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OPTIMAL CONTROL OF STOCHASTIC SYSTEMS WITH INTERRUPTED OBSERVATION†

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Abstract—An optimal control problem is considered for a nonlinear stochastic system with an interrupted observation mechanism that is characterized in terms of a jump Markov process taking on the values 0 or 1. The state of the system is described by a diffusion process, but the observation has components modulated by the jump process. The admissible control laws are smooth functions of the observation. Using the calculus of variations, necessary conditions on optimal controls are derived. These conditions amount to solving a set of four coupled nonlinear partial differential equations. A numerical procedure for solving these equations is suggested and an example dealt with numerically.

1. INTRODUCTION

In this paper consideration is given to a stochastic optimal control problem for a nonlinear stochastic system with an interrupted observation mechanism that is characterized in terms of a jump Markov process taking on the values 0 or 1. The state of the system is described by a diffusion process, but the observation has the component modulated by the jump process. Such a problem arises in systems with observation devices where the signal process is subjected to random attenuation or fading.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider the controlled dynamical system represented by the stochastic differential equation

$$dx_i = [f_i(x) + v_i(y)]\,dt + \sigma_i(x)\,dW_i, \quad t > 0,\ i = 1, \ldots, m \tag{1}$$

and let the interrupted observation be given by

$$dy_i = zx_i\,dt + \gamma_i(x)\,dB_i, \quad t > 0,\ i = 1, \ldots, m \tag{2}$$

where f_i: $\mathbf{R}^m \to \mathbf{R}$, σ_i: $\mathbf{R}^m \to \mathbf{R}$, γ_i: $\mathbf{R}^m \to \mathbf{R}$, $i = 1, \ldots, m$ are given functions. v_i: $\mathbf{R}^m \to \mathbf{R}$, $i = 1, \ldots, m$ are the control functions and $W = \{W(t) = (W_1(t), \ldots, W_m(t)),\ t \geq 0\}$ and $B = \{B(t) = (B_1(t), \ldots, B_m(t)),\ t \geq 0\}$ are two $\mathbf{R}^m$- valued standard Wiener processes on $(\Omega, \mathcal{F}, P)$. $Z = \{z(t),\ t \geq 0\}$ is a homogeneous jump Markov process on $(\Omega, \mathcal{F}, P)$ with state space $S = \{0, 1\}$ and transition probabilities

$$P(z(t+\Delta) = j | z(t) = i) = \begin{cases} q\Delta + o(\Delta) & \text{if } j \neq i \\ 1 - q\Delta + o(\Delta) & \text{if } j = i \end{cases} \tag{3}$$

$i, j = 0, 1$

where $\pi_i = P(z(0) = i)$, $i = 0, 1$, and $q > 0$ are given. It is assumed that the processes W, B and Z are mutually independent.

It is further assumed that σ_i and γ_i, $i = 1, \ldots, m$ are twice continuously differentiable for all $x \in \mathbf{R}^m$ and that f_i, $i = 1, \ldots, m$ are continuously differentiable for all $x \in \mathbf{R}^m$. In addition it is assumed that

$$|f(x)|^2 + |\sigma(x)|^2 + |\gamma(x)|^2 \leq \alpha + \beta|x|^2, \quad \alpha > 0, \quad \beta > 0 \tag{4}$$

and

$$|f(x) - f(x')|^2 + |\sigma(x) - \sigma(x')|^2 + |\gamma(x) - \gamma(x')|^2 \leq k_0|x - x'|^2, \quad k_0 \geq 0 \tag{5}$$

for all $x, x' \in \mathbf{R}^m$, where $|x|^2 = \sum_{i=1}^m x_i^2$, $|f(x)|^2 = \sum_{i=1}^m f_i^2(x)$, $|\sigma(x)|^2 = \sum_{i=1}^m \sigma_i^2(x)$ and $|\gamma(x)|^2 = \sum_{i=1}^m \gamma_i^2(x)$.

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Denote by U the class of all the control functions $v = v(y)$, $v = (v_1, \ldots, v_m)$: $\mathbf{R}^m \to \mathbf{R}^m$ that satisfy

$$|v(y)|^2 \leq \alpha + \beta|y|^2, \quad \alpha > 0, \quad \beta > 0 \tag{6}$$

and

$$|v(y) - v(y')|^2 \leq k_0|y - y'|^2, \quad k_0 > 0 \tag{7}$$

for all $y, y' \in \mathbf{R}^m$.

Let $v \in U$ and $\bar{x} = (x, y) \in \mathbf{R}^{2m}$. Then in the same manner as in[1] and [2] it can be shown that equations (1) and (2) have a unique solution $\zeta^v_{\bar{x}} = \{\zeta^v_{\bar{x}}(t) = (X_1{}^v(t), \ldots, X_m{}^v(t), Y^v_1(t), \ldots, Y^v_m(t)), t \geq 0\}$ which is such that $\zeta^v_{\bar{x}}(0) = \zeta_{\bar{x}}(0) = \bar{x}$. Also, in the same manner as in[1] it can be shown that $(\zeta^v_{\bar{x}}, z)$ is a Markov process on $(\Omega, \mathcal{F}, P)$. Furthermore, by following the same reasoning as in[3] (Section 5, Chap. 1) and using Theorem 3.10 of[4] it can be shown that $(\zeta^v_{\bar{x}}, z)$ is a strong Markov process. Note that the sample functions of $\{\zeta^v_{\bar{x}}(t), t \geq 0\}$ are continuous with probability 1.

Let

$$D_0 = \{\bar{x} = (x, y): |x_i| < l \text{ and } |y_i| < l, \quad i = 1, \ldots, m\} \tag{8}$$

$$D = D_0 - D_c \tag{9}$$

where D_c is a closed domain in $\mathbf{R}^{2m}$, and $D_c \subset D_0$. Define

$$\tau_i(\bar{x}; v) \triangleq \begin{cases} \inf\{t: \quad (\zeta^v_{\bar{x}}(t), z(t)) \in \partial D \times S \quad \text{when} \quad (\zeta_{\bar{x}}(0), z(0)) = (\bar{x}, i) \in D \times S\} \\ 0 \qquad \text{if} \quad \zeta_{\bar{x}}(0) = \bar{x} \notin D \quad \text{and} \quad z(0) = i. \\ \infty \quad \text{if} \quad \zeta^v_{\bar{x}}(t) \in D \quad \text{for all} \quad t \geq 0 \quad \text{when} \quad (\zeta_{\bar{x}}(0), z(0)) = (\bar{x}, i) \in D \times S \end{cases} \tag{10}$$

$i = 0, 1$ where ∂D denotes the boundary of D.

In the sequel the following notations will be used:

$$P_{\bar{x},i}(\cdot) = P(\cdot|(\zeta_{\bar{x}}(0), z(0)) = (\bar{x}, i)), i = 0, 1, \tag{11}$$

and

$$E_{\bar{x},i} = E[\,\cdot\,|(\zeta_{\bar{x}}(0), z(0)) = (\bar{x}, i), i = 0, 1. \tag{12}$$

where E denotes the expectation operator.

Define the following functionals

$$V(\bar{x}, i; v) \triangleq P_{\bar{x},i}(\{\zeta^v_{\bar{x}}(\tau_i(\bar{x}; v)) \in D_c\}) - E_{\bar{x},i} \int_0^{\tau_i(\bar{x}; v)} \sum_{j=1}^m \lambda_j(z(t)) v_j^2(Y^v(t))\, dt$$
$$v \in U, \quad i = 0, 1 \tag{13}$$

and

$$V(\bar{x}; v) \triangleq \sum_{i=0}^{1} P(z(0) = i) V(\bar{x}, i; v) = \sum_{i=0}^{1} \pi_i V(\bar{x}, i; v) \tag{14}$$

where $Y^v = (Y^v_1, \ldots, Y^v_m)$. $\lambda_j(\cdot)$, $j = 1, \ldots, m$, are given functions satisfying $\lambda_j(0) \geq \lambda_j(1) > 0$, $j = 1, \ldots, m$.

In this paper the following optimal control problem is treated. Find a control law $v^* \in U$ such that

$$V(\bar{x}; v^*) \geq V(\bar{x}; v) \text{ for any } v \in U \text{ and all } \bar{x} \in D. \tag{15}$$

A control law $v^* \in U$ for which equation (15) is satisfied will here be called an *optimal control.*

An optimal control $v^* \in U$, whenever it exists, is supposed, rougly speaking, to steer the state $\zeta_{\bar{x}}^v(t)$ in such a manner as to maximize the probability that the state reaches the set D_c before reaching ∂D_0 and subject to 'soft' constraints on the control function given by the second term in equation (13).

Optimal control problems for systems with jump Markov disturbances were considered by several authors during the last two decades (see for example[5–8]. and the references cited there). In these references the process $(\zeta_{\bar{x}}^v, z)$ is completely observable and the admissible control laws are of the form $v_t = v(\zeta_{\bar{x}}^v(t), z(t))$. These make it possible to find dynamic programming type of conditions on optimal control laws. In the case dealt with here the admissible control laws are of the form $v_t = v(Y^v(t))$, where Y^v is given by (2). This excludes the possibility of deriving implementable conditions of dynamic programming type for the problem considered here. Even in the case of linear systems the problem of estimating $\zeta_{\bar{x}}^v(t)$ from $\{Y^v(s),\ 0 \leq s \leq t\}$ results in an infinite-dimensional filter[9]. This excludes the possibility of applying control laws of the form $v_t = v(\hat{\zeta}_{\bar{x}}^v(t), \hat{z}(t))$, where $\hat{\zeta}_{\bar{x}}^v(t) = E[\zeta_{\bar{x}}^v(t) | Y^v(s), 0 \leq s \leq t]$ and $\hat{z}(t) = E[z(t) | Y^v(s), 0 \leq s \leq t]$.

In the present paper the problem is formulated in terms of a pair of coupled partial differential equations, the coefficients of which involve the control function v. By varying v we vary the coefficients of the infinitesimal generator of $(\zeta_{\bar{x}}^v(t), z(t))$, and this makes it possible to deduce necessary conditions for an optimal control (Section 3, Theorem 1). These necessary conditions amount to solving a set of four coupled nonlinear partial differential equations.

2. THE GOVERNING EQUATIONS

Let $\mathscr{D}$ denote the class of all pairs $(V(\bar{x}, 0),\ V(\bar{x}, 1))$ such that $V(\bar{x}, i)$, $i = 0, 1$, are continuous on $\bar{D}$ ($\bar{D}$ denotes the closure of D), twice continuously differentiable on D, and such that $\partial V(\bar{x}, k)/\partial x_i$, $\partial V(\bar{x}, k)/\partial y_i$, $\partial^2 V(\bar{x}, k)/\partial x_i^2$ and $\partial^2 V(\bar{x}, k)/\partial y_i^2$ are in $L_2(D)$ for $i = 1, \ldots, m$ and $k = 0, 1$.

By using the same method as in[3] (Chap. 1, Section 5) for deriving the weak infinitesimal operator of $(\zeta_{\bar{x}}^v, z)$ (see also[5]) and using the fact that $(\zeta_{\bar{x}}^v, z)$ is a strong Markov process, the following equations are obtained

$$E_{\bar{x},i} V(\zeta_{\bar{x}}^v(\tau_i(\bar{x}; v)),\ z(\tau_i(\bar{x}; v))) = V(\bar{x}, i) + E_{\bar{x},i} \int_0^{\tau_i(\bar{x}; v)} \mathscr{L}_{z(t)}(v) V(\zeta_{\bar{x}}^v(t), z(t))\, \mathrm{d}t \tag{16}$$

$$i = 0, 1$$

where

$$\begin{aligned} \mathscr{L}_0(v) V(\bar{x}, 0) = &\sum_{i=1}^m [f_i(x) + v_i(y)] \partial V(\bar{x}, 0)/\partial x_i \\ &+ \left(\frac{1}{2}\right) \sum_{i=1}^m (\sigma_i^2(x) \partial^2 V(\bar{x}, 0)/\partial x_i^2 + \gamma_i^2(x) \partial^2 V(\bar{x}, 0)/\partial y_i^2 \\ &- q V(\bar{x}, 0) + q V(\bar{x}, 1) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \mathscr{L}_1(v) V(\bar{x}, 1) = &\sum_{i=1}^m [f_i(x) + v_i(y)] \partial V(\bar{x}, 1)/\partial x_i + \sum_{i=1}^m x_i \partial V(\bar{x}, 1)/\partial y_i \\ &+ \left(\frac{1}{2}\right) \sum_{i=1}^m (\sigma_i^2(x) \partial^2 V(\bar{x}, 1)/\partial x_i^2 + \gamma_i^2(x) \partial^2 V(\bar{x}, 1)/\partial y_i^2) \\ &- q V(\bar{x}, 1) + q V(\bar{x}, 0), \quad (V(\bar{x}, 0),\ V(\bar{x}, 1)) \in \mathscr{D}. \end{aligned} \tag{18}$$

We introduce the notation $V_i(\bar{x}; v) = V(\bar{x}, i; v)$, $i = 0, 1$. In the sequel the following lemma will be used, the proof of which is based on equations (16)–(18) and is given in the Appendix.

LEMMA 1

Given $v \in U$. Let $(V_0, V_1) \in \mathscr{D}$ satisfy

$$\mathscr{L}_0(v)V_0(\bar{x}) = \sum_{j=1}^{m} \lambda_j(0)v_j^2(y) \qquad \bar{x} \in D \tag{19}$$

$$\mathscr{L}_1(v)V_1(\bar{x}) = \sum_{j=1}^{m} \lambda_j(1)v_j^2(y) \qquad \bar{x} \in D \tag{20}$$

$$V_0(\bar{x}) = V_1(\bar{x}) = 1 \;\; \bar{x} \in D_c; \quad V_0(\bar{x}) = V_1(\bar{x}) = 0 \;\; \bar{x} \in \partial D_0 \tag{21}$$

then

$$V_i(\bar{x}) = V_i(\bar{x}; v) = P_{\bar{x},i}(\{\zeta_{\bar{x}}^v(\tau_i(\bar{x}; v)) \in D_c\}) - E_{\bar{x},i}\int_0^{\tau_i(\bar{x};v)} \sum_{j=1}^{m} \lambda_j(z(t))v_j^2(Y^v(t))\,dt \qquad i = 0, 1. \tag{22}$$

3. NECESSARY CONDITIONS ON OPTIMAL CONTROLS

Let $v \in U$. Define

$$J(v) = \int_{D_0} V(\bar{x}; v)\,d\bar{x} \quad (d\bar{x} = dx_1 \dots dx_m\, dy_1 \dots dy_m). \tag{23}$$

Suppose that $v^*, v^0 \in U$ are control laws such that $V(\bar{x}; v^*) \geq V(\bar{x}; v)$ for any $v \in U$ and all $\bar{x} \in D$ and $J(v^0) \geq J(v)$ for any $v \in U$. Then it can be shown that $J(v^0) = J(v^*)$ and consequently that $V(\bar{x}; v^0) = V(\bar{x}; v^*)$ a.e. in D_0. Hence a control law $v^* \in U$ that maximizes $J(v)$ on U, whenever it exists, can be interpreted as an optimal control in some weak sense. In this section conditions are derived for the maximization of $J(v)$ on U.

Suppose that $v^0 \in U$ is a control law for which $J(v^0) \geq J(v)$ for any $v \in U$. Let $v^\alpha(y) = v^0(y) + \alpha\psi(y)$, where $\psi \in U$ and $\alpha \in [0, \alpha_0]$ for some $\alpha_0 > 0$. Assume that for each $\alpha \in [0, \alpha_0]$ there exists a solution $(V_0^\alpha, V_1^\alpha) \in \mathscr{D}$ to equations (24)

$$\begin{aligned} &\mathscr{L}_0(v^\alpha)V_0^\alpha(\bar{x}) = \sum_{j=1}^{m} \lambda_j(0)(v_j^\alpha(y))^2 && \bar{x} \in D \\ &\mathscr{L}_1(v^\alpha)V_1^\alpha(\bar{x}) = \sum_{j=1}^{m} \lambda_j(1)(v_j^\alpha(y))^2 && \bar{x} \in D \\ &V_0^\alpha(\bar{x}) = V_1^\alpha(\bar{x}) = 1 \;\; \bar{x} \in D_c; \quad V_0^\alpha(\bar{x}) = V_1^\alpha(\bar{x}) = 0 && \bar{x} \in \partial D_0. \end{aligned} \tag{24}$$

Define, for $v \in U$ and $(V_0, V_1) \in \mathscr{D}$

$$L_0(v)V_0 \triangleq \mathscr{L}_0(v)V_0 - qV_1 \tag{25}$$

$$L_1(v)V_1 \triangleq \mathscr{L}_1(v)V_1 - qV_0 \tag{26}$$

and

$$\begin{aligned} L_0^*(v)Q_0(\bar{x}) \triangleq -\sum_{i=1}^{m} [\partial(f_i(x)Q_0(\bar{x}))/\partial x_i + v_i(y)\partial Q_0(\bar{x})/\partial x_i] \\ + \left(\frac{1}{2}\right)\sum_{i=1}^{m} [\partial^2(\sigma_i^2(x)Q_0(\bar{x}))/\partial x_i^2 + \gamma_i^2(x)\partial^2 Q_0(\bar{x})/\partial y_i^2] \\ - qQ_0(\bar{x}) \end{aligned} \tag{27}$$

$$L_1^*(v)Q_1(\bar{x}) \triangleq -\sum_{i=1}^{m} [\partial(f_i(x)Q_1(\bar{x}))/\partial x_i + v_i(y)\,\partial Q_1(\bar{x})/\partial x_i + x_i\partial Q_1(\bar{x})/\partial y_i]$$
$$+\left(\frac{1}{2}\right)\sum_{i=1}^{m} [\partial^2(\sigma_i^2(x)Q_1(\bar{x}))/\partial x_i^2 + \gamma_i^2(x)\partial^2 Q_1(\bar{x})/\partial y_i^2] \qquad (28)$$
$$-qQ_1(\bar{x}),$$

for any (Q_0, Q_1) such that $L_i^*(v)Q_i \in L_2(D_0)$, $i = 0, 1$.

From equations (24) it follows that

$$\mathscr{L}_0(v^0)(V_0^\alpha - V_0^0) + (\mathscr{L}_0(v^\alpha) - \mathscr{L}_0(v^0))V_0^\alpha - \sum_{i=1}^{m} \lambda_i(0)[(v_i^\alpha)^2 - (v_i^0)^2]I_d = 0 \text{ a.e. in } D_0 \qquad (29)$$

$$\mathscr{L}_1(v^0)(V_1^\alpha - V_1^0) + (\mathscr{L}_1(v^\alpha) - \mathscr{L}_1(v^0))V_1^\alpha - \sum_{i=1}^{m} \lambda_i(1)[(v_i^\alpha)^2 - (v_i^0)^2]I_D = 0 \text{ a.e. in } D_0, \qquad (30)$$

where $I_D = 1$ if $\bar{x} \in D$ and $I_D = 0$ if $\bar{x} \notin D$.

Let Q_0 and Q_1 be weak solutions of the following equations:

$$L_0^*(v^0)Q_0(\bar{x}) = -1 \qquad \bar{x} \in D_0 \qquad (31)$$

$$L_1^*(v^0)Q_1(\bar{x}) = -1 \qquad \bar{x} \in D_0 \qquad (32)$$

$$Q_0(\bar{x}) = Q_1(\bar{x}) = 0 \qquad \bar{x} \in \partial D_0. \qquad (33)$$

Multiplying equation (29) by $\pi_0 Q_0$ and equation (30) by $\pi_1 Q_1$, adding the two expressions, integrating their sum over D_0, and then using equations (17)–(18), (25)–(26) and (31)–(33), the following equation is obtained:

$$J(v^\alpha) - J(v^0) = q\int_{D_0} \{\pi_0 Q_0(\bar{x})(V_1^\alpha(\bar{x}) - V_1^0(\bar{x})) + \pi_1 Q_1(\bar{x})(V_0^\alpha(\bar{x}) - V_0^0(\bar{x}))\}\,d\bar{x}$$
$$+\alpha\sum_{i=1}^{m}\int_{D_y} \psi_i(y)\int_{D_x} \{\pi_0 Q_0(\bar{x})\partial V_o^\alpha(\bar{x})/\partial x_i + \pi_1 Q_1(\bar{x})\partial V_1^\alpha(\bar{x})/\partial x_i \qquad (35)$$
$$-2v_i^0(y)(\lambda_i(0)\pi_0 Q_0(\bar{x}) + \lambda_i(1)\pi_1 Q_1(\bar{x}))I_D(\bar{x})\}\,dx\,dy$$
$$-\alpha^2\sum_{i=1}^{m}\int_{D_0} \{\pi_0\lambda_i(0)Q_0(\bar{x}) + \pi_1\lambda_i(1)Q_1(\bar{x})\}\psi_i^2(y)I_D(\bar{x})\,d\bar{x},$$

where

$$D_x = \{x: |x_i| < l \quad i = 1, \ldots, m\}\; D_y = \{y: |y_i| < l \quad i = 1, \ldots, m\}. \qquad (35)$$

THEOREM 1

Suppose there exists a control law $v^0 \in U$ such that

$$J(v^0) \geq J(v) \text{ for all } v \in U \qquad (36)$$

and assume that (i) equations (31)–(33) have weak solutions Q_0 and Q_1; (ii) for each $\alpha \in [0, \alpha_0]$, $(V_0^\alpha, V_1^\alpha) \in \mathscr{D}$ satisfy equations (24), where $v^\alpha = v^0 + \alpha\psi$, and $\psi \in U$; (iii) $(V_i^\alpha - V_i^0)/\alpha$, $i = 0, 1$, converge weakly (in $L_2(D_0)$) as $\alpha \downarrow 0$ too; (iv) $\partial V_i^\alpha/\partial x_j$, $j = 1, \ldots, m$, $i = 0, 1$ converge weakly (in $L_2(D_0)$), as $\alpha \downarrow 0$ to $\partial V_i^0/\partial x_j$, $j = 1, \ldots, m$, $i = 0, 1$, respectively.

Then

$$v_i^0(y) = \left(\frac{1}{2}\right)\int_{D_x} \{\pi_0 Q_0(\bar{x})\partial V_0^0(\bar{x})/\partial x_i$$
$$+\pi_1 Q_1(\bar{x})\partial V_1^0(\bar{x})/\partial x_i\}\,dx \Big/ \int_{D_x} [\lambda_i(0)\pi_0 Q_0(\bar{x}) + \lambda_i(1)\pi_1 Q_1(\bar{x})]I_D(\bar{x})\,dx \qquad (37)$$

$i = 1, \ldots, m$.

Proof. From assumptions (i)–(iv) and equations (34) it follows that

$$\delta J(v^0; \psi) = \lim_{\alpha \to 0} (J(v^\alpha) - J(v^0))/\alpha$$
$$= \sum_{i=1}^{m} \int_{D_y} \psi_i(y) \left\{ \int_{D_x} [\pi_0 Q_0(\bar{x}) \partial V_0^0(\bar{x})/\partial x_i + \pi_1 Q_1(\bar{x}) \partial V_1^0(\bar{x})/\partial x_i] \, dx \right. \tag{38}$$
$$\left. - 2 v_i^0(y) \int_{D_x} [\lambda_i(0)\pi_0 Q_0(\bar{x}) + \lambda_i(1)\pi_1 Q_1(\bar{x})] I_D(\bar{x}) \, dx \right\} dy,$$

where $\delta J(v^0; \psi)$ is the Gateaux differential of J at v^0 with increment ψ, [10]. From Theorem 1 (p. 178) of [10] it follows that if equation (36) is satisfied then $\delta J(v^0; \psi) = 0$ for all $\psi \in U$. Hence v^0 has to satisfy equation (37).

Thus if it is assumed that an optimal control law exists and that all the conditions stated in Theorem 1 are satisfied, then in order to determine an optimal control law the following system of equations has to be solved:

$$\mathscr{L}_0(v) V_0(\bar{x}) = \sum_{j=1}^{m} \lambda_j(0) v_j^2(y) \qquad \bar{x} \in D \tag{39}$$

$$\mathscr{L}_1(v) V_1(\bar{x}) = \sum_{j=1}^{m} \lambda_j(1) v_j^2(y) \qquad \bar{x} \in D \tag{40}$$

$$L_0^*(v) Q_0(\bar{x}) = -1 \qquad \bar{x} \in D_0 \tag{41}$$

$$L_1^*(v) Q_1(\bar{x}) = -1 \qquad \bar{x} \in D_0 \tag{42}$$

$$V_0(\bar{x}) = V_1(\bar{x}) = 1 \quad \bar{x} \in D_c; \quad V_0(\bar{x}) = V_1(\bar{x}) = Q_0(\bar{x}) = Q_1(\bar{x}) = 0 \quad \bar{x} \in \partial D_0 \tag{43}$$

where

$$v_i(y) = \left(\frac{1}{2}\right) \int_{D_x} \{\pi_0 Q_0(\bar{x}) \partial V_0(\bar{x})/\partial x_i + \pi_1 Q_1(\bar{x}) \partial V_1(\bar{x})/\partial x_i\} \, dx \Big/ \int_{D_x} [\lambda_i(0)\pi_0 Q_0(\bar{x}) + \lambda_i(1)\pi_1 Q_1(\bar{x})] I_D(\bar{x}) \, dx \tag{44}$$

$i = 1, \ldots, m$, $(dx = dx_1, \ldots, dx_m)$.

Equations (39)–(44) are a set of nonlinear partial differential equations. Since these constitute necessary conditions for optimality, it appears that the problem of the existence and uniqueness of solutions to these equations is crucial to the optimal control problem. Owing to the state of the art of the theory of nonlinear partial differential equations no efforts are made here to establish such conditions. Instead, a finite difference scheme for the solution of this set of equations is suggested, and a numerical example will be solved for various cases.

4. THE FINITE DIFFERENCE SCHEME

Let $\mathbf{R}_h^{2m}$ be a finite difference grid on $\mathbf{R}^{2m}$, with a constant mesh size h along all axes. Define $D_{oh} \triangleq \mathbf{R}_h^{2m} \cap D_0$, $D_{ch} \triangleq \mathbf{R}_h^{2m} \cap D_c$, $D_h \triangleq \mathbf{R}_h^{2m} \cap D$ and $\partial D_{oh} \triangleq \mathbf{R}_h^{2m} \cap \partial D_0$. Denote by e^i the unit vector along the ith axis, $\bar{x} = \sum_{i=1}^{2m} \bar{x}_i e^i$ ($\bar{x}_i = x_i$, $i = 1, \ldots, m$ and $\bar{x}_i = y_{i-m}$, $i = m+1, \ldots, 2m$). Let $\bar{x}$ be an internal point of D_{oh}. Using the approximations

$$g(\bar{x}) \partial F(\bar{x})/\partial \bar{x}_i \to \begin{cases} g(\bar{x})(F(\bar{x} + e^i h) - F(\bar{x}))/h & \text{if } g(\bar{x}) \geq 0 \\ g(\bar{x})(F(\bar{x}) - F(\bar{x} - e^i h))/h & \text{if } g(\bar{x}) < 0 \end{cases} \tag{45}$$

$i = 1, \ldots, 2m$

and

$$\partial^2 F(\bar{x})/\partial \bar{x}_i^2 \to (F(\bar{x}+e^i h)+F(\bar{x}-e^i h)-2F(\bar{x}))/h^2 \qquad i=1,\ldots,2m \tag{46}$$

equations (39)–(43) are replaced by

$$V_0(\bar{x}) = \sum_{i=1}^{2m} (P_{0,i}(\bar{x})V_0(\bar{x}+e^i h) + P_{0,-i}(\bar{x})V_0(\bar{x}-e^i h)) + qh^2 V_1(\bar{x})/R_0(\bar{x}) - \sum_{i=1}^{m} \lambda_i(0)v_i^2(y)h^2/R_0(\bar{x},) \qquad \bar{x}\in D_h \tag{47}$$

$$V_1(\bar{x}) = \sum_{i=1}^{2m} (P_{1,i}(\bar{x})V_1(\bar{x}+e^i h) + P_{1,-i}(\bar{x})V_1(\bar{x}-e^i h)) + qh^2 V_0(\bar{x})/R_1(\bar{x}) - \sum_{i=1}^{m} \lambda_i(1)v_i^2(y)h^2/R_1(\bar{x}), \qquad \bar{x}\in D_h \tag{48}$$

$$Q_0(\bar{x}) = \sum_{i=1}^{2m} (S_{0,i}(\bar{x})Q_0(\bar{x}+e^i h) + S_{0,-i}(\bar{x})Q_0(\bar{x}-e^i h)) + h^2/S_0(\bar{x}), \qquad \bar{x}\in D_{0h} \tag{49}$$

$$Q_1(\bar{x}) = \sum_{i=1}^{2m} (S_{1,i}(\bar{x})Q_1(\bar{x}+e^i h) + S_{1,-i}(\bar{x})Q_1(\bar{x}-e^i h)) + h^2/S_1(\bar{x}), \qquad \bar{x}\in D_{0h} \tag{50}$$

$$V_0(\bar{x}) = V_1(\bar{x}) = 1 \quad \bar{x}\in D_{ch}; \quad V_0(\bar{x}) = V_1(\bar{x}) = Q_0(\bar{x}) = Q_1(\bar{x}) = 0, \qquad x\in \partial D_{0h} \tag{51}$$

$$R_0(\bar{x}) = \sum_{i=1}^{m} (\sigma_i^2(x)+\gamma_i^2(x)) + h\sum_{i=1}^{m} (|f_i(x)|+|v_i(y)|) + qh^2 \tag{52}$$

$$R_1(\bar{x}) = \sum_{i=1}^{m} (\sigma_i^2(x)+\gamma_i^2(x)) + h\sum_{i=1}^{m} (|f_i(x)|+|v_i(y)|+|x_i|) + qh^2 \tag{53}$$

$$P_{0,i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^+(x)+v_i^+(y))]/R_0(\bar{x}), & i=1,\ldots,m \\ \gamma_{i-m}^2(x)/(2R_0(\bar{x}) & i=m+1,\ldots,2m \end{cases} \tag{54}$$

$$P_{0,-i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^-(x)+v_i^-(y))]/R_0(\bar{x}), & i=1,\ldots,m \\ \gamma_{i-m}^2(x)/(2R_0(\bar{x})), & i=m+1,\ldots,2m \end{cases} \tag{55}$$

$$P_{1,i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^+(x)+v_i^+(y))]/R_1(\bar{x}), & i=1,\ldots,m \\ [\gamma_{i-m}^2(x)/2 + hx_{i-m}^+]/R_1(\bar{x}), & i=m+1,\ldots,2m \end{cases} \tag{56}$$

$$P_{1,-i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^-(x)+v_i^-(y))]/R_1(\bar{x}), & i=1,\ldots,m \\ [\gamma_{i-m}^2(x)/2 + hx_{i-m}^-]/R_1(\bar{x}), & i=m+1,\ldots,2m \end{cases} \tag{57}$$

$$S_0(\bar{x}) = \sum_{i=1}^{m} (\sigma_i^2(x)+\gamma_i^2(x)+h^2c_i(x)-h^2a_i(x)/2) + h\sum_{i=1}^{m} (|f_i(x)|+|v_i(y)|+|b_i(x)|) + qh^2 \tag{58}$$

$$S_1(\bar{x}) = \sum_{i=1}^{m} (\sigma_i^2(x)+\gamma_i^2(x)+h^2c_i(x)-h^2a_i(x)/2) + h\sum_{i=1}^{m} (|f_i(x)|+|v_i(y)|+|b_i(x)|+|x_i|) + qh^2 \tag{59}$$

$$a_i(x) \triangleq \partial^2\sigma_i^2(x)/\partial x_i^2; \quad b_i(x) \triangleq \partial\sigma_i^2(x)/\partial x_i; \quad c_i(x) \triangleq \partial f_i(x)/\partial x_i \quad i=1,\ldots,m \tag{60}$$

$$S_{0,i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^-(x)+v_i^-(y)+b_i^+(x))]/S_0(\bar{x}), & i=1,\ldots,m \\ \gamma_{i-m}^2(x)/(2S_0(\bar{x})), & i=m+1,\ldots,2m \end{cases} \tag{61}$$

$$S_{0,-i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^+(x)+v_i^+(y)+b_i^-(x))]/S_0(\bar{x}), & i=1,\ldots,m \\ \gamma_{i-m}^2(x)/(2S_0(\bar{x})), & i=m+1,\ldots,2m \end{cases} \tag{62}$$

$$S_{1,i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^-(x) + v_i^-(y) + b_i^+(x))]/S_1(\bar{x}), & i = 1, \ldots, m \\ [\gamma_{i-m}^2(x)/2 + hx_{i-m}^-]/S_1(\bar{x}), & i = m+1, \ldots, 2m \end{cases} \tag{63}$$

$$S_{1,-i}(\bar{x}) = \begin{cases} [\sigma_i^2(x)/2 + h(f_i^+(x) + v_i^+(y) + b_i^-(x))]/S_1(\bar{x}), & i = 1, \ldots, m \\ [\gamma_{i-m}^2(x)/2 + hx_{i-m}^+]/S_1(\bar{x}), & i = m+1, \ldots, 2m \end{cases} \tag{64}$$

and for any real λ, $\lambda^+ = \max(0, \lambda)$; $\lambda^- = -\min(0, \lambda)$. v is a control law given by (44).

Equations (47)–(51) are solved by an iterative procedure using the underrelaxation technique with an acceleration factor w_0, until the difference between two consecutive iterations does not exceed a given tolerance ϵ_0.

Henceforward we assume that equations (47)–(51) have a unique solution $(V_0^h, V_1^h, Q_0^h, Q_1^h, v^h)$.

5. PROBABILISTIC INTERPRETATION

In this section we show that equations (47)–(48) and (51) (and under certain assumptions equations (47)–(51)) have a probabilistic interpretation.

Let $C_{0h} = \{\eta_{\bar{x}} \in \mathbf{R}_h^{2m}: \bar{x} \in D_{0h}\}$, $\partial C_{0h} = \{\eta_{\bar{x}} \in \mathbf{R}_h^{2m}: \bar{x} \in \partial D_{0h}\}$, $C_h = \{\eta_{\bar{x}} \in \mathbf{R}_h^{2m}: \bar{x} \in D_h\}$ and $T_{ch} = \{\eta_{\bar{x}} \in \mathbf{R}_h^{2m}: \bar{x} \in D_{ch}\}$ be such that (i) $C_{0h} \cup \partial C_{0h}$ and $D_{0h} \cup \partial D_{0h}$ are disjoint; (ii) $\eta_{\bar{x}} \neq \eta_{\bar{x}'}$, iff $\bar{x} \neq \bar{x}'$; $\bar{x} \pm e^i h \Leftrightarrow n_{\bar{x}} \pm e^i h$, $i = 1, \ldots, 2m$; (iii) $T_{ch} \subset C_{0h}$; $C_h \subset C_{0h}$; $C_h = C_{0h} - T_{ch}$. Define the function Γ^h as follows:

$$\Gamma^h(\bar{x}) = V_0^h(\bar{x}) \qquad \bar{x} \in D_h \tag{65}$$

$$\Gamma^h(\eta_{\bar{x}}) = V_1^h(\bar{x}) \qquad \bar{x} \in D_h \tag{66}$$

$$\Gamma^h(\bar{x}) = 1 \quad \bar{x} \in D_{ch};\ \Gamma^h(\eta_{\bar{x}}) = 1 \quad \eta_{\bar{x}} \in T_{ch};\quad \Gamma^h(\bar{x}) = 0 \quad \bar{x} \in \partial D_{0h};$$
$$\Gamma^h(\eta_{\bar{x}}) = 0 \quad \eta_{\bar{x}} \in \partial C_{0h}. \tag{67}$$

In a similar manner to [11], we define, for every $\bar{x} \in \mathbf{R}_h^{2m}$:

$$p^h(\bar{x}, \bar{x} + e^i h) \triangleq P_{0,i}(\bar{x}), \qquad i = 1, \ldots, 2m, \quad \bar{x} \in D_h \tag{68}$$

$$p^h(\bar{x}, \bar{x} - e^i h) \triangleq P_{0,-i}(\bar{x}), \qquad i = 1, \ldots, 2m, \quad \bar{x} \in D_h \tag{69}$$

$$p^h(\bar{x}, \eta_{\bar{x}}) \triangleq qh^2/R_0(\bar{x}),\ \bar{x} \in D_h \tag{70}$$

$$p^h(\eta_{\bar{x}}, \eta_{\bar{x}} + e^i h) \triangleq P_{1,i}(\bar{x}), \qquad i = 1, \ldots, 2m, \quad \eta_{\bar{x}} \in C_h \tag{71}$$

$$p^h(\eta_{\bar{x}}, \eta_{\bar{x}} - e^i h) \triangleq P_{1,-i}(\bar{x}), \qquad i = 1, \ldots, 2m, \quad \eta_{\bar{x}} \in C_h \tag{72}$$

$$p^h(\eta_{\bar{x}}, \bar{x}) \triangleq qh^2/R_1(\bar{x}),\ \eta_{\bar{x}} \in C_h \tag{73}$$

$$p^h(\bar{x}, \bar{x}') \triangleq 0 \quad \text{if} \quad \bar{x}' \notin \{\bar{x} \pm e^i h,\ \ i = 1, \ldots, 2m\} \cup C_h \quad \text{or} \quad \bar{x} \notin D_h \tag{74}$$

$$p^h(\eta_{\bar{x}}, \eta_{\bar{x}'}) \triangleq 0 \quad \text{if} \quad \eta_{\bar{x}'} \notin \{\eta_{\bar{x}} \pm e^i h,\ \ i = 1, \ldots, 2m\} \cup D_h \quad \text{or} \quad \eta_{\bar{x}} \notin C_h. \tag{75}$$

Then equations (47)–(48), for $V_i = V_i^h$, $i = 0, 1$, can be written as

$$\Gamma^h(\bar{x}) = \sum_{i=1}^{2m} (p^h(\bar{x}, \bar{x} + e^i h)\Gamma^h(\bar{x} + e^i h) + p^h(\bar{x}, \bar{x} - e^i h)\Gamma^h(\bar{x} - e^i h))$$
$$+ p^h(\bar{x}, \eta_{\bar{x}})\Gamma^h(\eta_{\bar{x}}) - \sum_{i=1}^{m} \lambda_i(0)v_i^2(y)\Delta t^h(\bar{x}), \quad \bar{x} \in D_h \tag{76}$$

$$\Gamma^h(\eta_{\bar{x}}) = \sum_{i=1}^{2m} (p^h(\eta_{\bar{x}}, \eta_{\bar{x}} + e^i h)\Gamma^h(\eta_{\bar{x}} + e^i h) + p^h(\eta_{\bar{x}}, \eta_{\bar{x}} - e^i h)\Gamma^h(\eta_{\bar{x}} - e^i h))$$

$$+ p^h(\eta_{\bar{x}}, \bar{x})\Gamma^h(\bar{x}) - \sum_{i=1}^{m} \lambda_i(1) v_i^2(y)\Delta t^h(\eta_{\bar{x}}), \quad \eta_{\bar{x}} \in C_h \tag{77}$$

where

$$\Delta t^h(\bar{x}) \triangleq h^2/R_0(\bar{x}); \quad \Delta t^h(\eta_{\bar{x}}) \triangleq h^2/R_1(\bar{x}). \tag{78}$$

The set of functions $\{p^h(\bar{x}, \bar{\chi}),\ \bar{x}, \bar{\chi} \in \mathbf{R}_h^{2m}\}$ can be regarded as a family of transition probabilities for a Markov chain $\{Z_n^h,\ n = 0, 1, 2, \ldots\}$ defined on the discrete state space $\mathbf{R}_h^{2m}$.

Define

$$N^h(\bar{x}) \triangleq \min \{n\colon Z_n^h \notin D_h \cup C_h \quad \text{when} \quad Z_0^h = \bar{x}\}, \quad \bar{x} \in D_h \tag{79}$$

$$N^h(\eta_{\bar{x}}) \triangleq \min \{n\colon Z_n^h \notin D_h \cup C_h \quad \text{when} \quad Z_0^h = \eta_{\bar{x}}\}, \eta_{\bar{x}} \in C_h \tag{80}$$

and assume that $E_{\bar{x}}N^h(\bar{x}) < \infty$ and $E_{\eta_{\bar{x}}}N^h(\eta_{\bar{x}}) < \infty$ for all $\bar{x} \in D_h$, and $\eta_{\bar{x}} \in C_h$.

Then it can be shown, in the same manner as in[11], that

$$\Gamma^h(\bar{\chi}) = E_{\bar{\chi}}\Phi(Z^h_{N^h(\bar{\chi})}) - E_{\bar{\chi}} \sum_{k=0}^{N^h(\bar{\chi})-1} \sum_{i=1}^{m} \lambda_i(\bar{k}) v_i^2(Y_k^h)\Delta t^h(Z_k^h), \quad \bar{\chi} \in D_h \cup C_h \tag{81}$$

where

$$\lambda_i(\bar{k}) = \begin{cases} \lambda_i(0) & \text{if} \quad Z_k^h \in D_h \\ \lambda_i(1) & \text{if} \quad Z_k^h \in C_h \end{cases}, \tag{82}$$

$$Y_k^h = \begin{cases} (Z^h_{k,m+1}, \ldots, Z^h_{k,2m}) & \text{if} \quad Z_k^h \in D_h \\ (y_1, \ldots, y_m) & \text{if} \quad Z_k^h = \eta_{\bar{x}} \in C_h \end{cases} \tag{83}$$

($\bar{x} = (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$), and $\Phi\colon \mathbf{R}^{2m} \to \mathbf{R}$ is a bounded and continuous function such that: $\Phi(\bar{\chi}) = 1\ \bar{\chi} \in D_{ch} \cup T_{ch}$; $\Phi(\bar{\chi}) = 0\ \bar{\chi} \notin D_{0h} \cup C_{0h}$.

In the same manner as in the Appendix it can be shown that

$$\Gamma^h(\bar{\chi}) = P_{\bar{\chi}}(\{Z^h_{N^h(\bar{\chi})} \in D_{ch} \cup T_{ch}\}) - E_{\bar{\chi}} \sum_{k=0}^{N^h(\bar{\chi})-1} \sum_{i=1}^{m} \lambda_i(\bar{k}) v_i^2(Y_k^h)\Delta t^h(Z_k^h), \quad \bar{\chi} \in D_h \cup C_h. \tag{84}$$

As in[11], define the time sequence $\{t_n^h\}$ and the interpolated continuous parameter process $\{\zeta^h(t),\ t \geq 0\}$, by

$$t_0^h = 0; \quad t_n^h = \sum_{k=0}^{n-1} \Delta t^h(Z_k^h) \quad 0 \leq n \leq N^h(\bar{\chi}) + 1 \tag{85}$$

$$\zeta^h(t) = Z_n^h \quad t \in [t_n^h, t^h_{n+1}). \tag{86}$$

Then equation (84) can be written as

$$\Gamma^h(\bar{\chi}) = P_{\bar{\chi}}(\{\zeta^h(N^h(\bar{\chi})) \in D_{ch} \cup T_{ch}\}) - E_{\bar{\chi}} \int_0^{N^h(\bar{\chi})} \sum_{i=1}^{m} \lambda_i(t) v_i^2(Y_t^h)\, \mathrm{d}t, \quad \bar{\chi} \in D_h \cup C_h. \tag{87}$$

where

$$\lambda_i(t) = \begin{cases} \lambda_i(0) & \text{if} \quad \zeta^h(t) \in D_h \\ \lambda_i(1) & \text{if} \quad \zeta^h(t) \in C_h \end{cases} \tag{88}$$

and

$$Y_t^h = \begin{cases} (\zeta_{m+1}^h(t), \ldots, \zeta_{2m}^h(t)) & \text{if} \quad \zeta^h(t) \in D_h \\ (y_1, \ldots, y_m) & \text{if} \quad \zeta^t(t) = \eta_{\bar{\chi}} \in C_h \end{cases} \tag{89}$$

Note that $\Gamma^h(\bar{\chi}) \cong V_0(\bar{\chi}; v^0)$ if $\bar{\chi} \in D_h$, and $\Gamma^h(\bar{\chi}) \cong V_1(\bar{\chi}; v^0)$ if $\bar{\chi} \in C_h$.

The probabilistic interpretation of equations (47), (48) and (51), given here by means of the Markov chain $\{Z_n^h, n = 0, 1, 2, \ldots\}$ and equations (84) and (87), can be extended in the following case.

Consider the system given by (1) where $a_i(x) = c_i(x) = 0, i = 1, \ldots, m, x \in \mathbf{R}^m$ (equations (60)). In this case equations (47)–(51) have a probabilistic interpretation by means of three Markov chains. These are $\{Z_n^h, n = 0, 1, 2, \ldots\}$, described above the two additional Markov chains $\{\bar{X}_h^{(i)}(n), n = 0, 1, 2, \ldots\}$, $i = 0, 1$, described below.

Let $\eta^{(0)}, \eta^{(1)} \in \mathbf{R}_h^{2m}$ be such that $\eta^{(0)}, \eta^{(1)} \notin D_{0h} \cup \partial D_{0h}$, define the values of Q_0^h and Q_1^h on $\eta^{(0)}$ and $\eta^{(1)}$ respectively by

$$Q_i^h(\eta^{(i)}) = 0, \quad i = 0, 1. \tag{90}$$

We define, for every $\bar{x} \in \mathbf{R}_h^{2m}$ and each $i \in \{0, 1\}$

$$q_i^h(\bar{x}, \bar{x} + e^j h) \triangleq S_{i,j}(\bar{x}) \qquad j = 1, \ldots, 2m \qquad \bar{x} \in D_{0h} \tag{91}$$

$$q_i^h(\bar{x}, \bar{x} - e^j h) \triangleq S_{i,-j}(\bar{x}), \qquad j = 1, \ldots, 2m \qquad \bar{x} \in D_{0h} \tag{92}$$

$$q_i^h(\bar{x}, \eta^{(i)}) \triangleq q h^2 / S_i(\bar{x}), \quad \bar{x} \in D_{0h} \tag{93}$$

$$q_i^h(\bar{x}, \eta) \triangleq 0 \text{ if } \eta \notin \{\bar{x} \pm e^j h, j = 1, \ldots, 2m\} \cup \{\eta^{(i)}\} \text{ or } \bar{x} \notin D_{0h} \tag{94}$$

Then equations (49)–(51), for $Q_i = Q_i^h$ $i = 0, 1$, can be written as

$$\begin{aligned} Q_i^h(\bar{x}) = \sum_{j=1}^{2m} (q_i^h(\bar{x}, \bar{x} + e^j h) Q_i^h(\bar{x} + e^j h) + q_i^h(\bar{x}, \bar{x} - e^j h) Q_i^h(\bar{x} - e^j h)) \\ + q_i^h(\bar{x}, \eta^{(i)}) Q_i^h(\eta^{(i)}) + \Delta T_i^h(\bar{x}), \quad \bar{x} \in D_{0h}, \quad i = 0, 1 \end{aligned} \tag{95}$$

$$Q_0^h(\bar{x}) = Q_1^h(\bar{x}) = 0 \qquad \bar{x} \in \partial D_{0h}, \tag{96}$$

where

$$\Delta T_i^h(\bar{x}) \triangleq h^2 / S_i(\bar{x}), \qquad i = 0, 1. \tag{97}$$

Again, for each $i \in \{0, 1\}$, the set of functions $\{q_i^h(\bar{x}, \eta), \bar{x}, \eta \in \mathbf{R}_h^{2m}\}$ can be regarded as a family of transition probabilities for a corresponding Markov chain $\{\bar{X}_h^{(i)}(n), n = 0, 1, 2, \ldots\}$ defined on the discrete state space $\mathbf{R}_h^{2m}$.

Define

$$M_h^{(i)}(\bar{x}) \triangleq \min\{n: \bar{X}_h^{(i)}(n) \notin D_{0h} \text{ when } \bar{X}_h^{(i)}(0) = \bar{x}\}, \quad i = 0, 1 \tag{98}$$

and assume that $E_{\bar{x}} M_h^{(i)}(\bar{x}) < \infty$ for all $\bar{x} \in D_{0h}$, $i = 0, 1$. Then, as before, it can be shown that

$$Q_i^h(\bar{x}) = E_{\bar{x}} M_h^{(i)}(\bar{x}), \quad i = 0, 1. \tag{99}$$

Hence in the case where $a_i(\cdot) = c_i(\cdot) = 0$, $i = 1, \ldots, m$ (equations (60)) the control law

$v^0 \in U$, given by equation (37), is represented by

$$v_i^0(y) \cong \left(\frac{1}{2}\right) \int_{D_x} \sum_{k=0}^{1} \pi_k E_{\bar{x}} M_h^{(k)}(\bar{x})(\partial V_k^h(\bar{x})/\partial x_i)\,\mathrm{d}x \Big/ \int_{D_x} \sum_{k=0}^{1} \lambda_i(k) \pi_k E_{\bar{x}} M_h^{(k)}(\bar{x}) I_D(\bar{x})\,\mathrm{d}x$$
$$i = 1, \ldots, m \qquad y \in \mathbf{R}_h^m \cap D_y \tag{100}$$

where the operation $\int_{D_x} \mathrm{d}x$ here represents an appropriate numerical integration operation.

The results described in the next section are to be interpreted in the light of equations (87) and (98)–(99), where $\Gamma^h(\bar{x})$ and $\Gamma^h(\eta_{\bar{x}})$ are approximations to $V(\bar{x}, 0; v^0)$ and $V(\bar{x}, 1; v^0)$(eqns. (13)) respectively.

6. EXAMPLE

Consider the dynamical system given by

$$\mathrm{d}x = v(y)\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1, \quad t > 0, x \in \mathbf{R} \tag{101}$$

with the observation

$$\mathrm{d}y = zx\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2, \quad t > 0, y \in \mathbf{R} \tag{102}$$

where $W = \{W(t) = (W_1(t), W_2(t)), t \geq 0\}$ is an $\mathbf{R}^2$-valued standard Wiener process and $Z = \{z(t), t \geq 0\}$ is a homogeneous jump Markov process with state space $S = \{0, 1\}$ as described in Section 1. We assume that W and Z are mutually independent. Let U, D_0, and D be as defined in Section 1 (but with $m = 1$) and let

$$D_c = \{\bar{x} = (x, y): \quad |x| \leq \rho \quad \text{and} \quad |y| \leq l - \epsilon\} \tag{103}$$

where ϵ is a given number, $l \gg \epsilon > 0$.

Assuming that there exists a control law $v^0 \in U$ such that $V(\bar{x}; v^0) \geq V(\bar{x}; v)$ for all $v \in U$ and $\bar{x} \in D$ (and consequently $J(v^0) \geq J(v)$ for any $v \in U$), and that all the conditions stated in Theorem 1 are satisfied, it follows from equations (39)–(44) and (101)–(102) that the following set of equations has to be solved:

$$v(y)\partial V_0(\bar{x})/\partial x + \left(\frac{1}{2}\right)(\sigma_1^2 \partial^2 V_0(\bar{x})/\partial x^2 + \sigma_2^2 \partial^2 V_0(\bar{x})/\partial y^2) - qV_0(\bar{x}) + qV_1(\bar{x})$$
$$= \lambda(0) v^2(y) \qquad \bar{x} \in D \tag{104}$$

$$v(y)\partial V_1(\bar{x})/\partial x + x\partial V_1(\bar{x})/\partial y + \left(\frac{1}{2}\right)(\sigma_1^2 \partial^2 V_1(\bar{x})/\partial x^2 + \sigma_2^2 \partial^2 V_1(\bar{x})/\partial y^2)$$
$$- qV_1(\bar{x}) + qV_0(\bar{x}) = \lambda(1) v^2(y) \qquad \bar{x} \in D \tag{105}$$

$$- v(y)\partial Q_0(\bar{x})/\partial x + \left(\frac{1}{2}\right)(\sigma_1^2 \partial^2 Q_0(\bar{x})/\partial x^2 + \sigma_2^2 \partial^2 Q_0(\bar{x})/\partial y^2) - qQ_0(\bar{x}) = -1$$
$$\bar{x} \in D_0 \tag{106}$$

$$- v(y)\partial Q_1(\bar{x})/\partial x - x\partial Q_1(\bar{x})/\partial y + \left(\frac{1}{2}\right)(\sigma_1^2 \partial^2 Q_1(\bar{x})/\partial x^2 + \sigma_2^2 \partial^2 Q_1(\bar{x})/\partial y^2)$$
$$- qQ_1(\bar{x}) = -1 \qquad \bar{x} \in D_0 \tag{107}$$

$$V_0(\bar{x}) = V_1(\bar{x}) = 1 \quad \bar{x} \in D_c; \quad V_0(\bar{x}) = V_1(\bar{x}) = Q_0(\bar{x}) = Q_1(\bar{x}) = 0 \quad \bar{x} \in \partial D_0 \tag{108}$$

$$v(y) = \left(\frac{1}{2}\right) \int_{-l}^{l} \{\pi_0 Q_0(\bar{x}) \partial V_0(\bar{x})/\partial x$$
$$+ \pi_1 Q_1(\bar{x}) \partial V_1(\bar{x})/\partial x\}\,\mathrm{d}x \Big/ \int_{-l}^{l} [\lambda(0)\pi_0 Q_0(\bar{x}) + \lambda(1)\pi_1 Q_1(\bar{x})] I_D(\bar{x})\,\mathrm{d}x. \tag{109}$$

Note that in this example $a(x) = \partial^2\sigma_1^2(x)/\partial x^2 \equiv 0$, $b(x) = \partial\sigma_1^2(x)/\partial x \equiv 0$ and $c(x) = \partial f(x)/\partial x \equiv 0$.

Let $(V_0^h, V_1^h, Q_0^h, Q_1^h, v^h)$ be the solution to equations (47)–(51) for a given set of parameters. Using the values of v^h in (52)–(57), and solving equations (47)–(48) and (51), for the same set of parameters but with $\lambda_i(0) = \lambda_i(1) = 0$, $i = 1, \ldots, m$, equations (47)–(48) and (51) yield a solution (P_0^h, P_1^h) with following probabilistic interpretation:

$$P_0^h(\bar{x}) = P_{\bar{x}}(\{\zeta^h(N^h(\bar{x})) \in D_{ch} \cup T_{ch} \text{ and the control law } v^h \text{ is being applied}\}) \quad (110)$$

$$P_1^h(\bar{x}) = P_{\eta_{\bar{x}}}(\{\zeta^h(N^h(\eta_{\bar{x}}) \in D_{ch} \cup T_{ch} \text{ and the control law } v^h \text{ is being applied}\}). \quad (111)$$

Once the values of (P_0^h, P_1^h) have been computed, it is possible to compute the values of P^h

$$P^h = \pi_0 P_0^h + \pi_1 P_1^h. \quad (112)$$

Using the finite difference scheme described in Section 4, equations (104)–(109) were solved and the values of $(V_0^h, V_1^h, Q_0^h, Q_1^h, v^h, P_0^h, P_1^h, P^h)$ computed for the following set of parameters: $l = 1$, $\epsilon = 10^{-4}$, $\rho = 0.1$, $\sigma_1^2 = 5 \cdot 10^{-4}$, $5 \cdot 10^{-10} \leq \sigma_2^2 \leq 0.5$, $\pi_0 = \pi_1 = 0.5$, $0.5 \leq \lambda(0) \leq \lambda(1) \leq 8$, $0.5 \leq q \leq 16$, $\epsilon_0 = 10^{-4}$ (ϵ_0 is the tolerance between two consecutive iterations, see the end of Section 4), and $h = 0.05$.

Table 1. The values of $\int_{D_0} V_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} Q_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} P_i^h(\bar{x})\,d\bar{x}$, $i = 0, 1$, for various values of h, where: $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$ and $q = 1$

h	$\int_{D_0} V_0^h(\bar{x})d\bar{x}$	$\int_{D_0} V_1^h(\bar{x})d\bar{x}$	$\int_{D_0} Q_0^h(\bar{x})d\bar{x}$	$\int_{D_0} Q_1^h(\bar{x})d\bar{x}$	$\int_{D_0} P_0^h(\bar{x})d\bar{x}$	$\int_{D_0} P_1^h(\bar{x})d\bar{x}$
0.1	.3164	.3084	.8806	.6997	.3238	.3144
0.05	.3143	.3080	.9258	.7352	.3224	.3147
0.025	.3133	.3083	.9463	.7527	.3213	.3149

Table 2. The values of $\int_{D_0} V_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} Q_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} P_i^h(\bar{x})\,d\bar{x}$, $i = 0, 1$, for various values of h, where: $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 8$ and $q = 1$

h	$\int_{D_0} V_0^h(\bar{x})d\bar{x}$	$\int_{D_0} V_1^h(\bar{x})d\bar{x}$	$\int_{D_0} Q_0^h(\bar{x})d\bar{x}$	$\int_{D_0} Q_1^h(\bar{x})d\bar{x}$	$\int_{D_0} P_0^h(\bar{x})d\bar{x}$	$\int_{D_0} P_1^h(\bar{x})d\bar{x}$
0.1	.1879	.1854	.8959	.6931	.1892	.1864
0.05	.1780	.1757	.9401	.7277	.1792	.1767
0.025	.1734	.1713	.9589	.7446	.1747	.1724

Table 3. The values of $\int_{D_0} V_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} Q_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} P_i^h(\bar{x})\,d\bar{x}$, $i = 0, 1$ for various values of σ_2^2, where: $\lambda(0) = \lambda(1) = 1$, $q = 1$ and $h = 0.05$

σ_2^2	$\int_{D_0} V_0^h(\bar{x})d\bar{x}$	$\int_{D_0} V_1^h(\bar{x})d\bar{x}$	$\int_{D_0} Q_0^h(\bar{x})d\bar{x}$	$\int_{D_0} Q_1^h(\bar{x})d\bar{x}$	$\int_{D_0} P_0^h(\bar{x})d\bar{x}$	$\int_{D_0} P_1^h(\bar{x})d\bar{x}$
$5 \cdot 10^{-10}$	.3156	.3091	.9299	.7359	.3237	.3157
$5 \cdot 10^{-8}$	.3156	.3091	.9299	.7359	.3237	.3157
$5 \cdot 10^{-6}$	.3156	.3091	.9298	.7359	.3237	.3157
$5 \cdot 10^{-4}$	.3143	.3080	.9256	.7352	.3224	.3147
$5 \cdot 10^{-2}$	.2298	.2289	.8062	.6823	.231C	.2299
$5 \cdot 10^{-1}$	.1387	.1385	.5009	.4643	.1387	.1385

In order to assess the accuracy of the numerical method, the values of $(V_0^h, V_1^h, Q_0^h, Q_1^h, v^h, P_0^h, P_1^h)$ were computed for $h = 0, 1, 0.05, 0.025$; the results are given in Tables 1 and 2.

The results given in these tables, and other results as well suggest that $V_i^h \to V_i$, $i = 0, 1$; $P_i^h \to P_i$, $i = 0, 1$ as $h \downarrow 0$ where

$$P_i(\bar{x}) \stackrel{\Delta}{=} P_{\bar{x},i}(\{\zeta_{\bar{x}}^{v^0}(\tau_i(\bar{x}; v^0)) \in D_c\}), \quad i = 0, 1. \tag{113}$$

The sensitivity of $(V_0^h, V_1^h, Q_0^h, Q_1^h, P_0^h, P_1^h)$ to variations in the value of σ_2^2, the noise factor in the observation, demonstrated in Table 3.

In Fig. 1 four samples of the control law $v^h = v^h(y)$ are given. These are the cases where: (i) $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = 1$, $\lambda(1) = 0.1$, $q = 1$ and $h = 0.05$. (ii) $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 0.8$, $q = 1$ and $h = 0.05$. (iii) $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$, $q = 1$ and $h = 0.05$. (iv) $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$, $q = 8$ and $h = 0.05$.

Figures. 2(a) and 2(b) show $\int_{D_0} V_i^h(\bar{x})\,d\bar{x}$, $\int_{D_0} P_i^h(\bar{x})\,d\bar{x}$, $i = 0, 1$, as functions of $\lambda = \lambda(0) = \lambda(1)$ for the cases where: $\sigma_2^2 = 5 \cdot 10^{-4}$, $q = 1$ and $h = 0.05$. The plots show that $\int_{D_0} V_i^h(\bar{x})\,d\bar{x}$ and $\int_{D_0} P_i^h(\bar{x})\,d\bar{x}$, $i = 0, 1$, decrease when λ increases. Figure 3 shows $\int_{D_0} P^h(\bar{x})\,d\bar{x}$ as a function of q for the case where: $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$ and $h = 0.05$. The plot shows that $\int_{D_0} P^h(\bar{x})\,d\bar{x}$ decreases when q increases.

Figure 1 shows the typical form of the control law v^h, and Figs. 2 and 3 shows the trend of all the results obtained here, which is that V_i^h and P_i^h, $i = 0, 1$, decrease when $\lambda = \lambda(0) = \lambda(1)$ or q increase.

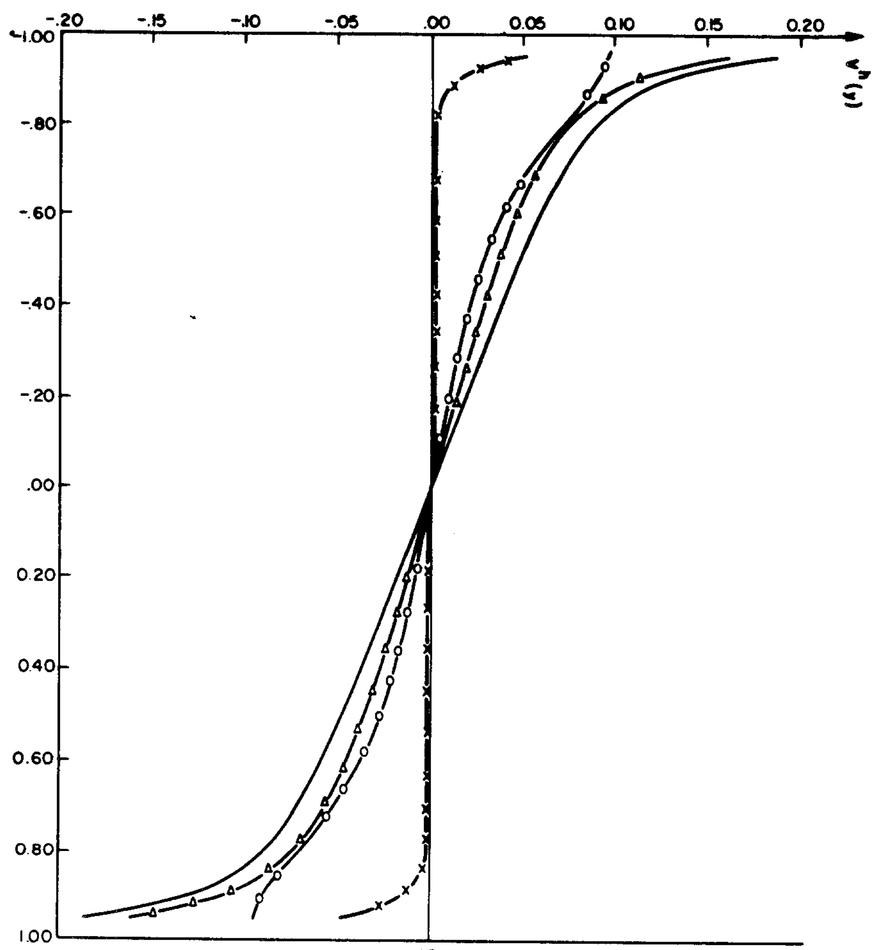

Fig. 1. The control law $v^h = v^h(y)$ for four cases. (i) ———, $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = 1$, $\lambda(1) = 0.1$, $q = 1$ and $h = 0.05$; (ii) -△-△-△-△, $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 0.8$, $q = 1$ and $h = 0.05$; (iii) -O-O-O-O, $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$, $q = 1$ and $h = 0.05$. (iv) -x-x-x-x, $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$, $q = 8$ and $h = 0.05$.

From equations (13), (65)–(66), (87) and (110)–(111), it follows that

$$P_0^h(\bar{x}) - V_0^h(\bar{x}) = E_{\bar{x}} \int_0^{N^h(\bar{x})} \sum_{i=1}^m \lambda_i(t) v_i^2(Y_t^h)\, dt \cong E_{\bar{x},0} \int_0^{\tau_0(\bar{x};\, v^0)} \sum_{j=1}^m \lambda_j(z(t)) v_j^2(Y^v(t))\, dt \quad (114)$$

$$P_1^h(\bar{x}) - V_1^h(\bar{x}) = E_{\eta_{\bar{x}}} \int_0^{N^h(\eta_{\bar{x}})} \sum_{i=1}^m \lambda_i(t) v_i^2(Y_t^h)\, dt \cong E_{\bar{x},1} \int_0^{\tau_1(\bar{x};\, v^0)} \sum_{j=1}^m \lambda_j(z(t)) v_j^2(Y^v(t))\, dt. \quad (115)$$

For the example solved here, the results given in Tables 1–3 and in Figs. 2(a) and 2(b), and other results as well, indicate that the numbers K_i^h, $i = 0, 1$, where

$$K_i^h \triangleq \int_{D_0} (P_i^h(\bar{x}) - V_i^h(\bar{x}))\, d\bar{x} = \int_{D_0} \left(E_{\bar{x},i} \int_0^{\tau_i(\bar{x};\, v^0)} \sum_{j=1}^m \lambda_j(z(t)) v_j^2(Y^v(t))\, dt \right) d\bar{x} \qquad i = 0, 1 \quad (116)$$

are very small in comparison with $\int_{D_0} V_i^h(\bar{x})\, d\bar{x}$ or $\int_{D_0} P_i^h(\bar{x})\, d\bar{x}$, $i = 0, 1$, respectively.

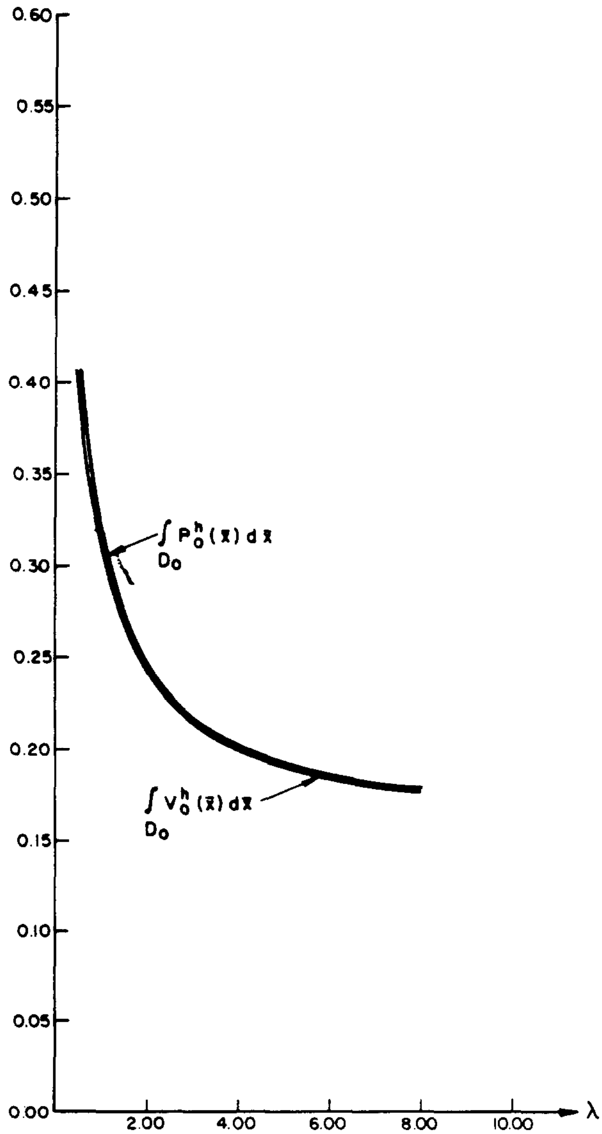

Fig. 2(a). $\int_{D_0} V_0^h(\bar{x})\, d\bar{x}$ and $\int_{D_0} P_0^h(\bar{x})\, d\bar{x}$ as functions of $\lambda = \lambda(0) = \lambda(1)$, where $\sigma_2^2 = 5 \cdot 10^{-4}$, $q = 1$ and $h = 0.05$.

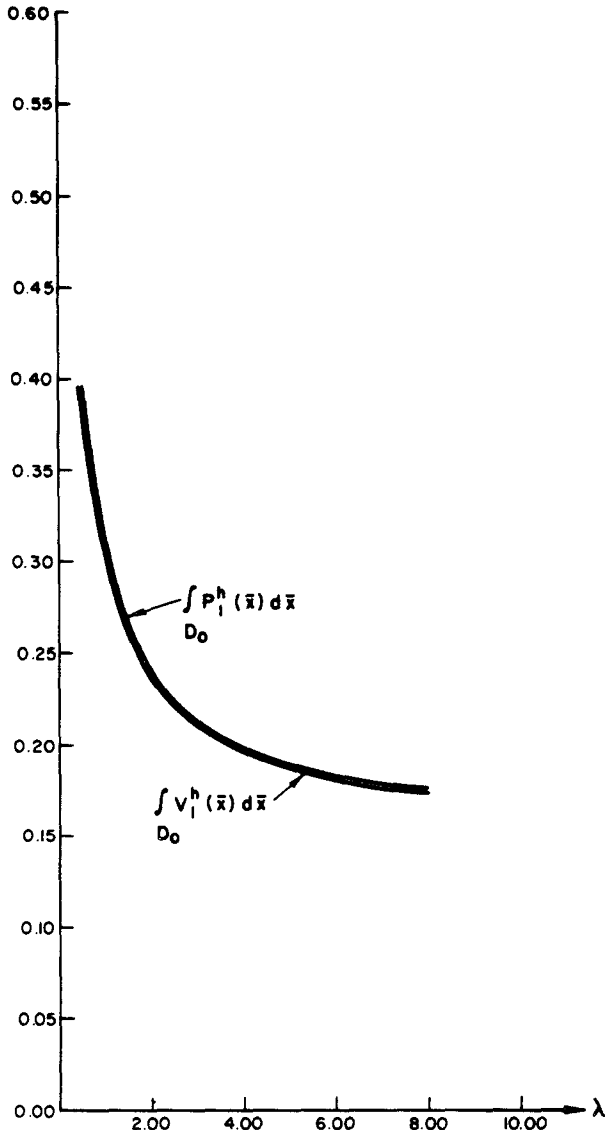

Fig. 2(b). $\int_{D_0} V_1{}^h(\bar{x})\,d\bar{x}$ and $\int_{D_0} P_1{}^h(\bar{x})\,d\bar{x}$ as functions of $\lambda = \lambda(0) = \lambda(1)$, where $\sigma_2{}^2 = 5 \cdot 10^{-4}$, $q = 1$ and $h = 0.05$.

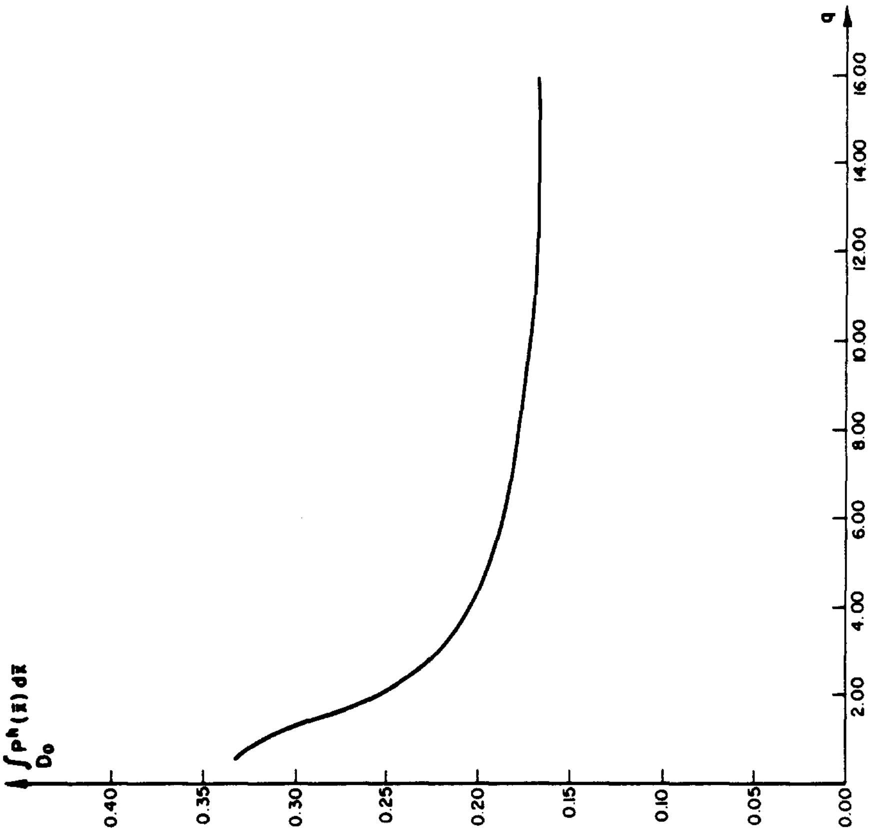

Fig. 3. $\int_{D_0} P^h(\bar{x})\,d\bar{x}$ as a function of q, $P^h = \pi_0 P_0^h + \pi_1 P_1^h$, where $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$ and $h = 0.05$.

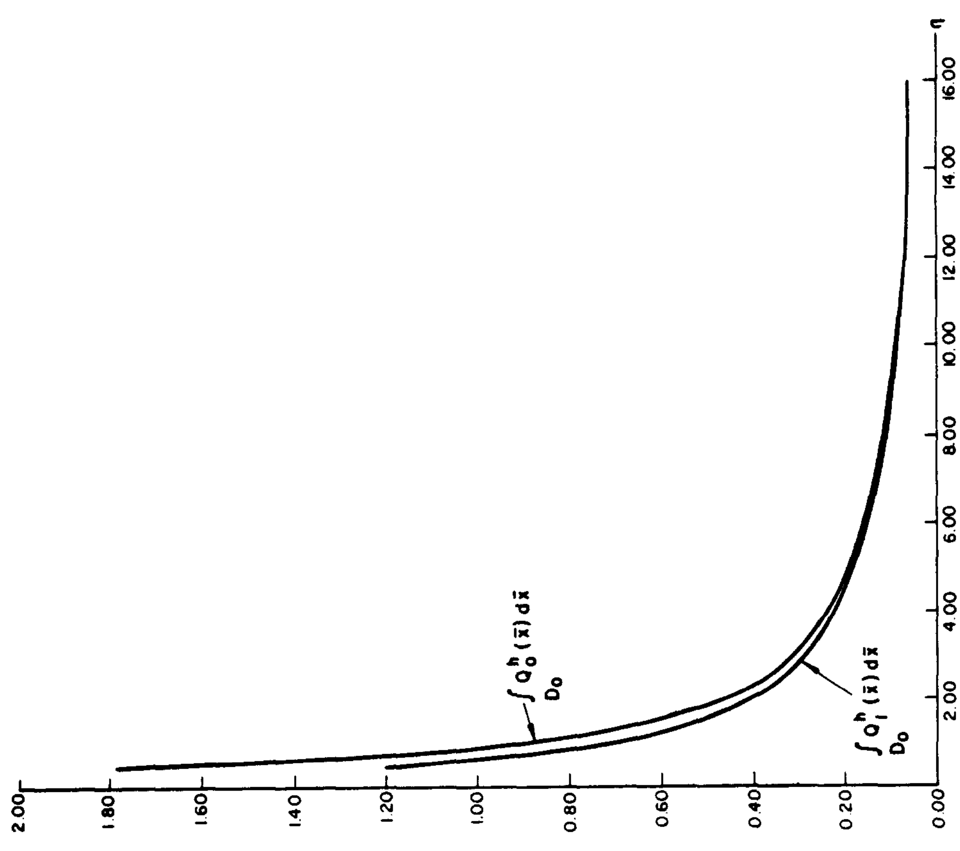

Fig. 4. $\int_{D_0} Q_i^h(\bar{x})\,d\bar{x}$, $i = 0, 1$, as functions of q, where $\sigma_2^2 = 5 \cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$ and $h = 0.05$.

Finally, the values of $\int_{D_0} Q_i^h(\bar{x})\,d\bar{x} = \int_{D_0} E_{\bar{x}} M_h^{(i)}(\bar{x})\,d\bar{x}$, $i = 0, 1$, as functions of q, in the case where $\sigma_2^2 = 5\cdot 10^{-4}$, $\lambda(0) = \lambda(1) = 1$ and $h = 0.05$, have been plotted in Fig. 4. Again, these values decrease when q increases.

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APPENDIX

Proof of Lemma 1

Let $v \in U$ and assume that $(V_0, V_1) \in \mathscr{D}$ satisfy equations (19)–(20). Then equation (16) yields

$$\begin{aligned} V_i(\bar{x}) &= E_{\bar{x},i} V_{z(\tau_i(\bar{x};\,v))}(\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v))) \\ &- E_{\bar{x},i} \int_0^{\tau_i(\bar{x};\,v)} \sum_{j=1}^{m} \lambda_j(z(t)) v_j^2(Y^v(t))\,dt, \quad i = 0, 1. \end{aligned} \tag{117}$$

Assume next that (V_0, V_1) also satisfy equations (21). Then

$$\begin{aligned} &E_{\bar{x},i} V_{z(\tau_i(\bar{x};\,v))}(\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v))) \\ &= \int_{\{\omega:\,\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v))(\omega) \in D_c\}} V_{z(\tau_i(\bar{x};\,v))}(\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v)))(\omega) P_{\bar{x},i}(d\omega) \\ &+ \int_{\{\omega:\,\zeta_{\bar{x}}^v(\tau_i(\bar{x};v))(\omega) \in \partial D_0\}} V_{z(\tau_i(\bar{x};v))}(\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v)))(\omega) P_{\bar{x},i}(d\omega) \\ &= \int_{\{\omega:\,\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v))(\omega) \in D_c\}} P_{\bar{x},i}(d\omega) = P_{\bar{x},i}(\{\zeta_{\bar{x}}^v(\tau_i(\bar{x};\,v)) \in D_c\}), \end{aligned} \tag{118}$$

which completes the proof of Lemma 1.