ELSEVIER

BULLETIN

www.elsevier.com/locate/bulsci

Construction of the fundamental solution of disturbed parabolic equation

Victor Bondarenko

The National Technical University of Ukraine of 'Kiev Polytechnic Institute', The Institute of Applied System Analysis, prospect Peremogy, 37; building 35, IPSA, 03056 Kiev, Ukraine

Received 23 July 2002; accepted 20 October 2002

Abstract

In present paper the parabolic equation solution is built. The construction is reduced to iterative procedure. And convergence of the latter is proven.

Résumé

Dans un travail presenté on a construit la solution d'equation parabolique. Cette construction est reduite a procédé d'iteration. On demonstre la convergence de le dernier.

MSC: 58J35; 58J65

Keywords: Riemannian manifold; Heat kernel; Curvature tensor

Mots-clés : Variété Riemannienne ; Noyau de chaleur ; Tenseur de courbure

1. Introduction

In presented work the problem of construction of fundamental solution $\mathrm{e}^{t(L+L_1)}$ of equation

$$\frac{du}{dt} = (L + L_1)u$$

E-mail addresses: victorb@region.scourt.gov.ua, victorb@mmsa.ntu-kpi.kiev.ua (V. Bondarenko).

0007-4497/03/$ – see front matter
doi:10.1016/S0007-4497(03)00006-X

is considered, where L and L_1 are elliptic operators, and the properties of evolution operator e^{tL} of a non-disturbed equation

$$\frac{du}{dt} = Lu$$

are assumed to be known.

If the coefficients L and L_1 are constant, then

$$e^{t(L+L_1)} = e^{tL}e^{tL_1}. \tag{1}$$

For the non-constant coefficients, satisfying some set of restrictions, we will prove that the right part of (1) under small t is quite good approximation for the evolution operator in the left part, that is,

$$e^{t(L+L_1)} = e^{tL}e^{tL_1} + A(t), \quad A(0) = 0,$$

and for the construction of family of operators $A(t)$ the iteration procedure will be proposed. In this work two examples of parabolic equations are considered:

1. L is the elliptic operator with constant coefficients,
$$Lu = \frac{1}{2}a^{jk}\frac{\partial^2 u}{\partial x^j \partial x^k},$$
the perturbation L_1 has a form
$$L_1 u = \frac{1}{2}b^{jk}(x)\frac{\partial^2 u}{\partial x^j \partial x^k}.$$
2. $L = \frac{1}{2}\Delta$, where Δ is Laplace–Beltrami operator on complete simply connected Riemann manifold M of non-positive curvature with dimension n, and metrics tensor $g_{jk}(x)$, distance ρ and volume σ, that is
$$Lu(x) = \frac{1}{2}\operatorname{div}\operatorname{grad} u(x),$$
and perturbation has a form
$$L_1 u(x) = \frac{1}{2}\operatorname{div} b(x)\operatorname{grad} u(x).$$

In the both examples

$$\left(e^{tL} f\right)(x) = \int f(y) p_0(t, x, y)\, dy,$$

where p_0 is a fundamental solution of non-disturbed equation. The aim of presented paper is to propose and to ground well a constructing of the function p, obtained from the equality

$$\left(e^{t(L+L_1)} f\right)(x) = \int f(y) p(t, x, y)\, dy.$$

Above-mentioned function $p(t, x, y)$ will be searched in the form

$$p(t, x, y) = m(t, x, y) + \int_0^t d\tau \int m(t-\tau, x, z) r(\tau, z, y)\, dz, \tag{2}$$

where the initial approximation

$$m(t,x,y)=\int p(t,z,y)p_1(t,x,z)\,\sigma(dz), \tag{3}$$

the function p_1 is an approximation of the kernel of integral operator e^{tL_1}, and $r(t,x,y)$ is a function being subject to be obtained.

A procedure of constructing of fundamental solution is analogous to the same procedure in parametrix method, namely: Eq. (2) is reduced to Volterra's integral equation for the function r

$$r(t,x,y)=M(t,x,y)+\int_0^t d\tau\int M(t-\tau,x,z)r(\tau,z,y)\,dz, \tag{4}$$

where an error $M(t,x,y)=(L+L_1)m-\frac{\partial m}{\partial t}$, and solution r of Eq. (3) has a structure $r(t,x,y)=\sum_{n=0}^{\infty}r_n(t,x,y)$, and each iteration is calculated with respect to recurrent formula:

$$r_0(t,x,y)=M(t,x,y),$$
$$r_{n+1}(t,x,y)=\int_0^t d\tau\int M(t-\tau,x,z)r_n(\tau,z,y)\,dz.$$

Convergence of the last integral and the series $\sum r_n$ is determined by properties of the error $M(t,x,y)$: it must have an integrable with respect to t singularity. Above-mentioned property holds under some restrictions on coefficients of the operators L and L_1.

Since the initial approximation $m(t,x,y)$ (and hence, the error $M(t,x,y)$ as well) is defined as an integral, for transforming of the error and its estimating the integration by the parts will be applied:

$$\int \operatorname{div}V(z)\,\mu(dz)=-\int\big(\Lambda(z),V(z)\big)\,\mu(dz),$$

where the role of the measure μ the relation $p_1(t,x,z)\sigma(dz)$ has:

$$\int \operatorname{div}V(z)p_1(t,x,z)\,\sigma(dz)=-\int\big(\Lambda(z),V(z)\big)p_1(t,x,z)\,\sigma(dz). \tag{5}$$

Here a logarithmic derivative is $\Lambda(t,x,z)=\operatorname{grad}_\varepsilon\ln p_1(t,x,z)$.

1.1. Perturbation of the constant operator

Let's demonstrate the described method in particular case, considering a parabolic equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\operatorname{tr}\big(A+B(x)\big)u'',\quad x\in R^n, \tag{6}$$

where A is a constant operator in R^n, and a positive operator $B(x)$ satisfies the conditions:

(1) a positive operator K_0 exists, such that

$$K(x) = A^{-1/2}B(x)A^{-1/2} \leqslant K_0 \leqslant \delta I, \quad \delta < 1;$$

(2) $\|B(x)B^{-1}(y)\| < \text{Const}$;

(3) the fist two derivatives of the operator $B(x)$ are bounded:

$$\|B'(x)h\| \leqslant c_1\|h\|,$$
$$\|B''(x)kh\| \leqslant c_2\|k\| \cdot \|h\|.$$

From condition 2 the boundedness of the ratio $\frac{\det B(x)}{\det B(y)}$ follows.

The solution of non-disturbed equation is

$$p_0(t,x,y) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det A}} \exp\left\{-\frac{(A^{-1}(x-y), x-y)}{2t}\right\}.$$

Put

$$p_1(t,x,y) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det B(y)}} \exp\left\{-\frac{(B^{-1}(x)(x-y), x-y)}{2t}\right\},$$

then zero approximation is

$$m(t,x,y) = \int_{R^n} p(t,z,y)\frac{1}{(2\pi t)^{n/2}\sqrt{\det B(z)}} \exp\left\{-\frac{(B^{-1}(x)(x-z), x-z)}{2t}\right\} dz.$$

From conditions 1 and 2 an estimate follows:

$$\begin{aligned} m(t,x,y) &< \frac{c}{\sqrt{\det(I+K(x))}} \\ &\quad \times \exp\left\{\frac{1}{2t}\big(A^{-1/2}K(x)A^{-1/2}(x-y), x-y\big)\right\} p_0(t,x,y) \\ &< c\exp\left\{\frac{1}{2t}\big(A^{-1/2}K_0A^{-1/2}(x-y), x-y\big)\right\} p_0(t,x,y). \end{aligned}$$

Error

$$M(t,x,y) = \frac{1}{2}\operatorname{tr}\big(A+B(x)\big)m''_{xx} - \frac{\partial m}{\partial t} = I_1 + I_2,$$

where

$$I_1 = \int_{R^n} \left(\frac{1}{2}\operatorname{tr} B(x)p_1''(t,x,z) - \frac{\partial p_1}{\partial t}(t,x,z)\right) p_0(t,z,y)\,dz,$$

$$I_2 = \int_{R^n} \left(\frac{1}{2}\operatorname{tr} Ap_1''(t,x,z) \cdot p(t,z,y) - \frac{\partial p_0}{\partial t}(t,z,y)p_1(t,x,z)\right) dz,$$

and differentiation is done with respect to variable x. For the next transformations let's reduce I_2 to the form

$$I_2 = \frac{1}{2}\int_{R^n} \big(\operatorname{tr} Ap_1''(t,x,z)p_0(t,z,y) - \operatorname{tr} Ap''_{0xx}(t,z,y)p_1(t,x,z)\big)\,dz,$$

and integrate the second item twice by the parts, see (5):

$$-\frac{1}{2}\int\limits_{R^n} \operatorname{tr}\big(Ap'_{0x}(t,z,y)\big)'_x p_1(t,x,z)\,dz$$
$$= -\frac{1}{2}\int\limits_{R^n} \operatorname{div}_x\big(Ap'_{0x}(t,z,y)\big)p_1(t,x,z)\,dz$$
$$= \frac{1}{2}\int\limits_{R^n} \big(p'_{0x}(t,z,y), A\Lambda(t,x,z)\big)p_1(t,x,z)\,dz$$
$$= -\frac{1}{2}\int\limits_{R^n} \big[\big(A\Lambda(t,x,z), \Lambda(t,x,z)\big) + \operatorname{div}_z A\Lambda(t,x,z)\big]p_0(t,z,y)p_1(t,x,z)\,dz.$$

Lemma 1. *A logarithmic derivative $\Lambda(t,x,z)$ of the measure $p_1(t,x,z)\,dz$ is defined by the equality*:

$$\big(\Lambda(t,x,z),h\big) = -\frac{1}{2}\operatorname{tr} B'(z)hB^{-1}(z) + \frac{1}{t}\big(B^{-1}(x)(x-z),h\big).$$

Proof. From definition

$$\big(\Lambda(t,x,z),h\big) = -\frac{1}{2}d_h \ln\det B(z) + \frac{1}{t}\big(B^{-1}(x)(x-z),h\big),$$

where d_h is a differential of function along vector h with respect to variable z.

Differentiating of the equality implies

$$\ln\det B(z) = \operatorname{tr}\int\limits_0^1 \big(B(z)-I\big)\big(I+\tau(B(z)-I)\big)^{-1}\,d\tau,$$

and we get

$$d_h \ln\det B(z) = \operatorname{tr} B'(z)h\big(B(z)-I\big)^{-1}\int\limits_0^1 \big(B(z)-I\big)\big(I+\tau(B(z)-I)\big)^{-2}\,d\tau,$$

thus, the statement of the lemma follows from the fact that the relation under integral sign equals

$$-\frac{d}{d\tau}\big(I+\tau(B(z)-I)\big)^{-1}. \qquad \square$$

Corollary.

$$I_2 = \frac{1}{2}\int\limits_{R^n}\left[\frac{\operatorname{tr} Ap''_1(t,x,z)}{p_1(t,x,z)} - \big(A\Lambda(t,x,z), \Lambda(t,x,z)\big) - \operatorname{div}_x A\Lambda(t,x,z)\right]$$
$$\times\, p_0(t,z,y)p_1(t,x,z)\,dz.$$

Lemma 2. *Error $M(t,x,y)$ satisfies the following estimate*:

$$\left|M(t,x,y)\right| \leqslant \frac{c}{\sqrt{t}} \int\limits_{R^n} \left(1 + \frac{\|x-z\|^4}{t^2}\right) p_0(t,z,y) p_1(t,x,z)\,dz.$$

Proof. Differentiating p_1, we reduce I_1 to the form ($\{e_k\}$ is an orthogonal basis in R^n):

$$\begin{aligned}
I_1 = \frac{1}{2}\int\limits_{R^n} \sum_k \bigg[&\frac{1}{4t^2}\big((B^{-1})' e_k(x-z), x-z\big)\big((B^{-1})' B e_k(x-z), x-z\big) \\
&+ \frac{1}{2t^2}\big((B^{-1})' e_k(x-z), x-z\big)(x-z, e_k) \\
&+ \frac{1}{2t^2}\big((B^{-1})' B e_k(x-z), x-z\big)\big(B^{-1}(x-z), e_k\big) \\
&- \frac{1}{2t}\big((B^{-1})'' e_k B e_k(x-z), x-z\big) - \frac{1}{t}\big((B^{-1})' e_k(x-z), B e_k\big) \\
&- \frac{1}{t}\big((B^{-1})' B e_k(x-z), e_k\big)\bigg] p_0(t,z,y) p_1(t,x,z)\,dz.
\end{aligned}$$

From lemma's conditions the relation in brackets satisfies the estimate:

$$c\left(\frac{\|x-z\|^4 + \|x-z\|^3}{t^2} + \frac{\|x-z\|}{t}\right) < \frac{c}{\sqrt{t}}\left(1 + \frac{\|x-z\|^4}{t^2}\right),$$

that is,

$$I_1 < \frac{c}{\sqrt{t}} \int\limits_{R^n} \left(1 + \frac{\|x-z\|^4}{t^2}\right) p_0(t,z,y) p_1(t,x,z)\,dz.$$

For estimating I_2 (see corollary to Lemma 1) we note that

$$\begin{aligned}
\operatorname{div}_z A\Lambda(t,x,z) &= \operatorname{tr} A\Lambda_z'(t,x,z) \\
&= \operatorname{tr} A\bigg[-\frac{1}{2}\operatorname{tr} B''(z)(\cdot)B^{-1}(z) \\
&\qquad + \frac{1}{2}\operatorname{tr} B'(z)(\cdot)B^{-1}(z)B'(z)(\cdot)B^{-1}(z) - \frac{1}{t}B^{-1}(x)\bigg] \\
&= -\frac{1}{t}\operatorname{tr} AB^{-1}(x) + \phi(z),
\end{aligned}$$

where the item ϕ is bounded in lemma's statement. The relation under integral sign in $I_2(A\Lambda(t,x,z), \Lambda(t,x,z))$ allows a presentation:

$$\big(A\Lambda(t,x,z), \Lambda(t,x,z)\big) = \frac{1}{t^2}\big(B^{-1}(x)AB^{-1}(x)(x-z), x-z\big) + \psi(t,x,z),$$

where

$$\left|\psi(t,x,z)\right| < \frac{c}{\sqrt{t}} \int\limits_{R^n} \left(1 + \frac{|x-z|}{\sqrt{t}}\right).$$

Calculating $Ap_1''(t,x,z)$, we see that under integral sign in I_2 the relation satisfying desired estimate, remains. □

Lemma 3. *For error $M(t,x,y)$ the following estimate holds*:

$$\left|M(t,x,y)\right| < \frac{c}{\sqrt{t}} \exp\left\{\frac{1}{2t}\left(A^{-1/2}K_0A^{-1/2}(x-y), x-y\right)\right\} p_0(t,x,y).$$

Proof. Let's use the equality

$$p_0(t,x,y) = \phi(t,z,x,y)p(t,x,y),$$

where

$$\begin{aligned}\phi(t,z,x,y) &= \exp\left\{\frac{1}{t}\left(A^{-1}(y-x), z-x\right)\right\}\exp\left\{-\frac{1}{2t}\left(A^{-1}(z-x), z-x\right)\right\}\\ &= \phi_1(t,x,y,z)\phi_2(t,x,z).\end{aligned}$$

From Lemma 2 the inequality follows:

$$\begin{aligned}\left|M(t,x,y)\right| &< \frac{c}{\sqrt{t}} p_0(t,x,y) \int_{R^n} \left(1 + \frac{\|x-z\|^4}{t^2}\right)\\ &\quad \times \phi_2(t,x,z)\phi_1(t,x,y,z)p_1(t,x,z)\,dz,\end{aligned}$$

and from boundedness of the product $\frac{\|x-z\|^4}{t^2}\phi_2(t,x,z)$ the estimate for error gets the form

$$\left|M(t,x,y)\right| < \frac{c}{\sqrt{t}} p_0(t,x,y) \int_{R^n} \exp\left\{\frac{1}{t}\left(A^{-1}(z-x), y-x\right)\right\} p_1(t,x,z)\,dz.$$

Substituting $z \to u$, $u = \frac{1}{\sqrt{t}} B^{-1/2}(x)(z-x)$, $z = x + \sqrt{t}\, B^{1/2}(x)u$, we get an inequality

$$\begin{aligned}\left|M(t,x,y)\right| &< \frac{c}{\sqrt{t}} p_0(t,x,y)\\ &\quad \times \int_{R^n} \exp\left\{\left(u, \frac{1}{\sqrt{t}} B^{1/2}(x)A^{-1}(y-x)\right)\right\}\sqrt{\frac{\det B(x)}{\det B(z)}}\,\mu(du),\end{aligned}$$

where μ is a canonical Gaussian measure in R^n.

Calculation of the last integral (after estimating of the ratio of determinants) leads to an inequality

$$M(t,x,y) < \frac{c}{\sqrt{t}} p_0(t,x,y) \exp\left\{\frac{1}{2t}\left(A^{-1}B(x)A^{-1}(y-x), y-x\right)\right\},$$

and the statement of lemma follows. □

Theorem 1. *Fundamental solution $p(t,x,y)$ of disturbed equation* (6) *for $t \in (0,T]$ satisfies an inequality*

$$p(t,x,y) < c\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(x-y), x-y)}{2t}\right\} p_0(t,x,y).$$

Proof. Let us construct $p_0(t,x,y)$ as a solution of integral equation (2). Estimating an iteration $r_1(t,x,y)$ of Eq. (4):

$$\begin{aligned}\left|r_1(t,x,y)\right| &< \int_0^t d\tau \int_{R^n} \left|M(\tau,z,y)M(t-\tau,x,z)\right| dz \\ &< c^2 \int_0^t \frac{d\tau}{\sqrt{\tau(t-\tau)}} \int_{R^n} \exp\left\{ \frac{(A^{-1/2}K_0A^{-1/2}(z-y), z-y)}{2\tau} \right. \\ &\quad \left. + \frac{(A^{-1/2}K_0A^{-1/2}(z-x), z-x)}{2(t-\tau)} \right\} p_0(\tau,z,y)p_0(t-\tau,x,z)\,dz.\end{aligned}$$

We substitute

$$A^{-1/2}\left(z\sqrt{\frac{t}{\tau(t-\tau)}} - x\sqrt{\frac{\tau}{t(t-\tau)}} - y\sqrt{\frac{t-\tau}{t\tau}} \right) = u,$$

and hence relations

$$\begin{aligned}\frac{z-x}{\sqrt{t-\tau}} &= \sqrt{\frac{\tau}{t}}\,A^{1/2}u + \frac{\sqrt{t-\tau}}{t}(y-x), \\ \frac{z-y}{\sqrt{\tau}} &= \sqrt{\frac{t-\tau}{t}}\,A^{1/2}u + \frac{\sqrt{\tau}}{t}(x-y),\end{aligned} \tag{7}$$

transform a space integral in form

$$\begin{aligned}& p_0(t,x,y)\exp\left\{ \frac{(A^{-1/2}K_0A^{-1/2}(z-y), z-y)}{2t} \right\} \int_{R^n} \exp\left\{ \frac{(K_0u,u)}{2} \right\} \mu(du) \\ &\quad = \frac{1}{\sqrt{\det(I-K_0)}}\, p_0(t,x,y)\exp\left\{ \frac{(A^{-1/2}K_0A^{-1/2}(z-y), z-y)}{2t} \right\},\end{aligned}$$

and from this

$$\left|r_1(t,x,y)\right| < c^2 c_1 \pi\, p_0(t,x,y)\exp\left\{ \frac{(A^{-1/2}K_0A^{-1/2}(z-y), z-y)}{2t} \right\},$$

where $c_1 = 1/\sqrt{\det(I-K_0)}$.

It's easy to get by induction that the following estimate is true:

$$\begin{aligned}&\left|r_n(t,x,y)\right| \\ &\quad < \int_0^t d\tau \int_{R^n} \left|r_{n-1}(\tau,z,y)M(t-\tau,x,z)\right| dz \\ &\quad < c^{n+1}c_1^n \frac{\pi(n+1)/2}{\Gamma((n+1)/2)} \exp\left\{ \frac{(A^{-1/2}K_0A^{-1/2}(x-y), x-y)}{2t} \right\} p_0(t,x,y),\end{aligned}$$

c is a new constant, therefore

$$
\begin{aligned}
\left|r(t,x,y)\right| &< \sum_{n=0}^{\infty}\left|r_n(t,x,y)\right| \\
&< \frac{c}{\sqrt{t}}\,\mathrm{e}^{ct}\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(z-y),x-y)}{2t}\right\}p_0(t,x,y).
\end{aligned}
$$

Estimating an integral in (2) by means plugging in (6), we get

$$
q(t,x,y) < c\bigl(1+\sqrt{t}\bigr)\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(x-y),x-y)}{2t}\right\}p_0(t,x,y),
$$

and the statement of theorem follows. □

Notation 1. In this example the well-known explicit form of solution $p_0(t,x,y)$ of non-disturbed equation was used for:

1) calculating of the function ϕ in relation $p_0(t,z,y) = \phi(t,x,z,y)p_0(t,x,y)$;

2) solving and then estimating the integrals by means substitution (7).

Under disturbing the equation with variable coefficients (on manifold) an explicit form of $p_0(t,x,y)$ is unknown, but the estimate of the function ϕ will be obtained and the analogous substitution (7) will be presented.

2. Scalar perturbation of variable operator

Let $p_0(t,x,y)$ be a fundamental solution (heat kernel) of parabolic equation

$$
\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u,
$$

where Δ is Laplace–Beltrami operator on complete simply connected Riemann manifold of non-positive curvature M, $\dim M = n$. Denote as $\gamma(s)$ geodezical, parametrized by natural parameter (as a rule, $\gamma(0) = y$, $\gamma(\rho(x,y)) = x$), and put $e(x,y) = -\dot{\gamma}(x)$.

The equation on the Riemann manifold is studied in many papers, some of them are [5,6].

Assume, that curvature tensor satisfies the following conditions:

2.1. For arbitrary $x \in M$, $U, V \in T_xM$:

$$
\sum_k \left|(R(x)(U,e_k)V,\phi_k)\right| < c\sqrt{\mathrm{Ric}(x)(U,U)\,\mathrm{Ric}(V,V)},
$$

where

$$
\mathrm{Ric}(x)(U,U) = \sum_k \bigl(R(x)(U,e_k)V,e_k\bigr),
$$

$\{e_k\}$, $\{\phi_k\}$ are arbitrary orthogonal basises in T_xM, and the constant c doesn't depend on x.

2.2. Along any geodezical the scalar curvature $r(x) = \mathrm{tr}\,\mathrm{Ric}(x)$ decreases quite fast, that is: $\int_0^\infty s r(\gamma(s))\,ds < c$, where c doesn't depend on γ.

2.3. Co-variant derivative of the curvature tensor satisfies estimates

$$\begin{aligned}
&\big\|(\nabla_{X(s)}R)\big(\gamma(s)\big)\big(Y(s),\dot\gamma(x)\big)Z(s)\big\| \\
&\quad\leqslant f_1\big(\gamma(s)\big)\big\|X(s)\big\|\big\|Y(s)\big\|\big\|Z(s)\big\|; \\
&\big\|(\nabla_{U(s)}\nabla_{X(s)}R)\big(\gamma(s)\big)\big(Y(s),\dot\gamma(x)\big)Z(s)\big\| \\
&\quad\leqslant f_2\big(\gamma(s)\big)\big\|X(s)\big\|\big\|Y(s)\big\|\big\|Z(s)\big\|\big\|U(s)\big\|,
\end{aligned}$$

where f_1 and f_2 are such functions, that along any geodezical γ: $\int_0^\infty s^2 f_k(\gamma(s))\,ds < c$, c doesn't depend on γ.

Let us define along $\gamma(s)$ an operator D on $T_{\gamma(s)}M$

$$D\big(\gamma(s)\big)U = \nabla_U\rho\big(y,\gamma(s)\big)\dot\gamma(x), \quad \dot\gamma(x) = y,$$

and functions

$$\begin{aligned}
&a(x,y) = \operatorname{tr}(D(x) - I), \\
&q(t,x,y) = (2\pi t)^{n/2}\exp\left\{-\frac{\rho^2(x,y)}{2t}\right\}. \qquad (8)
\end{aligned}$$

As it was shown in [1–4], under satisfaction conditions 1–3, the following results hold:

1) heat kernel $p_0(t,x,y)$ satisfies two-sided estimate

$$\exp\{-\phi(x,y) - kt\} \leqslant \frac{p_0(t,x,y)}{q(t,x,y)} \leqslant 1,$$

where

$$\phi(x,y) = \frac{1}{2}\int_0^{\rho(x,y)}\big(\rho(x,y) - \tau\big)\operatorname{Ric}\big(\gamma(\tau)\big)\big(\dot\gamma(\tau),\dot\gamma(\tau)\big)\,d\tau,$$

k is some constant.

2) the relation holds:

$$\operatorname{grad}_x \ln p_0(t,x,y) = \frac{\rho(x,y)}{t}e(x,y) + w(t,x,y),$$

where $\|w(t,x,y)\| < c$, $x, y \in M$, $0 < t \leqslant T$.

3) the function $a(x,y)$ satisfies the estimate

$$0 \leqslant a(x,y) \leqslant \int_0^{\rho(x,y)} s\operatorname{Ric}\big(\gamma(s)\big)\big(\dot\gamma(s),\dot\gamma(s)\big)\,ds,$$

$\operatorname{grad}_x a(x,y) < c$, $|\Delta_x(x,y)| < c$.

As a consequence of the relation for logarithmic gradient (condition 2) we have

Lemma 4. *The following inequality is true*:

$$p_0(t,z,y) \leqslant p_0(t,x,y)\phi_1(t,x,y,z)\phi_2(t,x,z),$$

where

$$\phi_1 = \exp\bigl\{\rho(x,y)\rho(x,y)\bigl(e(x,y),e(x,z)\bigr)\bigr\},$$
$$\phi_2 = \exp\left\{-\frac{\rho^2(x,z)}{2t} + c\rho(x,z)\right\}.$$

Proof. Let $\sigma(s)$ be geodezicals, connecting z and x, $\sigma(0)=x$, $\sigma(\rho(x,z))=z$. Integrating equality (2), we have

$$p_0(t,z,y) = p_0(t,x,y)\left\{\frac{1}{t}\int_0^{\rho(x,z)} \rho\bigl(y,\sigma(s)\bigr)\bigl(e\bigl(\sigma(s),y\bigr),\dot\sigma(s)\bigr)\,ds + \int_0^{\rho(x,z)} \bigl(w\bigl(t,\sigma(s)\bigr),\dot\sigma(s)\bigr)\,ds\right\}.$$

Notice that

$$\rho\bigl(y,\sigma(s)\bigr)\bigl(e(\sigma(s),y),\dot\sigma(s)\bigr) = -\frac{1}{2}\frac{d}{ds}\rho^2\bigl(y,\sigma(s)\bigr),$$

and, thus, we have an inequality

$$p_0(t,z,y) \leqslant p_0(t,x,y)\exp\left\{\frac{\rho^2(x,y)-\rho^2(z,y)}{2t} + c\rho(x,z)\right\}.$$

We can make stronger, using cosines theorem for manifold of non-positive curvature $\rho^2(y,z) \geqslant \rho^2(x,y)+\rho^2(x,z)-2\rho(x,z)\rho(x,y)(e(x,y),e(x,z))$, and this leads to desired result. □

Let us consider a perturbation

$$L_1 u = \frac{1}{2}\operatorname{div} b(x)\operatorname{grad} u,$$

where a scalar function $b(x)$ satisfies conditions:

1) $0 < b_1 \leqslant b(x) \leqslant b_2 < 1$;

2) $\|\operatorname{grad} b(x)\| < c$, $\nabla_U \operatorname{grad} b(x) < c\|U\|$.

Put

$$p_1(t,x,y) = \bigl(2\pi t b(x)\bigr)^{-n/2}\exp\left\{\frac{\rho^2(x,z)}{2tb(x)}\right\},$$

and define function $m(t,x,y)$ via equality (3). Then error of disturbed equation be

$$M(t,x,y) = \frac{1}{2}\operatorname{div}\bigl(1+b(x)\bigr)\operatorname{grad} m(t,x,y) - \frac{\partial}{\partial t}m(t,x,y) = I_1 + I_2, \tag{9}$$

where

$$I_1 = \int_M \left(\frac{1}{2}\operatorname{div}_x b(x)\operatorname{grad}_x p_1(t,x,z) - \frac{\partial p_1(t,x,z)}{\partial t}\right)p_0,$$

and the item I_2 after applying of formula (5) and integrating by the parts has a form

$$I_2 = \int\limits_M \big(\Delta_x p_1(t,x,z) - \big\|\Lambda(t,x,z)\big\|^2 p_1(t,x,z)$$
$$- \operatorname{div}_z \Lambda(t,x,z) p_1(t,x,z)\big) p_0(t,z,y)\,\sigma(dz).$$

Lemma 5. *Functions $m(t,x,y)$ and I_1 are bounded by values*

$$c\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\} p_0(t,x,y) \quad \textit{and} \quad \frac{c}{\sqrt{t}}\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\} p_0(t,x,y)$$

correspondingly.

Proof. From the statement of Lemma 4 (since ϕ_2 is negative) an estimate follows

$$m(t,x,y) < cp(t,x,y) \int\limits_M \big(2\pi t b(x)\big)^{-n/2}$$
$$\times \exp\left\{\frac{\rho(x,y)\rho(x,z)(e(x,y),e(x,z))}{t} - \frac{\rho^2(x,z)}{2tb(x)}\right\}\sigma(dz)$$
$$= c\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\} p(t,x,y) \int\limits_M \big(2\pi t b(z)\big)^{n/2}$$
$$\times \exp\left\{-\frac{\|\rho(x,z)e(x,z) - b(x)\rho(x,y)e(x,y)\|}{2tb(x)}\right\}\sigma(dz).$$

A substitution

$$U = \frac{\rho(x,z)e(x,z) - b(x)\rho(x,y)e(x,y)}{\sqrt{tb(x)}},$$
$$z(U) = \operatorname{Exp}_x\big\{\sqrt{tb(x)}\,U + b(x)\rho(x,y)e(x,y)\big\}$$

transforms the last integral to

$$\int\limits_{T_xM} \left(\frac{b(x)}{b(z(U))}\right)^{n/2} J\big(z(U)\big)\,\mu_x(dU),$$

where μ_x is canonical measure on T_xM, and boundedness of Jacobian J is proved in [6]. Thus, the estimate of $m(t,x,y)$ is obtained. □

For estimating of I_1 we transform the function under integral sign.

$$\frac{\partial p_1}{\partial t}(t,x,z) = \left(-\frac{n}{2t} + \frac{\rho^2(x,z)}{2t^2 b(x)}\right) p_1(t,x,z);$$
$$\operatorname{grad}_x p_1(t,x,z) = \frac{1}{t}\left(\frac{\rho^2(x,z)\operatorname{grad} b(x)}{2b^2(x)} - \frac{\rho^2(x,z)\dot\gamma(\rho(x,y))}{b(x)}\right) p_1(t,x,z);$$
$$(\nabla_U b(x)\operatorname{grad}_x p_1(t,x,z), U)$$
$$= p_1(t,x,z)\left[\frac{1}{t^2 b(x)}\left(\rho^2(x,z)\big(\dot\gamma(\rho),U\big)^2 - \frac{\rho^3(x,z)(\operatorname{grad} b(x),U)(\dot\gamma(\rho),U)}{b(x)}\right.\right.$$

$$+ \frac{\rho^4(x,z)}{4b^2(x)}\big(\operatorname{grad} b(x), U\big)^2\bigg) + \frac{1}{t}\bigg(\frac{(\nabla_U \rho^2(x,z)\operatorname{grad} b(x), U)}{2b(x)} - \frac{\rho^2(x,z)(\operatorname{grad} b(x), U)^2}{2b^2(x)} - D(x)U\bigg)\bigg].$$

A summation with respect to orthogonal basis $\{e_k\}$ in T_xM $(e_1 = \dot\gamma(\rho))$ gives:

$$\frac{1}{2}\operatorname{div} b(x)\operatorname{grad} p_1(t,x,z) - \frac{\partial}{\partial t}p_1(t,x,z)$$
$$= \bigg(\frac{\rho(x,z)(\operatorname{grad} b(x), \dot\gamma)}{2t^2b^2(x)} + s\frac{\rho^4(x,z)\|\operatorname{grad} b(x)\|^2}{8t^2b^3(x)} - \frac{1}{2t}\operatorname{tr}\big(D(x) - I\big) + \frac{\rho(x,z)}{2tb(x)}\big(\operatorname{grad} b(x), \dot\gamma(\rho)\big) - \frac{\rho^2(x,z)}{4tb^2(x)}\big\|\operatorname{grad} b(x)\big\|^2 + \frac{\rho(x,z)}{4tb(x)}\,\Delta\, b(x)\bigg)p_1(t,x,z).$$

Obtained relation we estimate by the value $\frac{c}{\sqrt{t}}(1 + \frac{\rho^4(x,z)}{t^2})p_1(t,x,z)$, and as $(\frac{\rho(x,z)}{\sqrt{t}})^k\varphi_2(t,x,z)$ is bounded, then

$$(I_2) < \frac{c}{\sqrt{t}}\,p_0(t,x,y)\int_M \varphi_1(t,x,y,z)p_1(t,x,z)\,\sigma(dz).$$

The second statement of lemma follows.

We are coming to estimating of the item I_2, containing a logarithmic derivative

$$\Lambda(t,x,z) = -\frac{n}{2}\frac{\operatorname{grad} b(z)}{b(z)} - \frac{\rho(x,z)\dot\gamma(\rho(x,z))}{tb(x)} \in T_xM.$$

Lemma 6. *The following estimate is true*:

$$|I_2| \leqslant \frac{c}{\sqrt{t}}\exp\bigg\{\frac{\rho^2(x,y)b(x)}{2t}\bigg\}p_0(t,x,y).$$

Proof. Note, that

$$\big\|\Lambda(t,x,z)\big\|^2 + \operatorname{div}_z \Lambda(t,x,z)$$
$$= \bigg(\frac{n}{2} + \frac{n^2}{4}\bigg)\frac{\|\operatorname{grad} b(x)\|^2}{b^2(x)} + \frac{\rho^2(x,z)}{t^2b^2(x)} + \frac{n}{tb(x)b(z)}\rho(x,z)\big(\dot\gamma\big(\rho(x,z), \operatorname{grad} b(z)\big)\big) - \frac{n\Delta b(z)}{2b(z)} - \frac{\operatorname{tr} D(z)}{tb(x)}.$$

The second item under integral sign

$$\Delta_x p_1(t,x,z) = \bigg[\frac{1}{t^2}\bigg(\frac{\rho^4(x,z)\|\operatorname{grad} b(x)\|^2}{4b^4(x)} - \frac{\rho^3(x,z)(\operatorname{grad} b(x), \dot\gamma(\rho))}{b^3(x)} + \frac{\rho^2(x,z)}{b^2(x)}\bigg)$$

$$+\frac{1}{t}\left(\frac{2\rho(x,z)(\operatorname{grad} b(x),\dot{\gamma}(\rho))}{b^2(x)}+\frac{\rho^2(x,z)}{2b^2(x)}\Delta b(x)\right.$$
$$\left.\left.-\frac{\rho^2(x,z)}{b^3(x)}\|\operatorname{grad} b(x)\|^2-\frac{1}{b(x)}\operatorname{tr} D(x)\right)\right]p_1(t,x,z),$$

and difference

$$\Delta_x\, p_1(t,x,z)-p_1(t,x,z)\big(\big\|\Lambda(t,x,z)\big\|^2+\operatorname{div}_z \Lambda(t,x,z)\big)$$

is estimated by the value

$$\frac{c}{\sqrt{t}}\left(1+\frac{\rho^4(x,z)}{t^2}\right)p_1(t,x,z),$$

which has integrable with respect to t singularity. Desired statement is proved by the same way like in Lemma 5. □

Corollary 2. *Error $M(t,x,y)$, defined by* (9)*, satisfies an estimate*

$$M(t,x,y)<\frac{c}{\sqrt{t}}\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\}p_0(t,x,y).$$

Notation 3. Since the inequalities

$$b(x)\leqslant b_2<1 \quad \text{and} \quad p<q$$

hold, the function $\exp\{\frac{b(x)\rho^2(x,y)}{2t}\}$ is integrable with respect to measure $p_0(t,x,y)\,\sigma(dy)$.

Theorem 2. *Fundamental solution $p(t,x,y)$ of disturbed equation*

$$\frac{\partial u}{\partial t}=\frac{1}{2}\operatorname{div}\big(1+b(x)\big)\operatorname{grad} u$$

satisfies the estimate:

$$p(t,x,y) < c\exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\}q(t,x,y)$$
$$\leqslant c\exp\left\{\frac{b_2\rho^2(x,y)}{2t}+\varphi(x,y)+kt\right\}p_0(t,x,y),$$

where functions $q(t,x,y)$ and $\varphi(x,y)$ are determined by formulas (8) *and* (9) *correspondingly.*

Proof. We will prove a convergence of $\sum_{n=0}^{\infty} r_n$ under solution of Volterra's equation, estimating the items of series. From corollary of Lemmas 5 and 6 and the inequality $p\leqslant q$ an estimate follows:

$$|r_1(t,x,y)| = \left|\int_0^t d\tau\int_M M(t-\tau,x,z)M(\tau,z,y)\,\sigma(dz)\right|$$

$$< c^2 \int_0^t \frac{d\tau}{\sqrt{\tau(t-\tau)}} \int_M \bigl(4\pi^2\tau(t-\tau)\bigr)^{-n/2}$$
$$\times \exp\left\{-\frac{(1-b_2)}{2}\left(\frac{\rho^2(y,x)}{\tau} + \frac{\rho^2(x,z)}{t-\tau}\right)\right\}\sigma(dz).$$

Let us transform the inner integral by means substitution

$$u = \left(\sqrt{\frac{t}{\tau(t-\tau)}}\,\rho(x,z)e(x,z) - \sqrt{\frac{t-\tau}{t\tau}}\,\rho(x,y)e(x,y)\right)\sqrt{1-b_2},$$
$$z(u) = \mathrm{Exp}_x\left(\sqrt{\frac{\tau(t-\tau)}{t(1-b_2)}}u + \frac{t-\tau}{t}\rho(x,y)e(x,y)\right). \tag{10}$$

The argument of exponent equals here

$$-\frac{1}{2}\|u\|^2 - \frac{\rho^2(x,y)}{2t}(1-b_2) - \frac{(1-b_2)}{\tau}\bigl(\rho^2(y,z) - \rho^2(x,y) - \rho^2(x,z)$$
$$+ 2\rho(x,y)\rho(x,z)\bigl(e(x,y), e(x,z)\bigr)\bigr),$$

and the relation in brackets is non-negative because of curvature non-positivity. Thus,

$$\int_M (4\pi^2\tau(t-\tau))^{-n/2} \exp\left\{-\frac{(1-b_2)}{2}\left(\frac{\rho^2(y,z)}{\tau} + \frac{\rho^2(x,z)}{t-\tau}\right)\right\}\sigma(dz)$$
$$\leqslant (1-b_2)^{-n/2}\exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\}q(t,x,y)\int_{T_xM} J\bigl(z(u)\bigr)\,\mu_x(du).$$

So, we have

$$\bigl|r_1(t,x,y)\bigr| < c^2 c_1 \pi \exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\}q(t,x,y),$$

where $c_1 = (1-b_2)^{-n/2}\sup_{z\in M} J(z)$.

It's easy to obtain an estimate

$$\bigl|r_n(t,x,y)\bigr| < \frac{c^{n+1}c_1^n \pi^{(n+1)/2} t^{(n-1)/2}}{\Gamma((n+1)/2)} \exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\}q(t,x,y),$$

providing with absolute convergence of series $\sum_{n=0}^{\infty} r_n(t,x,y)$, $0 < t \leqslant T$ and for the sum of series

$$\bigl|r(t,x,y)\bigr| < \frac{c}{\sqrt{t}}\exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\}q(t,x,y).$$

Then

$$p(t,x,y) = m(t,x,y) + \int_0^t d\tau \int_M m(t-\tau,x,z) r(\tau,z,y)\,\sigma(dz)$$
$$< c\exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\} q(t,x,y)\big(1+\sqrt{t}\big),$$

and this implies, finally, the statement of theorem. □

References

[1] V. Bondarenko, Diffusion sur variete de courbure non positive, CR AS 324 (10) (1997) 1099–1103.
[2] V. Bondarenko, Estimates for heat Kernel on a manifold of nonpositive curvature, Ukrainian Math. J. 50 (8) (1998) 1129–1136.
[3] V. Bondarenko, Method of parametrix for parabolic equation on the Riemann manifold, Ukrainian Math. J. 51 (11) (1999) 1443–1448.
[4] V. Bondarenko, Logarithmic gradient of heat kernel on the Riemann manifold, Notes Math. 73 (5) (2003), in print.
[5] A. Debiard, D. Gaveau, E. Mazet, Theoremes de comparaison en geometrie Riemannienne, Publ. RIMS 12 (2) (1976) 391–425.
[6] S.T. Yau, On the heat kernel of complete Riemannian manifold, J. Math Pure Appl. Ser. 9 57 (2) (1978) 191–201.