Let b be an arbitrary optimal solution. Then $b - a$ is a solution of the homogeneous system (15). Since for this solution the inequalities $x_{n+1}, x_{n+2}, \ldots, x_{n+m} \geqslant 0$ hold, the values of $x_1, x_2, \ldots, x_l$ define a solution of the zero system. Therefore, $b = a + \tilde{x}$, i.e., the vector b is as in (16).

Conversely, if x is fixed, then for sufficiently small λ the expression (16) is an optimal solution. The value of λ can be increased until one of the variables $x_{n+k+1}, x_{n+k+2}, \ldots, x_{n+m}$ becomes negative. This defines the value of $\lambda(x)$. If we increase λ further this variable will remain negative. This concludes the proof. ■

From Lemma 3 one can obtain the following corollaries:

COROLLARY 1. *The dimension of the set of optimal solutions of an LPP is equal to the dimension of the solution set of the zero system of any of the optimal basic solutions of this LPP, i.e.,*

$$\dim V = \dim W. \tag{17}$$

The next statement follows from Corollary 1 and (3).

COROLLARY 2. *The dimension of the optimal solution set of the LPP can be calculated by the formula*

$$\dim V = p + s, \tag{18}$$

where p and s are the parameters of a normal form of any optimal basic solution.

It follows from Corollary 2 and (18) that

COROLLARY 3. *An optimal solution of an LPP is unique if and only if it is a basic solution and*

$$p = s = 0$$

in its normal form.

Corollary 3 yields

COROLLARY 4 (O. L. Mangasarian [3]). *An optimal solution of an LPP is unique if and only if it remains a solution to all LPPs obtained from the given LPP by arbitrary but sufficiently small perturbations of the objective form C.*

Indeed, for any optimal solution of the new LPP, all basic variables of the optimal basic solution of the given LPP have to be zero, again because of the negativity of the coefficients c_i for $x_{l+1}, \dots, x_n$ and because $p = 0$, $s = 0$ for $x_1, \dots, x_l$. In other words, for any other solution the value of the form C will be smaller or some variables will be negative. So the optimal solution of the given LPP remains the optimal solution of the new LPP. (Moreover it remains the unique optimal solution.)

The evident generalization of this corollary is

COROLLARY 5. *A set V of optimal solutions of an LPP is the whole optimal solution set if and only if the optimal solution sets of all LPPs obtained from the given LPP by arbitrary but sufficient small perturbations of the objective form C are subsets of the given set V.*

From the definition of the N-algorithm, Lemma 3, and Corollary 2 the next statement follows immediately.

THEOREM 2. *The N-algorithm maps any optimal basic solution into an optimal basic solution that has the normal form and changes neither values of variables nor values of the coefficients of the linear form. Then the solution set is described by Lemma 3 and its dimension is given by (18).*

4. THE TOTAL DIMENSION OF THE OPTIMAL SOLUTION SET FOR PRIMAL AND DUAL LPP'S

Consider the dual problem of linear programming. As is well known (see [1]), any optimal basic solution of the primal problem corresponds to an optimal basic solution of the dual problem such that the augumented matrix expressing the basic variables in terms of nonbasic ones for the primal solution is the negaitve transpose of the other matrix. A consequence of this fact is the following statement.

LEMMA 4. *The zero system of any optimal basic solution of the dual LPP is the dual system of restrictions for the zero system of the corresponding solution of the primal LPP.*

Recall that the *dual system of restrictions* for the system

$$\begin{aligned} x_{t+1} &= -a_{11}x_1 - a_{12}x_2 - \cdots - a_{1t}x_t, \\ x_{t+2} &= -a_{21}x_1 - a_{22}x_2 - \cdots - a_{2t}x_t, \\ &\vdots \\ x_{t+s} &= -a_{s1}x_1 - a_{s2}x_2 - \cdots - a_{st}x_t, \\ x_i &\geqslant 0, \qquad i = 1, 2, \ldots, s+t, \end{aligned} \tag{19}$$

is the system

$$\begin{aligned} y_{s+1} &= a_{11}y_1 + a_{21}y_2 + \cdots + a_{s1}y_s, \\ y_{s+2} &= a_{12}y_1 + a_{22}y_2 + \cdots + a_{s2}y_s, \\ &\vdots \\ y_{s+t} &= a_{1t}y_1 + a_{2t}y_2 + \cdots + a_{st}y_s, \\ y_i &\geqslant 0, \qquad i = 1, 2, \ldots, t+s. \end{aligned} \tag{20}$$

If the matrix A is as in (2), the next statement follows immediately.

LEMMA 5. *If a system of homogeneous equations is reduced to the normal form with the parameters p, q, r, s, then the dual system is simultaneously reduced to the normal form with the parameters*

$$\tilde{p} = q, \qquad \tilde{q} = p, \qquad \tilde{r} = s, \qquad \tilde{s} = r.$$

REMARK 4. From the last lemma one can immediately derive the theorems of alternatives by Ville [6] and Motzkin [4] and the theorem of nonrigidity by Tucker [7]. The usual proofs of these results are more complicated.

Denote the set of optimal solutions of a primal LPP by V_1 and of the dual LPP by V_2. Summarizing the contents of Lemmas 4 and 5 and Theorem 2, we obtain the following statement.

THEOREM 3. *If an optimal basic solution is reduced to the normal form, then*

$$\dim V_1 = p + s, \tag{21}$$

$$\dim V_2 = q + r, \tag{22}$$

$$\dim V_1 + \dim V_2 = p + q + r + s, \tag{23}$$

where p, q, r, s are the parameters of the normal form.

Next, we derive some consequences from the formulas (21), (22), and (23). We shall say that the set of optimal solutions of a primal (dual) LPP is *saturated* if $\dim V_1 = l$ ($\dim V_2 = k$). Recall that we denote the number of nonbasic variables in an optimal solution of the primal LPP that are equal to zero by k; the number of the coefficients of the linear form, expressed in terms of nonbasic variables, that are equal to zero, is denoted by l.

COROLLARY 1. *One of the two mutually dual LPPs has a unique optimal solution if and only if the set of all optimal solutions of the other problem is saturated.*

Proof. Let $\dim V_1 = 0$, i.e., $p + s = 0$, $p = 0$, $s = 0$. Since the matrix A has the form (2), we have $q + r = k$, or, in other words, $\dim V_2 = k$. The converse statement can be proved similarly. ■

COROLLARY 2 (O. L. Mangasarian [3]). *The optimal solutions of the primal and the dual LPP are both unique if and only if*

$$k = 0, \qquad l = 0,$$

i.e., if and only if none of the basic variables in an optimal solution vanishes and none of the coefficients of the linear form expressed in terms of nonbasic variables vanishes.

Proof. This follows from the proof of the previous corollary, because if each of the two solution is unique, then each of them is also saturated, which means that $k = 0$, $l = 0$. ■

COROLLARY 3. *The sum of dimensions of the sets of optimal solutions of the primal and dual LPPs is equal to one,*

$$\dim V_1 + \dim V_2 = 1, \tag{24}$$

if and only if the matrix of the zero system is one strictly negative row or one strictly positive column,

$$(-\alpha_1, -\alpha_2, \ldots, -\alpha_l) \quad \textit{or} \quad \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}, \qquad \alpha_i, \beta_j > 0$$

(the cases $k = 0$, $l = 0$ are included). In the remaining cases the inequality

$$\dim V_1 + \dim V_2 \geqslant 2 \tag{25}$$

holds.

Proof. If (24) holds, then either $\dim V_1 = 0$ or $\dim V_2 = 0$. Assuming that $\dim V_2 = 0$ and that the matrix A is in normal form, we obtain $q + r = 0$ and hence $p + s = 1$. The last relations imply $p = 1$, $s = 0$ or $p = 0$, $s = 1$. The relations $p = 1$, $s = 0$ mean that the matrix A is one strictly positive column, while the relations $p = 0$, $s = 1$ can be interpreted as $k = 0$ in the strictly positive column. The case $\dim V_1 = 0$ is treated similarly.

If neither the optimal solutions are unique nor (24) holds, then (25) takes place. This concludes the proof. ■

The following example shows that the inequality (25) cannot be improved for arbitrary k, l:

$$A = \begin{pmatrix} b_1 & \tilde{A} \\ 0 & c_1 \end{pmatrix}.$$

Here A is the matrix of the zero system with $p = q = 1$, $r = s = 0$.

PROPOSITION 2. *The inequality*

$$\dim V_1 + \dim V_2 \leqslant k + l - \operatorname{rank} A, \tag{26}$$

where A is the matrix of a zero system, holds.

Proof. Let t_p be the total number of rows in the columns $b_1, b_2, \ldots, b_p$, and t_q be the total number of columns in the rows $c_1, c_2, \ldots, c_q$. The form of the matrix A in (2) implies

$$k = t_p + r + q, \qquad l = p + s + t_q.$$

We now prove the inequality

$$\operatorname{rank} A \leqslant t_p + t_q.$$

Indeed, it becomes obvious if we express the matrix A as $A_1 + A_2$, where A_1 is the matrix consisting of the rows passing through the columns b_1, $b_2, \dots, b_p$ and A_2 is the matrix consisting of the remaining rows. The inequality

$$\operatorname{rank} A \leqslant \operatorname{rank} A_1 + \operatorname{rank} A_2 \leqslant t_p + t_q$$

holds. Taking into account the relations we just derived, we can deduce from (23) that

$$\begin{aligned} \dim V_1 + \dim V_2 &= p + q + r + s \\ &= (t_p + r + q) + (p + s + t_q) - (t_p + t_q) \\ &\leqslant k + l - \operatorname{rank} A, \end{aligned}$$

which is exactly the inequality (26). ■

REFERENCES

1 G. Dantzig, *Linear Programming and Extensions*, Princeton. U. P., Princeton, N.J., 1963.
2 V. Eckhardt, Theorems on the dimension of convex sets, *Linear Algebra Appl.* 12:63–76 (1975).
3 O. L. Mangasarian, Uniqueness of solutions in linear programming, *Linear Algebra Appl.* 25:151–162 (1979).
4 T. S. Motzkin, Beitrage zur Theorie der linearen Ungleichungen, Inagural Dissertation, Jerusalem, 1936.
5 K. G. Murty, *Linear Programming*, Wiley, 1983.
6 J. Ville, Sur la théorie générale des jeux où intervient l'habilité des joueurs, in *Traité du Calcul des Probabilities et des Applications* (E. Borel et al., Eds.), Paris, 1938, Vol. IV, Part 2, pp. 105–113.
7 A. W. Tucker, Dual systems of homogeneous linear relations, in *Linear Inequalities and Related Systems* (U. W. Kuhn and A. W. Tucker, Eds.), Ann. of Math. Stud. 38, Princeton U. P., Princeton, N.J., 1956, pp. 3–18.

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